1. For every positive integer n, let $f_n(x)$ be the function on $\mathbb{R}$ given by $f_n(x) := \frac{x^n}{n!}$.

(a) Determine the subset $S \subseteq \mathbb{R}$ consisting of all of real numbers a such that the sequence $\big(f_n(a)\big)_{n\geq 1}$ converges. For each $a \in S$, find the limit.

(b) Determine whether the sequence of functions $\big(f_n(x)\big)_{n\geq 1}$ converges uniformly on the subset $S \subseteq \mathbb{R}$ defined in (a).

Justify your assertions.

Solution. (a) $S = \mathbb{R}$, and $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ for every $a \in \mathbb{R}$: for every $a \neq 0$ and every $n \geq 2\,|a|$, we have $\left|\frac{f_{n+1}(a)}{f_n(a)}\right| \leq \frac{|a|}{n+1} < \frac{1}{2}$.

(b) No. Otherwise there exists a positive integer $n_0 > 0$ such that $\left|\frac{a^n}{n!}\right| < 1$ for all $n \geq n_0$ and all $a \in \mathbb{R}$, which is obviously false for $a = n_0$.

2. Let $f(x) = x^6 - 1$.

(a) Find all maximal ideals in the polynomial ring $\mathbb{Q}[x]$ that contain $f(x)$.

(b) Find all maximal ideals in the polynomial ring $\mathbb{C}[x]$ that contain $f(x)$.

Solution. (a) $f(x) = (x-1)(x+1)(x^2+x+1)(x^2-x+1)$, and these 4 factors are all irreducible in $\mathbb{R}[x]$, therefore irreducible in $\mathbb{Q}[x]$. There are 4 maximal ideals in $\mathbb{Q}[x]$ which contain $f(x)$, generated by these 4 irreducible factors.

(b) In $\mathbb{C}[x]$ we have

$$f(x) = (x-1)(x+1)(x - e^{\pi\sqrt{-1}/3})(x - e^{-\pi\sqrt{-1}/3})(x - e^{2\pi\sqrt{-1}/3})(x - e^{-2\pi\sqrt{-1}/3}),$$

and there are 6 maximal ideals in $\mathbb{C}[x]$ which contain $f(x)$, generated by these 6 irreducible factors.

3. Let f be the function on $[-1,1]$ defined by

$$f(x) = \begin{cases} 3x^2 & \text{if } \ 0 < x \leq 1 \\ -1+2x & \text{if } \ -1 \leq x \leq 0 \end{cases}$$

Let F be the function on $[-1,1]$ defined by

$$F(x) = \int_{-1}^{x} f(t)\,dt.$$

(a) Is the function F continuous at $x = 0$?

(b) Is the function F differentiable at $x = 0$?

Justify your assertions.

Solution. (a) Yes: Clearly $|f(x)| \leq 3$ for all $x \in [-1,1]$. From

$$|F(x) - F(0)| = |\int_{0}^{x} f(t)\,dt| \leq 3\,|x|$$

for all $x \in [-1,1]$, one sees that $\lim_{x\to 0} F(x) = F(0)$.

(b) No, F is not differentiable: f is piecewise continuous on $[-1,1]$, so by the fundamental theorem of calculus we have

$$\lim_{x\to 0+} \frac{F(x)-F(0)}{x} = \lim_{x\to 0+} f(x) = 0,$$

and

$$\lim_{x\to 0^-} \frac{F(x)-F(0)}{x} = \lim_{x\to 0^-} f(x) = -1.$$

So $\lim_{x\to 0} \frac{F(x)-F(0)}{x}$ does not exist.

4. Let $\vec{v}_0$ be the vector $(1,1,1)$ in $\mathbb{R}^3$. Let T be the linear operator on $\mathbb{R}^3$ defined by cross product with $\vec{v}_0$:

$$T(\vec{w}) := \vec{v}_0 \times \vec{w}$$

for every element $\vec{w} \in \mathbb{R}^3$. For every real number $x \in \mathbb{R}$, define a linear operator U_x on $\mathbb{R}^3$ by

$$U_x := \exp(xT) = \sum_{n \geq 0} \frac{x^n\, T^n}{n!}.$$

(a) Find the matrix representation A of T with respect to the standard basis of $\mathbb{R}^3$.

(b) Show that for every $x \in \mathbb{R}$, the operator U_x is orthogonal.

(Recall that a real 3×3 matrix B is *orthogonal* if $B \cdot B^t = \mathrm{I}_3 = B^t \cdot B$; and an operator on $\mathbb{R}^3$ is *orthogonal* if it is equal to "multiplication on the left by an orthogonal 3×3 matrix".)

Solution. (a) Let $\vec{i}, \vec{j}, \vec{k}$ be the standard basis of $\mathbb{R}^3$. We have

$$T(\vec{i}) = -\vec{k} + \vec{j},\ T(\vec{j}) = -\vec{i} + \vec{k},\ T(\vec{k}) = -\vec{j} + \vec{i},$$

so $A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$. Notice that $A^t = -A$.

(b) Let B be the matrix representation of A. Then $\exp(xA)$ is the matrix representation of U_x. Since $A^t = -A$, we see that

$$\exp(xA^t) \cdot \exp(xA) = \exp(xA) \cdot \exp(xA^t) = \exp(xA + xA^t) = \exp(0 \cdot \mathrm{I}_3) = \mathrm{I}_3$$

Therefore U_x is orthogonal for every $x \in \mathbb{R}$.

5. Consider the improper integral

$$\iint_{\mathbb{R}^2} \frac{dx\,dy}{(1+x^2+y^2)^\alpha}$$

on $\mathbb{R}^2$, with a real parameter $\alpha > 0$. Determine whether this improper integral converges for $\alpha = 1$ and whether it converges for $\alpha = 2$.

(Hint: Use polar coordinates.)

Solution. We will determine all parameters $\alpha > 0$ for which the improper integral converges. The condition turns out to be $\alpha > 1$. So the improper integral diverges for $\alpha = 1$ and converges for $\alpha = 2$.

Since the integrand is non-negative, it suffices to show that there exists a positive real number $M_\alpha > 0$ such that

$$\iint_{D_c} \frac{dx\,dy}{(1+x^2+y^2)^\alpha} \leq M_\alpha$$

for every circular disk $D_c := \{(a,b) \in \mathbb{R}^2 \mid a^2+b^2 \leq c^2\}$ of radius $c > 0$ centered about the origin. Since $2^{-\alpha} \leq \frac{1}{(1+x^2+y^2)^\alpha} \leq 1$ for all (x,y) in the closed unit disk, the above condition is equivalent to the existence of a positive real number $N_\alpha > 0$ such that

$$\iint_{D_c \smallsetminus D_1} \frac{dx\,dy}{(1+x^2+y^2)^\alpha} \leq N_\alpha$$

for all $c > 1$. Using polar coordinates, we have

$$\iint_{D_c \smallsetminus D_1} \frac{dx\,dy}{(1+x^2+y^2)^\alpha} = 2\pi \int_1^c \frac{r\,dr}{(1+r^2)^\alpha}.$$

Since $\frac{1}{2} \leq \frac{1}{1+r^2} \leq 1$ for all $r \geq 1$, the condition is equivalent to the condition that

$$\int_1^c \frac{r\,dr}{r^{2\alpha}}$$

is bounded as $c \to \infty$, which is easily seen to be equivalent to $\alpha > 1$.

Note: In this solution, it's not actually necessary to split off the unit disc; one can apply the polar coordinates argument for the integral from 0 to ∞ (or from 0 to c and take a limit).

6. Let $E := \{(x, y) \in \mathbb{R}^2 \,|\, x^2+xy+y^2 \leq 1\}$, with the topology given by the standard metric on $\mathbb{R}^2$. Let E^0 be the interior of the subset $E \subseteq \mathbb{R}^2$.

(Recall that E^0 is the subset consisting of all points $P \in E$ such that there exists $\epsilon > 0$ such that the open disk $D(P; \epsilon)$ in $\mathbb{R}^2$ of radius $\epsilon > 0$ centered at P is contained in E.)

(a) Determine whether E (respectively E^0) is connected, and whether E (respectively E^0) is compact. (Your answer should have 4 parts.)

(b) Prove that $E^0 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + xy + x^2 < 1\}$, and E is equal to the closure in $\mathbb{R}^2$ of E^0.

Solution. (a) The quadratic form $Q(x, y) := x^2+xy+y^2$ on $\mathbb{R}^2$ is positive definite, because $x^2 + xy + y^2 = (x + \frac{y}{2})^2 + \frac{3y^2}{4}$, therefore defines a metric d_Q on $\mathbb{R}^2$. Given any two points $P_1 = (x_1, y_1), P^2 = (x_2, y_2) \in E$, the line segment $\overline{P_1, P_2}$ is contained in E by the triangle inequality for d_Q. So E is connected. The same argument shows that E^0 is connected. (Alternatively, E is an ellipse together with the enclosed region, and so it is convex, as is its interior.)

The set E is a closed and bounded subset of $\mathbb{R}^2$, so it is compact. On the other hand E^0 is not a closed subset of $\mathbb{R}^2$, so E^0 is not compact.

(b) It is clear from the triangle inequality for d_Q (or by continuity of polynomials) that $\{(x, y) \in \mathbb{R}^2 \,|\, x^2 + xy + x^2 < 1\}$ is contained in the interior of E. On the other hand for any given point $P = (a, b) \in E$ with $a^2 + ab + b^2 = 1$ and any given $\epsilon > 0$, the point $\big((1 + \frac{\epsilon}{2})a, (1 + \frac{\epsilon}{2})b\big)$ is in the disk $D((a, b); \epsilon)$ but not in E. So E^0 is the interior of E.

Similarly, for every point $P = (a, b) \in E$ with $a^2 + ab + b^2 = 1$ and every $\epsilon > 0$, the point $\big((1 - \frac{\epsilon}{2})a, (1 - \frac{\epsilon}{2})b\big)$ is in $E^0 \cap D((a, b); \epsilon)$. So E is contained in the closure of E^0.

7. Let f be an increasing continuously differentiable function on the real line, and let $g = f'$. Let $a = f(0)$, $b = f(1)$, $c = g(0)$, $d = g(1)$. Let R be the region in the (x, y)-plane lying below the graph of $y = g(x)$, above the x-axis, and between the lines $x = 0$ and $x = 1$. Let C be the boundary of R, oriented counterclockwise. Evaluate

$$\oint_C (xy^2e^{x^2y^2} + 3x^2)dx + (x^2ye^{x^2y^2} + 5x)dy.$$

Solution. Let $P = xy^2e^{x^2y^2} + 3x^2$ and $Q = x^2ye^{x^2y^2} + 5x$. By Green's theorem, the contour integral is equal to

$$\iint_R (Q_x - P_y)\,dx\,dy = 5\iint_R dx\,dy = 5 \cdot \text{area}(R) = 5\int_0^1 f'(x)\,dx = 5(b - a).$$

8. Let $U \in \mathrm{M}_3(\mathbb{Q})$ be a 3×3 matrix with coefficients in $\mathbb{Q}$ such that $U^5 = \mathrm{I}_3$, where I_3 is the 3×3 identity matrix. Prove that $U = \mathrm{I}_3$.

(Hint: You may use the factorization $T^5 - 1 = (T-1)(T^4 + T^3 + T^2 + T + 1)$ in the polynomial ring $\mathbb{Q}[T]$, and the fact that $T^4 + T^3 + T^2 + T + 1$ is irreducible in $\mathbb{Q}[T]$.)

Solution. The minimal polynomial $f(x)$ of U is a monic polynomial in $\mathbb{Q}[x]$ which divides $x^5 - 1$ by assumption, and has degree at most 3 by Cayley–Hamilton. On the other hand $x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1)$, and $\Phi_4(x) := x^4 + x^3 + x^2 + x + 1$ is irreducible in $\mathbb{Q}[x]$ (by Eisenstein's criterion applied to $\Phi_4(x+1)$ for the prime number 5). So the minimal polynomial $f(x)$ is equal to $x - 1$, thus $U = \mathrm{I}_3$.

9. Let f be a piecewise continuous function on $\mathbb{R}$ such that $f(x+1) = -f(x)$ for all $x \in \mathbb{R}$. Determine whether the limit

$$\lim_{a\to\infty} \int_0^1 f(ax)\,dx$$

exists. If it does, find it.

Solution. The assumption implies that $\int_c^{c+n} f(x)\,dx = 0$ for every even positive integer n and every $c \in \mathbb{R}$. We have

$$\int_0^1 f(ax)\,dx = a^{-1} \cdot \int_0^a f(t)\,dt = a^{-1} \int_0^{a-n_a} f(t)\,dt$$

where n_a is the largest even positive integer not exceeding a. Therefore $|\int_0^1 f(ax)\,dx| \leq a^{-1} \int_0^2 |f(t)|\,dt$ for $a > 0$, and $\lim_{a\to\infty} \int_0^1 f(ax)\,dx = 0$.

10. Let $\mathrm{GL}_2(\mathbb{R})$ be the group of all invertible 2×2 matrices with entries in $\mathbb{R}$. Let H be the subgroup of $\mathrm{GL}_2(\mathbb{R})$ consisting of all diagonal 2×2 matrices $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ with $x, y \neq 0$.

(a) Determine explicitly the *center* $\mathrm{Z}(\mathrm{GL}_2(\mathbb{R}))$ of the group $\mathrm{GL}_2(\mathbb{R})$.

(b) Determine explicitly the *normalizer* subgroup $\mathrm{N}_{\mathrm{GL}_2(\mathbb{R})}(H)$ of $\mathrm{GL}_2(\mathbb{R})$, and the index $(\mathrm{N}_{\mathrm{GL}_2(\mathbb{R})}(H) : H)$.

(Recall that $\mathrm{N}_{\mathrm{GL}_2(\mathbb{R})}(H)$ consists of all elements $g \in \mathrm{GL}_2(\mathbb{R})$ such that $g \cdot H \cdot g^{-1} = H$.)

Solution. (a) Consider the condition $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for all $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$. Because $\mathrm{GL}_2(\mathbb{R})$ is dense in $\mathrm{M}_2(\mathbb{R})$, the above condition is equivalent to the condition that the same equality holds for all $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathrm{M}_2(\mathbb{R})$. This is four linear equations in variables x, y, z, w. For instance the $(1,1)$-entry of both sides gives $ax + bz = ax + cy$ for all $x, y, z, w \in \mathbb{R}$, therefore $b = c = 0$. Similarly the $(2,1)$-entry gives $cx + dz = az + cw$ for all $x, y, z, w \in \mathbb{R}$, therefore $d = a$. We have shown that $\mathrm{Z}(\mathrm{GL}_2(\mathbb{R})) \subseteq \mathbb{R}^\times \cdot \mathrm{I}_2$. On the other hand it is clear that $\mathbb{R}^\times \cdot \mathrm{I}_2 \subseteq \mathrm{Z}(\mathrm{GL}_2(\mathbb{R}))$. Therefore $\mathrm{Z}(\mathrm{GL}_2(\mathbb{R})) = \mathbb{R}^\times \cdot \mathrm{I}_2$, the group of non-zero constant 2×2 real matrices.

(b) The necessary and sufficient condition for an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $\mathrm{GL}_2(\mathbb{R})$ to be in $\mathrm{N}_{\mathrm{GL}_2(\mathbb{R})}(H)$ is that for all $x, y \in \mathbb{R}^\times$, the two off-diagonal entries of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ are both 0. An easy computation shows that this condition is: $ab = 0$ and $cd = 0$; equivalently either $b = c = 0$ or $a = d = 0$. The index of H in $\mathrm{N}_{\mathrm{GL}_2(\mathbb{R})}(H)$ is 2, with $\left\{ \mathrm{I}_2, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$ as a set of representatives.

11. For each of the following, give a proof or a counterexample.

(a) If $(a_n)_{n\geq 1}$ is a sequence of positive real numbers such that the series $\sum a_n$ converges, then the series $\sum a_n^2$ converges.

(b) If $(a_n)_{n\geq 1}$ is a sequence of arbitrary real numbers such that the series $\sum a_n$ converges, then the series $\sum a_n^2$ converges.

(c) If f is a continuous function on $\mathbb{R}$, and if $a_n = \frac{1}{n}\sum_{j=1}^{n} f(j/n)$ for all positive integers n, then the sequence $(a_n)_{n\geq 1}$ converges.

Solution. (a) Proof: $\sum a_n^2$ converges by the comparison test (with $\sum_n a_n$), since $0 < a_n < 1$ for all sufficiently large n.

(b) Counterexample: Let $a_n = (-1)^n n^{-1/2}$. Then the alternating series $\sum_n a_n$ converges, while the series $\sum_n a_n^2 = \sum_n \frac{1}{n}$ diverges.

(c) Proof: The sequence $(a_n)_{n\geq 1}$ of Riemann sums for the integral $\int_0^1 f(x)\,dx$ converges to $\int_0^1 f(x)\,dx$, because $f(x)$ is continuous on $[0, 1]$.

12. Let A be the 4×4 real matrix

$$A = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{pmatrix}.$$

(a) Show that the minimal polynomial of A is $x^2 - x + 1$.

(b) Let K be the $\mathbb{R}$-linear span in $\mathrm{M}_4(\mathbb{R})$ of A and the identity matrix I_4. Show that K is a subring of $\mathrm{M}_4(\mathbb{R})$, and is isomorphic to $\mathbb{C}$.

(Hint: Use (a).)

(c) Let $V \subseteq \mathrm{M}_4(\mathbb{R})$ be the subset of $\mathrm{M}_4(\mathbb{R})$ consisting of all real 4×4 matrices B such that $AB = BA$. Show that V is stable under left and right matrix multiplication by elements of K (i.e., kv, $vk \in V$ for $k \in K$ and $v \in V$), and that V is a vector space over K.

Solution. (a) An easy computation shows that $A^2 = A - \mathrm{I}_4$, so the minimal polynomial of A divides $x^2 - x + 1$. Since A is visibly not a scalar multiple of I_4, the minimal polynomial of A is $x^2 - x + 1$. Alternatively, it follows because $x^2 - x + 1$ is irreducible over $\mathbb{R}$.

(b) Clearly $K = \mathbb{R} \cdot \mathrm{I}_4 + \mathbb{R} \cdot A$ is closed under addition, and it is closed under multiplication because $A^2 = A - \mathrm{I}_4 \in K$. We have an obvious surjective ring homomorphism α from $\mathbb{R}[x]/(x^2 - x + 1)$ to K that sends x to A, and α is a bijection because both its source and target have dimension 2.

The existence of a ring isomorphism $\mathbb{R}[x]/(x^2-x+1) \cong \mathbb{C}$ follows from the fact that x^2-x+1 is irreducible in $\mathbb{R}[x]$ (e.g., because the discriminant of $x^2 - x + 1$ is negative). There are two such $\mathbb{R}$-linear ring isomorphisms, each sending x mod $x^2 - x + 1$ to $(1 \pm \sqrt{-3})/2$, the two primitive sixth roots of unity in $\mathbb{C}$.

(c) It is straightforward to see that a stronger statement holds: V is a subring of $\mathrm{M}_4(\mathbb{R})$, i.e. V is stable under matrix multiplication, and V contains K.

(*A side remark*: The matrix A is the matrix representation of the operator "multiplication on the left by the norm-one element $u = \frac{1}{2}(1 + i + j + k)$" on the ring of Hamilton's quaternions $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$. The element u has norm one, $u - \frac{1}{2}$ is purely imaginary, $u^2 = u - 1$, and $u^4 = u^2 - 2u + 1 = -u$. The fact that u has norm one implies that A is an orthogonal matrix, which is also clear by inspection.)