$\mathbb{A}^1$-Brouwer Degrees and Applications to Enriched Enumerative Geometry

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ABSTRACT

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$\mathbb{A}^1$-*enumerative geometry*, or *enriched enumerative geometry*, is a recent program of mathematics following work of Kass-Wickelgren, Levine and others, which wields tools from motivic homotopy theory in order to investigate enumerative geometry problems over arbitrary fields. One of the key constructions used in this program is an algebrao-geometric analogue of the Brouwer degree, called the $\mathbb{A}^1$-*Brouwer degree*, first defined by Morel. Early computational results for $\mathbb{A}^1$-Brouwer degrees include Cazanave's thesis, and work of Kass and Wickelgren comparing $\mathbb{A}^1$-Brouwer degrees at rational points with the Eisenbud–Khishiashvili–Levine signature formula. However a few years ago, the general question of computing an $\mathbb{A}^1$-Brouwer degree of an endomorphism of affine space with an isolated zero at an arbitrary closed point was largely open. We report on work which closes this gap, providing a suite of computational tools, and discussing applications to enriched enumerative geometry.

Contents

List of papers

This thesis is comprised of work spread across five thematically related papers: one expository, three with coauthors, and one solo.

Chapter 2: **An introduction to $\mathbb{A}^1$-enumerative geometry**, in *Homotopy Theory and Arithmetic Geometry — Motivic and Diophantine Aspects. Lecture Notes in Mathematics*, vol 2292. Springer, Cham. 2021.

Chapter 3: **The trace of the local $\mathbb{A}^1$-degree**, with R. Burklund, M. Montoro, S. McKean, M. Montoro, M. Opie. *Homology, Homotopy and Applications*, 23(1) 243—255, 2021.

Chapter 4: **Bézoutians and the $\mathbb{A}^1$-degree**, with S. McKean and S. Pauli. To appear in *Algebra & Number Theory*, 2023.

Chapter 5: **Lifts, transfers, and degrees of univariate maps**, with S. McKean. To appear in *Mathematica Scandinavica*, 2023.

Chapter 6: **An enriched degree of the Wronski map**. Submitted, 2023.

Chapter 1

Introduction

1.1 The enumerative backdrop

Enumerative geometry emerged in antiquity alongside the development of projective geometry. One of the earliest instances of an enumerative problem appears in writings of Pappus, attributing the original result to Apollonius: how many circles are tangent to any three drawn on the plane? Results which may later be called "enumerative geometry" include the study of intersections of planar curves, dating back to work of Newton and then Bézout, a century later. Poncelet is often credited with inspiring the resurrection of projective geometry in France during the early 19th century. His controversial "principle of continuity of number" postulates that answers to enumerative questions should be invariant under small parametrized changes in initial parameters. Despite backlash to this idea from established mathematicians like Cauchy, this idea was championed by Chasles and others, and it later appeared in work of Schubert under the name “principle of conservation of number.” While his work was undoubtedly more rigorous than that of Poncelet, Schubert’s “calculus of conditions” (now known as *Schubert calculus*) lacked justification, leading Hilbert to include its rationalization as the 15th entry in his famous list of problems for the 20th century. The calculations Schubert carried out were later formalized via the methods of singular cohomology, as developed by early 20th century mathematicians including van der Waerden and Lefschetz.

Enumerative geometry can be (and in general is) approached from the perspective of algebraic geometry, with computations carried out in the Chow ring and reliant upon the development of intersection theory following work of Fulton and MacPherson. While this modern perspective is the right one, many enumerative problems can be passed through the bridge between algebraic geometry and topology and solved in the topological world. Explicitly, the algebraic Chern classes of an algebraic vector bundle over a sufficiently nice complex variety is mapped, under a cycle class map, to the topological Chern classes of the underlying topological bundle over the associated complex analytic space. This perspective lends credence to Poncelet’s continuity of number — solutions to enumerative problems are invariant under small parametrized changes because they can *literally* be formulated as something which is homotopy invariant.

1.2 Enter motivic homotopy theory

One does not have to pass entirely to the world of topology to study invariance under a notion of algebraic homotopy, however. While the Yoneda philosophy tells us that studying varieties is equivalent to studying their representable functors valued in sets, Grothendieck had already in 1958 thought to look at presheaves valued in chain complexes over a ring, and had formulated a sheaf condition in this setting. A decade later, Quillen observed that the study of chain complexes up to quasi-isomorphism shared a lot structurally with the study of spaces up to weak homotopy equivalence. Quillen referred to this work as *homotopical algebra*, and we now know it as the theory of *model categories*.

From this vantage point, one may ask to develop a notion of a "homotopy theory of varieties," by studying sheaves of varieties valued in a homotopical category such as spaces or simplicial sets. Illusie outlined such a strategy in 1971, and this was deeply influential on Joyal's 1984 letter to Grothendieck, in which the first closed model structure on simplicial sheaves was constructed. In 1990, Morel and Voevodsky released their seminal paper on $\mathbb{A}^1$-*homotopy theory*, in which they extended the homotopical study of simplicial sheaves on varieties by formally inverting the affine line in order to force it to behave in an analogous way to the unit interval. This was one of the huge milestones in the recent history of homotopy theory, and this work famously led to the resolution of the Milnor conjectures, winning Voevodsky the Fields Medal.

In the motivic world, one can formulate hybrid theories that interpolate roughly between both algebraic and topological phenomena. One such example is *Chow-Witt groups*, following Barge and Morel — this is the motivic extension of Chow groups of a smooth k-variety, but should also be thought of in some sense as analogous to singular cohomology. In this setting, one may make precise notions such as Euler classes of vector bundles, wrong-way maps on cohomology, and a quadratic analogue of the Poincaré–Hopf theorem: that the Euler class of an appropriately oriented vector bundle is Poincaré dual to the zero locus of a generic section. We consider this as the jumping off point for an "enriched" approach to enumerative geometry.

1.3 Enriched enumerative geometry

Classically, the Poincaré–Hopf theorem states the following: given an oriented rank r vector bundle $E \to M$ over a smooth compact oriented r-manifold, and a section

$\sigma : M \to E$, we have that the Euler number $n(E)$ can be computed as the sum of the local contributions of the connected components of the zero locus of the section; that is $n(E) = \sum_{p \in Z(\sigma)} \mathrm{ind}_p \sigma$. We call $\mathrm{ind}_p\sigma$ the *local index* of σ at p. Generically, $Z(\sigma)$ will be zero-dimensional, and $\mathrm{ind}_p\sigma$ can be computed as the Brouwer degree of the intrinsic derivative of the section: $\mathrm{ind}_p\sigma = \deg(d_p\sigma)$.[1] The local index should be interpreted as the local intersection multiplicity of the coordinate functions of σ on a chart. This leads us to the slogan that *Euler numbers count things with multiplicity.* Some illustrative examples include:

- $n\left(\mathcal{O}_{\mathbb{C}\mathrm{P}^2}(n) \oplus \mathcal{O}_{\mathbb{C}\mathrm{P}^2}(m) \to \mathbb{C}\mathrm{P}^2\right) = nm$; that is, generic planar curves of degrees n and m intersect at nm points, counted with multiplicity (Bézout).

- $n\left(\mathcal{O}(1)^{\oplus 4} \to \mathrm{Gr}(2,4)\right) = 2$; there are two lines intersecting any four generic lines in $\mathbb{C}\mathrm{P}^3$ (Schubert).

- $n\left(\mathrm{Sym}^3\mathcal{S}^* \to \mathrm{Gr}_{\mathbb{C}}(2,4)\right) = 27$, there are 27 lines on a smooth cubic surface (Salmon, Cayley).

With motivic analogues of Euler classes in hand, Kass and Wickelgren, and independently Levine, commenced a study of enumerative geometry over arbitrary fields

[1] When $Z(\sigma)$ is not zero-dimensional we may still make sense of $\mathrm{ind}_p\sigma$ via reference to *excess/residual intersection.*

using techniques from motivic homotopy theory. We call this $\mathbb{A}^1$-*enumerative geometry* (this is also referred to as *enriched enumerative geometry* or more specifically *quadratically enriched enumerative geometry*).

The origin of the phrase "quadratically enriched" comes from the fact that Euler numbers in this setting are not integers, but generally are *quadratic forms*. More precisely, they are valued in the *Grothendieck–Witt ring* $\mathrm{GW}(k)$ of the ground field, defined to be the group completion of the semiring of isomorphism classes of non-degenerate symmetric bilinear forms (which agrees with quadratic forms in characteristic $\neq 2$).

A theorem of Kass and Wickelgren states that the "number" of lines on a smooth cubic surface over k is $15\langle 1\rangle + 12\langle -1\rangle \in \mathrm{GW}(k)$, that is, a symmetric bilinear form with Gram matrix given by a diagonal matrix with fifteen 1's and twelve -1's along the diagonal. The invariants needed to classify symmetric bilinear forms encode different information over other fields. The rank of this form is 27, recovering the classical result that there are 27 lines on any smooth complex cubic surface. Over the reals, another invariant is needed to classify symmetric bilinear forms, namely the *signature*. Lines on a real cubic surface break into two classes, called *hyperbolic* and *elliptic*, depending on whether the loop in the frame bundle induced by the line lifts to the double cover. The signature of the form computed by Kass and Wickelgren is

3, recovering a result of Segre (also Finashin-Kharlamov and Okonek-Teleman) that the number of real hyperbolic lines minus the number of real elliptic lines is always 3. For more on this result see Subsection 2.2.4.

This is a fantastic illustration of the power of this research program — as the field changes, conservation of number breaks, but it can be repaired if local indices to enumerative problems are permitted to encode some deeper geometry about the problem in question. An enriched analogue of the Poincaré-Hopf theorem appears in various forms throughout the literature, with the most general statement appearing as (BW21, Meta-Corollary 3.11)

Our capacity to solve enriched enumerative problems is then inhibited by our ability to carry out explicit computations — in particular, given an enumerative problem encoded as an appropriately oriented algebraic vector bundle $E \to M$, can we compute a local index $\mathrm{ind}_p\sigma$ at an isolated point? Such a computation makes use of the $\mathbb{A}^1$-*Brouwer degree*, first defined by Morel, valued in $\mathrm{GW}(k)$. However Morel's identification between the $(0,0)$th motivic stable stem and $\mathrm{GW}(k)$ does not inform us how to carry out computations on affine charts.

Lannes and Morel understood how to compute local degrees at the origin for endomorphisms of $\mathbb{A}^1_k$, while Cazanave's thesis gave a beautiful construction for understanding global $\mathbb{A}^1$-degrees of endomorphisms of the projective line. Kass and

Wickelgren provided the first result permitting computations in higher dimensions — they proved that for a morphism $f\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ with an isolated zero at the origin, its local $\mathbb{A}^1$-degree $\deg^{\mathbb{A}^1}_0(f)$ can be computed as the *EKL form* (referring to work of Eisenbud-Levine and Khimshiashvili in the 1970's computing local Brouwer degrees of maps of real manifolds). They further proved that if f is étale at a closed zero $p \in \mathbb{A}^n_k$, and the Jacobian of f in $k(p)$ is non-vanishing at p, then $\deg^{\mathbb{A}^1}_p(f)$ can be computed as $\mathrm{Tr}_{k(p)/k}\langle J(p)\rangle$, where $\mathrm{Tr}_{k(p)/k}\colon \mathrm{GW}(k(p)) \to \mathrm{GW}(k)$ is the separable field trace, and $J(p)$ is the Jacobian determinant of f evaluated at p in $k(p)$.

This leads to many questions. Can the étale assumption be dropped? To what extent is the $\mathbb{A}^1$-degree always traced down from its field of definition? Can we compute local $\mathbb{A}^1$-degrees at points whose residue fields are inseparable extensions of the base field, making enriched enumerative geometry accessible over imperfect fields? Can we provide easy efficient algorithms for computing $\mathbb{A}^1$-degrees?

The work that forms this thesis answers many of these questions.

1.4 Overview of results

Chapter 2 (Bra21) provides an expository introduction to $\mathbb{A}^1$-enumerative geometry, following lectures of Kirsten Wickelgren. We provide a more precise introduction to Grothendieck–Witt rings and outline the construction of motivic spaces and the stable motivic homotopy category. We provide an overview of the Eisenbud–Khimshiashvili–Levine signature formula in Subsection 2.2.1 (see Section 4.4 for more detail). Finally, we discuss $\mathbb{A}^1$-Milnor numbers and 27 lines on a cubic surface, following work of Kass and Wickelgren, and lines meeting for lines in three-space, following work of Srinivasan and Wickelgren.

Chapter 3 (BBM$^+$21) studies the situation where $f\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ has an isolated zero at a closed point $p \in \mathbb{A}^n_k$, without any assumption that f is étale at the point. We prove that

$$\deg_p^{\mathbb{A}^1}(f) = \mathrm{Tr}_{k(p)/k} \deg_{\widetilde{p}}^{\mathbb{A}^1} f_{k(p)},$$

where $\widetilde{p} \in \mathbb{A}^n_{k(p)}$ is a canonical rational point above p, and $f_{k(p)}$ is the base change of the morphism to $k(p)$. This extends the context in which $\mathbb{A}^1$-Brouwer degrees align with the so-called *Scheja–Storch form*.

Chapter 4 (BMP21b) provides the current state-of-the-art on $\mathbb{A}^1$-Brouwer degrees. We prove that the local $\mathbb{A}^1$-Brouwer degree $\deg_p^{\mathbb{A}^1}(f)$ can be computed at any closed point, independent of any assumption about residue fields. This is done by aligning the $\mathbb{A}^1$-Brouwer degree with the Scheja–Storch form in all cases, and indicating that this can be computed in terms of a multivariate Bézoutian. We provide computation rules for $\mathbb{A}^1$-Brouwer degrees and applications to $\mathbb{A}^1$-Brouwer degrees. Finally we provide Sage code for efficiently computing $\mathbb{A}^1$-degrees in (BMP21a).

Chapter 5 (BM23) explores $\mathbb{A}^1$-Brouwer degrees of univariate maps and investigates how local degrees at a closed point are transferred down from the field of definition of the point. Along inseparable field extensions, the field trace identically vanishes, nonetheless there are *geometric* and *cohomological* transfers $\mathrm{GW}(k(p)) \to \mathrm{GW}(k)$ arising as incarnations of Becker–Gottlieb transfers in the stable motivic setting. We define geometric and cohomological lifts of functions at a closed point, and argue that $\mathbb{A}^1$-degrees can be computed as a transfer of the appropriate lift. This leads to a structure theorem for univariate $\mathbb{A}^1$-Brouwer degrees.

Chapter 6 (Bra23) explores the $\mathbb{A}^1$-Brouwer degree of the *Wronski map*, computed by Schubert over the complex numbers and by Eremenko-Gabrielov over the reals. This has many enumerative applications, counting planes meeting planes of complementary dimension in projective space (generalizing lines meeting four lines in three-space), as well as counting rational curves with prescribed inflection. We provide a computation for the local $\mathbb{A}^1$-Brouwer degree at a plane as a determinantal relationship between Plücker coordinates of the complementary planes it intersects. In certain parities we compute a global $\mathbb{A}^1$-degree of the Wronski map.

Chapter 2

An introduction to $\mathbb{A}^1$-enumerative geometry

based on lectures by Kirsten Wickelgren

Abstract: We provide an expository introduction to $\mathbb{A}^1$-*enumerative geometry*, which uses the machinery of $\mathbb{A}^1$-homotopy theory to enrich classical enumerative geometry questions over a broader range of fields. Included is a discussion of enriched local degrees of morphisms of smooth schemes, following Morel, $\mathbb{A}^1$-Milnor numbers, as well as various computational tools and recent examples.

Introduction

In the late 1990's Fabien Morel and Vladimir Voevodsky investigated the question of whether techniques from algebraic topology, particularly homotopy theory, could be applied to study varieties and schemes, using the affine line $\mathbb{A}^1$ rather than the interval $[0, 1]$ as a parametrizing object. This idea was influenced by a number of preceding papers, including work of Karoubi and Villamayor (KV71) and Weibel (Wei89) on K-theory, and Jardine's work on algebraic homotopy theory (Jar81a; Jar81b). In work with Suslin developing an algebraic analog of singular cohomology which was $\mathbb{A}^1$-invariant (SV96), Voevodsky laid out what he considered to be the starting point of a homotopy theory of schemes parametrized by the affine line (Voe98). This relied upon Quillen's theory of model categories (Qui67), which provided the abstract framework needed to develop homotopy theory in broader contexts. Following this work, Morel (Mor99) and Voevodsky (Voe98) constructed equivalent unstable $\mathbb{A}^1$-homotopy categories, laying the groundwork for their seminal paper (MV99) which marked the genesis of $\mathbb{A}^1$-*homotopy theory*. Since its inception, this field of mathematics has seen far-reaching applications, perhaps most notably Voevodsky's resolution of the Bloch-Kato conjecture, a classical problem from number theory (Voe11).

The machinery of $\mathbb{A}^1$-homotopy theory works over an arbitrary field k (in fact over arbitrary schemes, and even richer mathematical objects), allowing enrichments of classical problems which have only been explored over the real and complex numbers. Recent work in this area has generalized classical enumerative problems over wider ranges of fields, forming a body of work which we are referring to as $\mathbb{A}^1$-*enumerative geometry*.

Beginning with a recollection of the topological degree for a morphism between manifolds in Section 2.1.1, we pursue an idea of Barge and Lannes to produce a notion of degree valued in the Grothendieck–Witt ring of a field k, defined in Section 2.1.2. We produce such a naive degree for endomorphisms of the projective line in Section 2.1.3, however in order to produce such a degree for smooth n-schemes in general, we will need to develop some machinery from $\mathbb{A}^1$-homotopy theory. A brief detour is taken to establish the setting in which one can study motivic spaces, defining the unstable motivic homotopy category in Section 2.1.4, and establishing some basic, albeit important computations involving colimits of motivic spaces in Section 2.1.5. This discussion culminates in the purity theorem of Morel and Voevodsky, stated in Section 5.2, which is requisite background for defining the local $\mathbb{A}^1$-degree following Morel.

In Section 2.2, we are finally able to define the local $\mathbb{A}^1$-degree of a morphism of schemes, which is a powerful, versatile tool in enriching enumerative geometry problems over arbitrary fields. At this point, we survey a number of recent results in $\mathbb{A}^1$-enumerative geometry. We discuss the Eisenbud–Khimshiashvili–Levine signature formula in Sections 2.2.1 and 2.2.2, and we see its relation to the $\mathbb{A}^1$-degree, as proved in (KW19). An enriched version of the $\mathbb{A}^1$-Milnor number is provided in Section 2.2.3, which provides an enriched count of nodes on a hypersurface, following (KW16). The problem of counting lines on a cubic surface, and the associated enriched results (KW21) are discussed in Section 2.2.4. Finally, in Section 2.2.5 we provide an arithmetic count of lines meeting four lines in three-space, following (SW21).

Throughout these conference proceedings, various exercises (most of which were provided by Wickelgren in her 2018 lectures) are included. Detailed solutions may be found on the author's website.

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2.1 Preliminaries

2.1.1 Enriching the topological degree

A continuous map $f : S^n \to S^n$ from the n-sphere to itself induces a homomorphism on the top homology group $f_* : H_n(S^n) \to H_n(S^n)$, which is of the form $f_*(x) = dx$ for some $d \in \mathbb{Z}$. This integer d defines the *(global) degree* of the map f. If f and g are homotopic as maps from the n-sphere to itself, they will induce the same homomorphism on homology groups. Therefore, taking $[S^n, S^n]$ to denote the set of homotopy classes of maps, we can consider degree as a function

$$\deg^{\mathrm{top}} : [S^n, S^n] \to \mathbb{Z}.$$

Throughout these notes, we will use the notation $\deg^{\mathrm{top}}$ to refer to the topological (integer-valued) degree.

For any continuous map of n-manifolds $f : M \to N$, we could define a naive notion of the "local degree" around a point $p \in M$ via the following procedure: suppose that $q \in N$ has the property that $f^{-1}(q)$ is discrete, and let $p \in f^{-1}(q)$. Pick a small ball W containing q, and a small ball $V \subseteq f^{-1}(W)$ satisfying $V \cap f^{-1}(q) = \{p\}$. Then we may see that the spaces $V/(V \setminus \{p\}) \simeq (V/\partial V) \simeq S^n$ are homotopy equivalent.

Similarly, we have that $W/(W \setminus \{q\}) \simeq S^n$. We obtain the following diagram:

$$\begin{tikzcd} S^n \arrow[rr, "g"] \arrow[d, "\simeq"'] & & S^n \\ V/(V \setminus \{p\}) \arrow[r, "f"'] & W/(W \setminus \{q\}). \arrow[ur, "\simeq"'] \end{tikzcd} \tag{2.1.1}$$

Thus we could define the local (topological) degree of f around our point p, denoted $\deg_p^{\mathrm{top}}(f)$, to be the induced degree map on the n-spheres, that is, $\deg_p^{\mathrm{top}}(f) := \deg^{\mathrm{top}}(g)$ in the diagram above. If $f^{-1}(q) = \{p_1, \ldots, p_m\}$ is a discrete set of isolated points, we may relate the global degree to the local degree via the following formula

$$\deg^{\mathrm{top}}(f) = \sum_{i=1}^{m} \deg_{p_i}^{\mathrm{top}}(f).$$

One may prove that the left hand side is independent of q, and thus that the choice of q is arbitrary in calculating the global degree from local degrees. In differential topology, when discussing the degree of a locally differentiable map f between n-manifolds, we have a simple formula for the local degree at a simple zero. We pick local coordinates $(x_1, \ldots, x_n)$ in a neighborhood of our point p_i, and local coordinates $(y_1, \ldots, y_n)$ around a regular value q. Then we can interpret f locally as a map $f = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$. Suppose that the Jacobian $\mathrm{Jac}(f)$ is nonvanishing at the point p_i. Then we define

$$\deg_{p_i}^{\mathrm{top}}(f) = \mathrm{sgn}(\mathrm{Jac}(f)(p_i)) = \begin{cases} +1 & \text{if } \mathrm{Jac}(f)(q_i) > 0 \\ -1 & \text{if } \mathrm{Jac}(f)(q_i) < 0. \end{cases}$$

When working over a field k, Barge and Lannes[1] defined a notion of degree for a map $\mathbb{P}^1_k \to \mathbb{P}^1_k$. Their insight was, rather than taking the sign of the Jacobian as in differential topology, to instead remember the value of $\mathrm{Jac}(f)(p_i)$ as a square class in $k^\times / (k^\times)^2$. Over the reals this recovers the sign, but over a general field we may have vastly more square classes. We encode this value as a rank one symmetric bilinear form over k, and we will soon see that this idea can be used to define a local degree at k-rational points, and that by using field traces we can extend the definition of local degree to hold for points with residue fields a finite separable extension of k. These degrees, rather than being integers, are elements of the *Grothendieck–Witt ring of* k, denoted $\mathrm{GW}(k)$, defined below.

2.1.2 The Grothendieck–Witt Ring

Over a field k, we may form a semiring of isomorphism classes of non-degenerate symmetric bilinear forms (or quadratic forms if we assume $\mathrm{char}(k) \neq 2$) on vector spaces over k, using the operations $\otimes_k$ and $\oplus$. Group completing this semiring with respect to $\oplus$, we obtain the Grothendieck–Witt ring $\mathrm{GW}(k)$. For any $a \in k^\times$, we let

[1] Unpublished. See the note by Morel on (Mor06, p.1037).

$\langle a \rangle \in \mathrm{GW}(k)$ denote the following rank one bilinear form:

$$\langle a \rangle : k \times k \to k$$
$$(x, y) \mapsto axy.$$

Symmetric bilinear forms are equivalent if they differ only by a change of basis. For example, if $b \neq 0$ we can see that $\langle ab^2 \rangle (x, y) = \langle a \rangle (bx, by)$, so we identify $\langle a \rangle = \langle ab^2 \rangle$ in $\mathrm{GW}(k)$, since these bilinear forms agree up to a vector space automorphism of k. We may describe $\mathrm{GW}(k)$ to be a ring generated by elements $\langle a \rangle$ for each $a \in k^\times / (k^\times)^2$, subject to the following relations (Mor12, Lemma 4.9)

1. $\langle a \rangle \langle b \rangle = \langle ab \rangle$
2. $\langle a \rangle + \langle b \rangle = \langle ab(a + b) \rangle + \langle a + b \rangle$, for $a + b \neq 0$
3. $\langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle$. We conventionally denote this element as $\mathbb{H} := \langle 1 \rangle + \langle -1 \rangle$, called the *hyperbolic element* of $\mathrm{GW}(k)$.

Exercise 2.1.2. In the statements above, (1) and (2) imply (3).

Proposition 2.1.3. We have a ring isomorphism $\mathrm{GW}(\mathbb{C}) \cong \mathbb{Z}$, given by taking the rank.

Proof. We remark that $\langle a \rangle = \langle b \rangle$ for any $a, b \in \mathbb{C}^\times$, thus we only have one isomorphism class of non-degenerate symmetric bilinear forms in rank one. □

The isomorphism $\mathrm{GW}(\mathbb{C}) \cong \mathbb{Z}$ relates to a general fact that the $\mathbb{A}^1$-degree of a morphism of complex schemes recovers the size of the fiber, counted with multiplicity.

Proposition 2.1.4. The rank and signature provide a group isomorphism $\mathrm{GW}(\mathbb{R}) \cong \mathbb{Z} \times \mathbb{Z}$.

Proof. The Gram matrix of a symmetric bilinear form on $\mathbb{R}^n$ is an $n \times n$ real symmetric matrix A. After diagonalizing our matrix A, we can always find a change of basis in which the eigenvalues lie in the set $\{-1, 0, 1\}$. A non-degenerate symmetric bilinear form guarantees that no eigenvalues will vanish, so all of these eigenvalues will be ± 1. We may define the *signature* of A as the number of 1's appearing on the diagonalized matrix minus the number of -1's, and by *Sylvester's law of inertia* this determines an invariant on our matrix A. Thus we obtain an injective map

$$\mathrm{GW}(\mathbb{R}) \to \mathbb{Z} \times \mathbb{Z}$$
$$A \mapsto (\mathrm{rank}(A), \mathrm{sig}(A)).$$

The image of this map is the subgroup $\{(a + b, a - b) \ : \ a, b \in \mathbb{Z}\}$, which one may verify is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. $\square$

The multiplication on $\mathrm{GW}(\mathbb{R})$ does not agree with that of $\mathbb{Z} \times \mathbb{Z}$, in the sense that $\mathrm{GW}(\mathbb{R}) \cong \mathbb{Z} \times \mathbb{Z}$ is not a ring isomorphism. However one may verify that the map

$$\mathrm{GW}(\mathbb{R}) \to \frac{\mathbb{Z}[t]}{(t^2-1)},$$

given by sending $\langle 1 \rangle \mapsto 1$ and $\langle -1 \rangle \mapsto t$, is in fact a ring isomorphism, and hence provides the multiplicative structure of $\mathrm{GW}(\mathbb{R})$.

Proposition 2.1.5. The rank and determinant provide a group isomorphism $\mathrm{GW}(\mathbb{F}_q) \cong \mathbb{Z} \times \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$.

Proof sketch. We may still use the rank of our matrix as an invariant for $\mathrm{GW}(\mathbb{F}_q)$. Additionally, we might use the determinant of our matrix to distinguish between symmetric bilinear forms. However note that, for any similar matrix C^TAC, it has determinant $\det(C^TAC) = \det(A)\det(C)^2$. Therefore, we can view the determinant as a well-defined map $\det : \mathrm{GW}(\mathbb{F}_q) \to \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$. After group completion, we obtain a map $\mathrm{GW}(\mathbb{F}_q) \xrightarrow{(\mathrm{rank},\det)} \mathbb{Z} \times \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$, which we verify is a group isomorphism. For more details, see (Lam05, II,Theorem 3.5). □

One may use $\mathrm{GW}(\mathbb{F}_q)$ to understand $\mathrm{GW}(\mathbb{Q}_p)$ by applying the following result.

Theorem 2.1.6. (Lam05, VI,Theorem 1.4) *(Springer's Theorem)* Let K be a complete discretely valued field, and κ be its residue field, with the assumption that $\mathrm{char}(\kappa) \neq 2$. Then there is an isomorphism of groups

$$\mathrm{GW}(K) \cong \frac{\mathrm{GW}(\kappa) \oplus \mathrm{GW}(\kappa)}{\mathbb{Z}[\mathbb{H}, -\mathbb{H}]}.$$

Corollary 2.1.7. We have a group isomorphism $\mathrm{GW}(\mathbb{C}((t))) = \mathbb{Z} \oplus \mathbb{Z}/2$.

We should see how the Grothendieck–Witt ring interacts with extensions of fields. For a separable field extension $K \subset L$, and an element $\beta \in \mathrm{GW}(L)$, we can view the composition

$$V \times V \xrightarrow{\beta} L \xrightarrow{\mathrm{Tr}_{L/K}} K$$

as an element of $\mathrm{GW}(K)$ by post-composing with the trace map $L \to K$, and considering V as a K-vector space. This provides us a natural homomorphism between Grothendieck–Witt rings[2]

$$\mathrm{Tr}_{L/K} : \mathrm{GW}(L) \to \mathrm{GW}(K).$$

At the level of $\mathbb{A}^1$-homotopy theory, this trace comes from a transfer on stable homotopy groups — for more detail see (Mor12, §4). Now that we have seen some computations involving the Grothendieck–Witt ring, we can develop in detail the notion of degree for maps of schemes.

2.1.3 Lannes' formula

Let $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be a non-constant endomorphism of the projective line over a field of characteristic 0. We can then pick a rational point $q \in \mathbb{P}^1_k$, with fiber $f^{-1}(q) =$

[2] When the field extension is assumed to be finite but the separability condition is dropped, a more general notion of transfer is given by *Scharlau's transfer* (Lam05, VII §1).

$\{p_1, \ldots, p_m\}$ such that $\mathrm{Jac}(f)(p_i) \neq 0$ for each i, where the Jacobian is computed by picking the same affine coordinates on both copies of $\mathbb{P}^1_k$. Since $\mathrm{Jac}(f)(p_i) \in k(p_i)$ is only defined in a residue field, we must precompose with the trace map in order to define the *local* $\mathbb{A}^1$*-degree*

$$\deg^{\mathbb{A}^1}_{p_i} f := \mathrm{Tr}_{k(p_i)/k} \langle \mathrm{Jac}(f)(p_i) \rangle . \tag{2.1.8}$$

We can then define the global $\mathbb{A}^1$-degree of f as the following sum, which is independent of our choice of rational point q with discrete fiber (this fact is attributable to Lannes and Morel, although a detailed proof may be found in (KW19, Proposition 14)):

$$\deg^{\mathbb{A}^1} f := \sum_{i=1}^{m} \mathrm{Tr}_{k(p_i)/k} \langle \mathrm{Jac}(f)(p_i) \rangle .$$

Exercise 2.1.9. Compute the $\mathbb{A}^1$-degrees of the following maps:

1. $\mathbb{P}^1_k \to \mathbb{P}^1_k$, given by $z \mapsto az$, for $a \in k^\times$.

2. $\mathbb{P}^1_k \to \mathbb{P}^1_k$, given by $z \mapsto z^2$.

Maps of schemes $\mathbb{P}^1_k \to \mathbb{P}^1_k$ are precisely rational functions $\frac{f}{g}$. Assuming that f and g are relatively prime, we can determine the classical topological (integer-valued) degree of this rational function as

$$\deg^{\mathrm{top}} \left(\frac{f}{g}\right) = \max\{\deg^{\mathrm{top}}(f), \deg^{\mathrm{top}}(g)\}.$$

To the rational function $\frac{f}{g}$, one may associated a bilinear form, called the *Bézout form*, which is denoted $\text{Béz}\left(\frac{f}{g}\right)$. This is done by introducing two variables X and Y, and remarking that we have the following equality

$$\frac{f(X)g(Y) - f(Y)g(X)}{X - Y} = \sum_{1 \leq i,j \leq n} B_{ij} X^{i-1} Y^{j-1},$$

where $n = \deg^{\text{top}}\left(\frac{f}{g}\right)$, and where $B_{ij} \in k$. We can see that this defines a symmetric bilinear form $k^n \times k^n \to k$, whose Gram matrix is given by the coefficients B_{ij}.

Exercise 2.1.10. Compute the Bézout bilinear forms of the maps given in Exercise 2.1.9.

Theorem 2.1.11. *(Cazanave)* We have that

$$\text{Béz}\left(\frac{f}{g}\right) = \deg^{\mathbb{A}^1}\left(\frac{f}{g}\right).$$

This is stated in (KW20, Theorem 2), but is attributable to (Caz12).

This provides us with an efficient way to compute the $\mathbb{A}^1$-degree of rational maps while circumventing the tedium of computing the local $\mathbb{A}^1$-degree at each point in a fiber.

2.1.4 The unstable motivic homotopy category

One of the primary ideas in $\mathbb{A}^1$-homotopy theory is to replace the unit interval in classical homotopy theory with the affine line $\mathbb{A}^1_k = \text{Spec}(k[t])$. To this end, one might develop a *naive* $\mathbb{A}^1$-*homotopy* of maps of schemes $f, g : X \to Y$ as a morphism

$$h : X \times \mathbb{A}^1_k \to Y,$$

such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for all $x \in X$. This was first introduced by Karoubi and Villamayor (KV71). This notion of naive $\mathbb{A}^1$-homotopy is not generally the most effective, partially due to the following observation.

Exercise 2.1.12. (Aso19) Prove that naive $\mathbb{A}^1$-homotopy fails to be a transitive relation on hom-sets by considering three morphisms $\operatorname{Spec} k \to \operatorname{Spec} k[x, y]/(xy)$ identifying the points $(0, 1)$, $(0, 0)$, and $(1, 0)$.

We will build a model category in which we have a class of $\mathbb{A}^1$-*weak equivalences*, and we will denote by $[-, -]_{\mathbb{A}^1}$ the weak equivalence classes of morphisms. In particular, naive $\mathbb{A}^1$-homotopy equivalences are tractable examples of $\mathbb{A}^1$-weak equivalences. Nonetheless, naive $\mathbb{A}^1$-homotopy generates an equivalence relation, and in practice the naive homotopy classes of maps $[X, Y]_N$ are often easier to compute than their genuine counterparts $[X, Y]_{\mathbb{A}^1}$. In fact, the naive homotopy classes of maps $[\mathbb{P}^1_k, \mathbb{P}^1_k]_N$ are equipped with an addition, induced by pinch maps, which endows this set with a monoid structure. It was demonstrated by Cazanave that the map

$$[\mathbb{P}^1_k, \mathbb{P}^1_k]_N \to [\mathbb{P}^1_k, \mathbb{P}^1_k]_{\mathbb{A}^1}$$

is a group completion (Caz12).

In order to study the homotopy theory of schemes, we must develop a model structure which encodes a notion of $\mathbb{A}^1$-weak equivalence. In particular we must force $\mathbb{A}^1$ to be contractible — as we have remarked, the initial motivation for forming such a model category was to treat $\mathbb{A}^1$ as if it were akin to the interval $[0,1]$ in the category of topological spaces. Morel and Voevodsky initially formulated the theory of the "homotopy category of a site with an interval"; for this classical treatment see (MV99, §2.3).

We remark that the category of smooth k-schemes $\mathtt{Sm}_k$ does not admit all colimits, and therefore cannot be endowed with a model structure. To rectify this issue, we pass to the category of the simplicial presheaves via the Yoneda embedding

$$\begin{aligned} \mathtt{Sm}_k &\to \mathtt{sPre}(\mathtt{Sm}_k) = \mathrm{Fun}(\mathtt{Sm}_k^{\mathrm{op}}, \mathtt{sSet}) \\ X &\mapsto \mathrm{Map}(-, X). \end{aligned}$$

This new category is *cocomplete* (it admits all small colimits), and moreover can be equipped with the *projective model structure* arising from the classical model structure on simplicial sets. Given our model structure, we are now permitted to identify a class of morphisms which we would like to call weak equivalences, and perform Bousfield localization in order to formally invert them. For exposition on Bousfield localization and related results, we refer the reader to (Law20).

The analog of open covers in a categorical setting is provided by a Grothendieck topology τ. The category $\mathtt{Sm}_k$ can be equipped with a Grothendieck topology in order to make it a site, after which, we will apply Bousfield localization to render the class of τ-hypercovers (our analog of open covers) into weak equivalences. We remark that by (DHI04, Theorem 6.2), this localization exists, and we denote it by $L_\tau : \mathtt{sPre}(\mathtt{Sm}_k) \to \mathrm{Sh}_{\tau,k}$. The fibrant objects in $\mathrm{Sh}_{\tau,k}$ are those simplicial presheaves which are *homotopy sheaves* in the τ topology (AE17, p. 20). We therefore think about the localization L_τ as a way to encode the topology τ into the homotopy theory of $\mathtt{sPre}(\mathtt{Sm}_k)$.

Due to the wealth of properties granted to us by simplicial presheaves, the category $\mathrm{Sh}_{\tau,k}$ inherits a left proper combinatorial simplicial model category structure, and in particular we are allowed to perform Bousfield localization again in order to force $\mathbb{A}^1$ to be contractible. We identify a set of maps $\{X \times \mathbb{A}^1 \to X\}$, indexed over the set of isomorphism classes of objects in $\mathtt{Sm}_k$, as our desired weak equivalences, then perform a final Bousfield localization $L_{\mathbb{A}^1}$ with respect to this set. Finally, we define

$$\mathtt{Spc}^{\mathbb{A}^1}_{\tau,k} := L_{\mathbb{A}^1}\mathrm{Sh}_{\tau,k} = L_{\mathbb{A}^1} L_\tau \mathtt{sPre}(\mathtt{Sm}_k).$$

This category has a model structure by construction, and we refer to its homotopy category as the *unstable motivic homotopy category*. Throughout these notes and in much of the literature, it is assumed we are using the *Nisnevich topology* (which is defined and contrasted with other choices of topologies below), and we will write $\mathtt{Spc}_k^{\mathbb{A}^1} := \mathtt{Spc}_{\mathrm{Nis},k}^{\mathbb{A}^1}$. Our primary objects of study in $\mathtt{Spc}_k^{\mathbb{A}^1}$ will be the fibrant objects of this category, which we refer to as $\mathbb{A}^1$-*spaces*. These admit a tangible recognition as precisely those presheaves which are valued in Kan complexes, satisfy Nisnevich descent, and are $\mathbb{A}^1$-invariant (AE17, Remark 3.58). For more detail, see (AE17, §3).

There are many equivalent constructions of $\mathtt{Spc}_k^{\mathbb{A}^1}$, one notable one arising from the *universal homotopy theory* on the category of smooth schemes, as described by (Dug01). By freely adjoining homotopy colimits, we obtain a universal category $U(\mathtt{Sm}_k)$ which we may localize at the collections of maps

$$\begin{aligned}&\{\mathrm{hocolim} U_\bullet \to X \ : \ \{U_\alpha\} \text{ is a hypercover of } X\}\\ &\{X \times \mathbb{A}^1 \to X\}.\end{aligned}$$

This procedure produces a model category $U(\mathtt{Sm}_k)_{\mathbb{A}^1}$ which is Quillen equivalent to $\mathtt{Spc}_k^{\mathbb{A}^1}$.

Remark 2.1.13. In more modern language, one may build $\mathtt{Spc}_k^{\mathbb{A}^1}$ using $(\infty, 1)$-categories rather than model categories. Such a perspective may be found throughout the literature, for example in (BH21; Rob15).

One may study the categories $\mathtt{Spc}_{k,\tau}^{\mathbb{A}^1}$ arising from other choices of Grothendieck topologies, and indeed the homotopy theories arising from each selection behave quite differently and merit individual study. A small inexhaustive list of possible topologies includes the Zariski, Nisnevich, and étale topologies.

Definition 2.1.14. Suppose that X and Y are smooth over a field k. Then we say $f : X \to Y$ is *étale* at x if the induced map on cotangent spaces

$$(f^*\Omega_{Y/k})_x \xrightarrow{\sim} \Omega_{X/k,x}$$

is an isomorphism (BLR90, §2.2, Corollary 10). If we have the additional structure of coordinates on our base and target spaces, this is equivalent to the condition that $\mathrm{Jac}(f) \neq 0$ in $k(x)$.

For example, any open immersion $X \hookrightarrow Y$ is a local isomorphism, and is therefore an étale map.

Definition 2.1.15. Let $\{f_\alpha : U_\alpha \to X\}$ be a family of étale morphisms. We say that it is

1. an *étale cover* if this is a cover of X, that is the underlying map of topological spaces is surjective

2. a *Nisnevich cover* if this is a cover of X, and for every $x \in X$ there exists an $\alpha \in A$ and $y \in U_\alpha$ such that $y \mapsto x$ and it induces an isomorphism on residue fields $k(y) \xrightarrow{\cong} k(x)$

3. a *Zariski cover* if this is a cover of X, and each f_α is an open immersion.

Remark 2.1.16. Every Zariski cover is a Nisnevich cover, and every Nisnevich cover is an étale cover, however the converses of these statements do not hold.

In the Nisnevich topology, we are also able to retain some of the advantages that the Zariski topology offers. One of the primary advantages is that algebraic K-theory satisfies Nisnevich descent. Additionally we are able to compute the Nisnevich cohomological dimension as the Krull dimension of a scheme (MV99, p.94). Finally, we refer the reader to (AE17, Proposition 7.2), which allows us to treat morphisms of schemes locally as morphisms of affine spaces, analogous to charts of Euclidean space in differential topology.

2.1.5 Colimits

Recall that the primary motivation in passing from $\mathtt{Sm}_k$ to $\mathtt{sPre}(\mathtt{Sm}_k)$ was the existence of colimits. Despite the fact that $\mathtt{Sm}_k$ does not admit all small colimits, it still admits some — as a class of examples, consider colimits of schemes arising from Zariski open covers. The problem is that the Yoneda embedding $y : \mathtt{Sm}_k \to \mathtt{sPre}(\mathtt{Sm}_k)$ does not preserve colimits in general, thus in our efforts to rectify the failure of $\mathtt{Sm}_k$ to admit colimits, we have essentially forgotten about the colimits that it did in fact possess. This is part of the motivation to localize at τ-hypercovers — we see that colimits of schemes correspond to hypercovers on the associated representable presheaves. By our discussion in the previous section, the localization L_τ can be considered as the localization precisely at the class of maps $\mathrm{hocolim} U_\bullet \to X$ for any τ-hypercover $U_\bullet \to X$. Thus colimits of schemes are recorded in the category $\mathtt{Spc}_k^{\mathbb{A}^1}$ as homotopy colimits corresponding to hypercovers. For ease of reference, we summarize this in the following slogan.

Slogan 2.1.17. Colimits of smooth schemes along τ-covers yield homotopy colimits of motivic spaces.

To illustrate this point, we consider the following example, where $\mathbb{G}_m := \mathrm{Spec}\, k\left[x, \frac{1}{x}\right]$ denotes the multiplicative group scheme.

Example 2.1.18. Let $f : \mathbb{G}_m \to \mathbb{A}_k^1$ be given by $z \mapsto z$, and $g : \mathbb{G}_m \to \mathbb{A}_k^1$ be given by $z \mapsto \frac{1}{z}$. Then the diagram

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, "f"] \arrow[d, "g"'] & \mathbb{A}_k^1 \arrow[d] \\ \mathbb{A}_k^1 \arrow[r] & \mathbb{P}_k^1 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is a homotopy pushout of motivic spaces.

Proof. We see that the two copies of the affine line form a Zariski open cover of $\mathbb{P}_k^1$, and hence a Nisnevich open cover of schemes. This corresponds to a hypercover on the representable simplicial presheaves, and after localization at Nisnevich hypercovers, we see that the homotopy pushout of $(\mathbb{A}_k^1 \leftarrow \mathbb{G}_m \to \mathbb{A}_k^1)$ is precisely $\mathbb{P}_k^1$. $\square$

For based topological spaces, recall we have a smash product, defined as

$$X \wedge Y = X \times Y \big/ ((X \times \{y\}) \cup (\{x\} \times Y)).$$

We can think about the category of based topological spaces as the slice category $*/\mathtt{Top}$, where $*$ denotes the one-point space, i.e. the terminal object. By similarly taking the slice category under the terminal object $* := \operatorname{Spec} k$, we obtain a pointed version of $\mathtt{Spc}_k^{\mathbb{A}^1}$, which is often denoted by $\mathtt{Spc}_{k,*}^{\mathbb{A}^1}$.[3] We can then define the smash product as the homotopy cofiber of the canonical map between the coproduct of two pointed motivic spaces into their product:

$$\begin{tikzcd} X \vee Y \arrow[r] \arrow[d] & X \times Y \arrow[d] \\ * \arrow[r] & X \wedge Y. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

One may define the *suspension* as $\Sigma X := S^1 \wedge X$, which we may verify is the same as the homotopy cofiber of $X \to *$. One may see that, since $\mathbb{A}_k^1 \simeq \operatorname{Spec} k$ is contractible, we have that Example 2.1.18 implies that $\mathbb{P}_k^1$ is the homotopy cofiber of the unique map $\mathbb{G}_m \to \operatorname{Spec} k$. Concisely, this example tells us that

$$\mathbb{P}_k^1 \simeq \Sigma \mathbb{G}_m.$$

Recall from topology that the spheres satisfy $S^n \wedge S^m \cong S^{n+m}$. In developing a homotopy theory of schemes, we would like to search for a class of objects satisfying

[3]We note that such a slice category must be taken at the level of model categories rather than homotopy categories in order to have a tractable pointed homotopy theory.

an analogous property. From this motivation, we uncover two types of spheres in $\mathtt{Spc}_k^{\mathbb{A}^1}$. The first, denoted S^1, is called the *simplicial sphere*, and can be thought of as the union of three copies of the affine line, enclosing a triangle. As a simplicial presheaf, we think of it as the constant presheaf at $S^1 = \Delta^1/\partial\Delta^1$. Our second sphere, often called the *Tate sphere*, is taken to be the projective line $\mathbb{P}_k^1 \simeq S^1 \wedge \mathbb{G}_m$.

There are various conventions for the notation on spheres in $\mathbb{A}^1$-homotopy theory, and in the literature one may see $S^{p+q\alpha}$, $S^{p,q}$ or $S^{p+q,q}$ to mean the same thing, depending on the context. In these notes, we will use the convention that

$$S^{p+q\alpha} := (S^1)^{\wedge p} \wedge (\mathbb{G}_m)^{\wedge q}.$$

Exercise 2.1.19. Show that the diagram

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \Sigma(X \wedge Y) \end{tikzcd}$$

is a homotopy pushout diagram. The context for this example is left ambiguous as the result holds in $\mathtt{Spc}_{k,*}^{\mathbb{A}^1}$ just as well as it does for pointed topological spaces.

Example 2.1.20. There is an $\mathbb{A}^1$-homotopy equivalence $\mathbb{A}_k^n \smallsetminus \{0\} \simeq (S^1)^{\wedge(n-1)} \wedge (\mathbb{G}_m)^{\wedge n}$.

Proof. Note that we may construct $\mathbb{A}^n_k \smallsetminus \{0\}$ as a homotopy pushout

$$\begin{tikzcd}
(\mathbb{A}^1_k \smallsetminus \{0\}) \times (\mathbb{A}^{n-1}_k \smallsetminus \{0\}) \arrow[r] \arrow[d] & \mathbb{A}^1_k \times (\mathbb{A}^{n-1}_k \smallsetminus \{0\}) \arrow[d] \\
(\mathbb{A}^1_k \smallsetminus \{0\}) \times \mathbb{A}^n_k \arrow[r] & \mathbb{A}^n_k \smallsetminus \{0\}\,.
\end{tikzcd}$$

Applying the exercise above, we see that

$$\mathbb{A}^n_k \smallsetminus \{0\} \simeq \Sigma(\mathbb{A}^{n-1}_k \smallsetminus \{0\}) \wedge (\mathbb{A}^1_k \smallsetminus \{0\}) = S^1 \wedge (\mathbb{A}^{n-1}_k \smallsetminus \{0\}) \wedge \mathbb{G}_m.$$

The result follows inductively. □

Notation 2.1.21. For a morphism of motivic spaces $f : X \to Y$, denote by Y/X the homotopy cofiber of the map f, that is, the homotopy pushout

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\
* \arrow[r] & Y/X.
\end{tikzcd}$$

Example 2.1.22. *(Excision)* Suppose that X is a smooth scheme over k, that $Z \hookrightarrow X$ is a closed immersion, and that $U \supseteq Z$ is a Zariski open neighborhood of Z inside of X. Then we have a Nisnevich weak equivalence (that is, a weak equivalence in the category $\mathrm{Sh}_{\mathrm{Nis},k}$)

$$\frac{U}{U \smallsetminus Z} \xrightarrow{\sim} \frac{X}{X \smallsetminus Z}.$$

We refer to this result informally as *excision* (not to be confused with excision in the sense of (AE17, Proposition 3.53)), as we regard this weak equivalence as excising the closed subspace $X \smallsetminus U$ from the top and bottom of the cofiber $X/(X \smallsetminus Z)$.

Proof. We remark that $(X \smallsetminus Z)$ and U form a Zariski open cover of X, and that their intersection is $(X \smallsetminus Z) \cap U = U \smallsetminus Z$. As Zariski covers are Nisnevich covers, one remarks that we have a homotopy pushout diagram of motivic spaces

$$\begin{tikzcd} U \smallsetminus Z \arrow[r] \arrow[d] & X \smallsetminus Z \arrow[d] \\ U \arrow[r] & X. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

The fact that the homotopy cofibers of the vertical maps in the diagram above are $\mathbb{A}^1$-weakly equivalent follows from the following diagram:

$$\begin{tikzcd} U \smallsetminus Z \arrow[r] \arrow[d] & X \smallsetminus Z \arrow[r] \arrow[d] & * \arrow[d] \\ U \arrow[r] & X \arrow[r] \arrow[ul, phantom, "\ulcorner", very near start] & \frac{X}{X \smallsetminus Z}. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

As the left and right squares are homotopy cocartesian, it follows formally that the entire rectangle is homotopy cocartesian. □

Example 2.1.23. There is an $\mathbb{A}^1$-homotopy equivalence $\mathbb{P}^n_k / \mathbb{P}^{n-1}_k \simeq (S^1)^{\wedge n} \wedge (\mathbb{G}_m)^{\wedge n}$

Proof. As $\mathbb{P}^n_k \smallsetminus \{0\}$ is the total space of $\mathcal{O}(1)$ on $\mathbb{P}^{n-1}_k$, we have an $\mathbb{A}^1$-equivalence $\mathbb{P}^n_k \smallsetminus \{0\} \simeq \mathbb{P}^{n-1}_k$. Therefore, one sees $\mathbb{P}^n_k / \mathbb{P}^{n-1}_k \simeq \mathbb{P}^n_k / (\mathbb{P}^n_k - \{0\})$. Via excision, we are able to excise everything away from a standard affine chart, from which we may see that $\mathbb{P}^n_k / (\mathbb{P}^n_k - \{0\}) \simeq \mathbb{A}^n_k / (\mathbb{A}^n_k - \{0\})$. Contracting $\mathbb{A}^n_k$, we obtain $* / (\mathbb{A}^n_k - \{0\}) \simeq \Sigma(\mathbb{A}^n_k - \{0\})$. Therefore $\mathbb{P}^n_k / \mathbb{P}^{n-1}_k \simeq \Sigma(\mathbb{A}^n_k - \{0\}) \simeq (S^1)^{\wedge n} \wedge (\mathbb{G}_m)^{\wedge n}$ after applying Example 2.1.20. □

This last example is of particular interest, as it exhibits the cofiber $\mathbb{P}^n_k/\mathbb{P}^{n-1}_k$ as a type of sphere in $\mathbb{A}^1$-homotopy theory. Given an endomorphism of such a motivic sphere, Morel defined a *degree homomorphism*

$$\deg^{\mathbb{A}^1} : \left[\mathbb{P}^n_k/\mathbb{P}^{n-1}_k, \mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right]_{\mathbb{A}^1} \to \mathrm{GW}(k),$$

which he proved was an isomorphism in degrees $n \geq 2$ (Mor06, Corollary 4.11).

Recall that to define a local Brouwer degree of an endomorphism between n-manifolds, we first had to pick a ball containing a point p, and then identify the cofiber $W/(W \smallsetminus \{p\})$ with the n-sphere S^n. This allowed us to construct Diagram 2.1.1, after which we could apply the degree homomorphism $[S^n, S^n] \to \mathbb{Z}$ to define a local degree. An analogous procedure will be available to us in $\mathbb{A}^1$-homotopy theory if, for a Zariski open neighborhood U around a k-rational point x, we are able to associate a canonical $\mathbb{A}^1$-weak equivalence between $U/(U \smallsetminus \{x\})$ and $\mathbb{P}^n_k/\mathbb{P}^{n-1}_k$. Indeed this is possible via the theorem of *purity*.

2.1.6 Purity

One of the major techniques in $\mathbb{A}^1$-homotopy theory comes from the purity theorem. In manifold topology, the tubular neighborhood theorem allows us to define a diffeomorphism between a tubular neighborhood of a smooth immersion and an open neighborhood around its zero section in the normal bundle. In $\mathbb{A}^1$ homotopy theory, the Nisnevich topology isn't fine enough to define such a tubular neighborhood, however we can still get an analog of the tubular neighborhood theorem which will allow us to define, among other things, local $\mathbb{A}^1$-degrees of maps.

Definition 2.1.24. A *Thom space* of a vector bundle $V \to X$ is the cofiber

$$V \big/ (V \smallsetminus X),$$

where $V \smallsetminus X$ denotes the vector bundle minus its zero section. In the literature, this may be denoted by

$$\mathrm{Thom}(V, X) = \mathrm{Th}(V) = X^V.$$

Remark 2.1.25. We may also describe the Thom space of a vector bundle via an $\mathbb{A}^1$-weak equivalence

$$\mathrm{Th}(V) \simeq \frac{\mathrm{Proj}\,(V \oplus \mathcal{O})}{\mathrm{Proj}\,(V)}.$$

Proof. We have a map $V \to V \oplus \mathcal{O}$ sending $v \mapsto (v, 1)$, and we may view this inside of projective space via the inclusion $V \oplus \mathcal{O} \subseteq \mathrm{Proj}\,(V \oplus \mathcal{O})$. Via excision (Example 2.1.22), we have a Nisnevich weak equivalence

$$\frac{\mathrm{Proj}\,(V \oplus \mathcal{O})}{\mathrm{Proj}\,(V \oplus \mathcal{O}) \smallsetminus 0} \simeq \frac{V}{V \smallsetminus 0},$$

where 0 denotes the image of the zero section. We remark that $\mathrm{Proj}\,(V \oplus \mathcal{O}) \smallsetminus 0$ is the total space of $\mathcal{O}(-1)$ on $\mathrm{Proj}\,(V)$, thus we have an $\mathbb{A}^1$-weak equivalence $\mathrm{Proj}\,(V \oplus \mathcal{O}) \smallsetminus 0 \simeq \mathrm{Proj}\,(V)$. The result follows from observing $\frac{\mathrm{Proj}\,(V \oplus \mathcal{O})}{\mathrm{Proj}\,(V \oplus \mathcal{O}) \smallsetminus 0} \simeq \frac{\mathrm{Proj}\,(V \oplus \mathcal{O})}{\mathrm{Proj}\,(V)}$. $\square$

Theorem 2.1.26. *(Purity theorem)* Let $Z \hookrightarrow X$ be a closed immersion in $\mathtt{Sm}_k$. Then we have an $\mathbb{A}^1$-equivalence

$$\frac{X}{X \smallsetminus Z} \simeq \mathrm{Th}(N_Z X),$$

where $N_Z X \to Z$ denotes the normal bundle of Z in X.

Proof. The proof uses the deformation to the normal bundle of Fulton and MacPherson (Ful98). Let f denote the composition of the maps

$$\mathrm{Bl}_{Z \times \{0\}}(X \times \mathbb{A}^1_k) \to X \times \mathbb{A}^1_k \to \mathbb{A}^1_k.$$

We define $D_Z X$ to be the scheme $\mathrm{Bl}_{Z\times\{0\}}(X \times \mathbb{A}^1_k) \smallsetminus \mathrm{Bl}_{Z\times\{0\}}(X \times \{0\})$, and note that f restricts to a map $f\Big|_{D_Z X} : D_Z X \to \mathbb{A}^1_k$. We may compute the fiber of $f|_{D_Z X}$ over 0 as

$$\begin{aligned} f\Big|_{D_Z X}^{-1}(0) &= \mathrm{Proj}\left(N_{Z\times\{0\}}(X \times \mathbb{A}^1_k)\right) \smallsetminus \mathrm{Proj}\left(N_{Z\times\{0\}}(X \times \{0\})\right) \\ &= \mathrm{Proj}(N_Z X \oplus \mathcal{O}) \smallsetminus \mathrm{Proj}\,(N_Z X) \\ &= N_Z X, \end{aligned}$$

and the fiber over 1 as $f\Big|_{D_Z X}^{-1}(1) = X$. Since $Z \times \mathbb{A}^1_k$ determines a closed subscheme in $D_Z X$, we have that the fiber over 0 is $Z \subseteq N_Z X$ and the fiber over 1 is $Z \subseteq X$. Thus we obtain morphisms of pairs

$$\begin{aligned} (Z, N_Z X) &\xrightarrow{i_0} (Z \times \mathbb{A}^1_k, D_Z X) \\ (Z, X) &\xrightarrow{i_1} (Z \times \mathbb{A}^1_k, D_Z X), \end{aligned} \tag{2.1.27}$$

corresponding to the inclusions of the fibers over the points 0 and 1, respectively. To prove the purity theorem, it now suffices to show that the induced morphisms on cofibers are weak equivalences:

$$\begin{aligned} \frac{N_Z X}{N_Z X \setminus Z} &\to \frac{D_Z X}{D_Z X \setminus Z \times \mathbb{A}^1_k} \\ \frac{X}{X \setminus Z} &\to \frac{D_Z X}{D_Z X \setminus Z \times \mathbb{A}^1_k}. \end{aligned}$$

Lemma 2.1.28. (AE17, Lemma 7.3) Suppose that $\mathbf{P}$ is a property of smooth pairs of schemes such that the following properties hold:

1. If (Z, X) is a smooth pair of schemes and $\{U_\alpha \to X\}_{\alpha \in A}$ is a Zariski cover of X such that $\mathbf{P}$ holds for the pair

   $$(Z \times_X U_{\alpha_1} \times_X \cdots \times_X U_{\alpha_n}, U_{\alpha_1} \times_X \cdots \times_X U_{\alpha_n})$$

   for each $(\alpha_1, \ldots, \alpha_n)$, then $\mathbf{P}$ holds for (Z, X)

2. If $(Z, X) \to (Z, Y)$ is a morphism of smooth pairs inducing an isomorphism on Z such that $X \to Y$ is étale, then $\mathbf{P}$ holds for (Z, X) if and only if $\mathbf{P}$ holds for (Z, Y)

3. **P** holds for the pair $(Z, \mathbb{A}^n_k \times Z)$,

then **P** holds for all smooth pairs.

To conclude the proof of purity, we let **P** be the property on the pair (Z, X) that the morphisms in Equation 2.1.27 induce homotopy pushout diagrams[4]

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & \frac{N_Z X}{N_Z X \smallsetminus Z} \arrow[d] \\ Z \times \mathbb{A}^1_k \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \frac{D_Z X}{D_Z X \smallsetminus Z \times \mathbb{A}^1_k} \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[r] \arrow[d] & \frac{X}{X \smallsetminus Z} \arrow[d] \\ Z \times \mathbb{A}^1_k \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \frac{D_Z X}{D_Z X \smallsetminus Z \times \mathbb{A}^1_k}. \end{tikzcd}$$

One may check that Lemma 2.1.28 holds for this property, and therefore since $Z \to Z \times \mathbb{A}^1_k$ is a weak equivalence, a homotopy pushout along this map is also a weak equivalence. Thus we obtain a sequence of $\mathbb{A}^1$-weak equivalences

$$\frac{X}{X \smallsetminus Z} \xrightarrow{\sim} \frac{D_Z X}{D_Z X \smallsetminus Z \times \mathbb{A}^1_k} \xleftarrow{\sim} \frac{N_Z X}{N_Z X \smallsetminus Z} = \mathrm{Th}(N_Z X).$$

$\square$

2.2 $\mathbb{A}^1$-enumerative geometry

As discussed above, Morel exhibited the global degree of maps of motivic spheres as

$$\deg^{\mathbb{A}^1} : [\mathbb{P}^n_k / \mathbb{P}^{n-1}_k, \mathbb{P}^n_k / \mathbb{P}^{n-1}_k]_{\mathbb{A}^1} \to \mathrm{GW}(k).$$

[4]Equivalently, one may say that i_0 and i_1 are *weakly excisive* morphisms of pairs (Hoy17, Definition 3.17).

Recall that, for a scheme X, we have functors to the category of topological spaces obtained by taking real and complex points, that is, $X \mapsto X(\mathbb{R})$ and $X \mapsto X(\mathbb{C})$. Morel's degree map satisfies a compatibility diagram with the degree maps we recognize from algebraic topology[5]

$$\begin{tikzcd}
{[S^n, S^n]} \arrow[d, "\deg^{\mathrm{top}}"'] & {[\mathbb{P}^n_{\mathbb{R}}/\mathbb{P}^{n-1}_{\mathbb{R}}, \mathbb{P}^n_{\mathbb{R}}/\mathbb{P}^{n-1}_{\mathbb{R}}]_{\mathbb{A}^1}} \arrow[l, "\mathbb{R}\text{-pts}"'] \arrow[r, "\mathbb{C}\text{-pts}"] \arrow[d, "\deg^{\mathbb{A}^1}"] & {[S^{2n}, S^{2n}]} \arrow[d, "\deg^{\mathrm{top}}"] \\
\mathbb{Z} & \mathrm{GW}(\mathbb{R}) \arrow[l, "\mathrm{sig}"] \arrow[r, "\mathrm{rank}"'] & \mathbb{Z}.
\end{tikzcd} \tag{2.2.1}$$

We can apply the purity theorem to develop a notion of local degree for a general map between schemes of the same dimension. Suppose that $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$, and $x \in \mathbb{A}^n_k$ is a k-rational preimage of a k-rational point $y = f(x)$. Further suppose that x is an isolated point in $f^{-1}(y)$, meaning that there exists a Zariski open set $U \subseteq \mathbb{A}^n_k$ such that $x \in U$ and $f^{-1}(y) \cap U = x$.

Definition 2.2.2. In the conditions above, the *local* $\mathbb{A}^1$*-degree* of f at x is defined to be the degree of the map

$$U \Big/ (U \smallsetminus \{x\}) \xrightarrow{\overline{f}} \mathbb{A}^n_k \Big/ (\mathbb{A}^n_k \smallsetminus \{y\}),$$

under the $\mathbb{A}^1$-weak equivalences $U \Big/ (U \smallsetminus \{x\}) \cong \mathrm{Th}(T_x \mathbb{A}^n_k) \cong \mathbb{P}^n_k \Big/ \mathbb{P}^{n-1}_k$ and $\mathbb{A}^n_k \Big/ (\mathbb{A}^n_k \smallsetminus$

[5] The commutativity of this diagram is one of the key features of Morel's $\mathbb{A}^1$-degree and is attributable to him (Mor06, p. 1037). We can provide an alternative justification of this fact following the discussion of the EKL form in Section 2.2.1.

$\{y\}) \cong \mathbb{P}^n_k \big/ \mathbb{P}^{n-1}_k$ provided to us by purity and by the canonical trivialization of the tangent space of affine space.

Dropping the assumption that $k(x) = k$, but still assuming that y is k-rational, we may equivalently define $\deg^{\mathbb{A}^1}_x f$ as the degree of the composite

$$\mathbb{P}^n_k \big/ \mathbb{P}^{n-1}_k \to \mathbb{P}^n_k \big/ (\mathbb{P}^n_k \smallsetminus \{x\}) \cong U \big/ (U \smallsetminus \{x\}) \xrightarrow{\bar{f}} \mathbb{A}^n_k \big/ (\mathbb{A}^n_k \smallsetminus \{y\}) \cong \mathbb{P}^n_k \big/ \mathbb{P}^{n-1}_k.$$

Proposition 2.2.3. These definitions of the local degree are equivalent. This was proven in (KW19, Prop. 12), which is a generalization of a proof of Hoyois (Hoy14, Lemma 5.5).

Equation 2.1.8 admits the following generalization to endomorphisms of affine space.

Proposition 2.2.4. (KW19, Proposition 15) Let $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$, assume that f is étale at a closed point $x \in \mathbb{A}^n_k$, and assume that that $f(x) = y$ is k-rational and that x is isolated in its fiber. Then the local degree is given by

$$\deg^{\mathbb{A}^1}_x (f) = \operatorname{Tr}_{k(x)/k} \langle \operatorname{Jac}(f)(x) \rangle .$$

Remark 2.2.5. At a non-rational point p whose residue field $k(p)|k$ is a finite separable extension of the ground field, the local $\mathbb{A}^1$-degree can be computed by base changing to $k(p)$ to compute the local degree rationally, and applying the field trace $\operatorname{Tr}_{k(p)/k}$ to obtain a well-defined element of $\operatorname{GW}(k)$ (BBM+21).

2.2.1 The Eisenbud–Khimshiashvili–Levine signature formula

Given a morphism $f = (f_1, \ldots, f_n) : \mathbb{A}^n_k \to \mathbb{A}^n_k$ with an isolated zero at the origin, we may associate to it a certain isomorphism class of bilinear forms $w_0^{\mathrm{EKL}}(f)$, called the *Eisenbud–Levine–Khimshiashvili (EKL) class.* This was studied by Eisenbud and Levine, and independently by Khimshiashvili, in the case where f is a smooth endomorphism of $\mathbb{R}^n$ (EL77; Him77). They ascertained that the degree $\deg_0^{\mathrm{top}} f$ can be computed as the signature of the form $w_0^{\mathrm{EKL}}(f)$. If f is furthermore assumed to be real analytic, the rank of this form recovers the degree of the complexification $f_{\mathbb{C}}$ (Pal67). This bilinear form $w_0^{\mathrm{EKL}}(f)$ can be defined over an arbitrary field k, and in this setting Eisenbud asked the following question: does $w_0^{\mathrm{EKL}}(f)$ have any topological interpretation? We will see that the answer is yes, via work of Kass and Wickelgren (KW19).

Suppose that $f = (f_1, \ldots, f_n) : \mathbb{A}^n_k \to \mathbb{A}^n_k$ has an isolated zero at the origin, and define the local k-algebra

$$Q_0(f) := \frac{k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}}{(f_1, \ldots, f_n)}.$$

We may pick polynomials a_{ij} so that, for each i, we have the equality

$$f_i(x_1, \ldots, x_n) = f_i(0) + \sum_{j=1}^{n} a_{ij} \cdot x_j.$$

By taking their determinant, we define $E_0(f) := \det(a_{ij})$ as an element of $Q_0(f)$, which we refer to as the *distinguished socle element* of the local algebra $Q_0(f)$. We remark that when $\mathrm{Jac}(f)$ is a nonzero element of $Q_0(f)$, one has the equality (SS75, 4.7 Korollar)

$$\mathrm{Jac}(f) = \dim_k(Q_0(f)) \cdot E_0(f).$$

We then pick η to be any k-linear vector space homomorphism $\eta : Q_0(f) \to k$ satisfying $\eta(E_0(f)) = 1$. One may check that the following bilinear form

$$Q_0(f) \times Q_0(f) \to k$$
$$(u, v) \mapsto \eta(u \cdot v)$$

is non-degenerate and its isomorphism class is independent of the choice of η (EL77, Propositions 3.4,3.5), (KW19, §3). The class of this form in $\mathrm{GW}(k)$ is referred to as the EKL class, and denoted by $w_0^{\mathrm{EKL}}(f)$.

Example 2.2.6. If $f : \mathbb{A}^1_k \to \mathbb{A}^1_k$ is given by $z \mapsto z^2$, we may see that $Q_0(f) = k[z]_{(z)} \big/ (z^2)$. We see that f has an isolated zero at the origin, and that

$$f = f(0) + x \cdot x,$$

hence $E_0(f) = x$. We determine $\eta : Q_0(f) \to k$ on a basis for $Q_0(f)$ by setting $\eta(x) = 1$ and $\eta(1) = 0$. Then we compute the EKL form via its Gram matrix as:

$$\begin{pmatrix} \eta(1 \cdot 1) & \eta(1 \cdot x) \\ \eta(x \cdot 1) & \eta(x \cdot x) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathbb{H}.$$

Theorem 2.2.7. If $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ is any endomorphism of affine space with an isolated zero at the origin, there is an equality $\deg_0^{\mathbb{A}^1} f = w_0^{\mathrm{EKL}}(f)$ in $\mathrm{GW}(k)$ (KW19).

In particular we observe that the compatibility stated in Diagram 2.2.1 is justified by this theorem, combined with the results of Eisenbud–Khimshiashvili–Levine and Palamodov. Moreover we remark that the EKL form can be defined at any k-rational point, and an analogous statement to Theorem 2.2.7 holds in this context.

Exercise 2.2.8.

1. Compute the degree of $f : \mathbb{A}^2_k \to \mathbb{A}^2_k$, given as $f(x, y) = (4x^3, 2y)$ in the case where $\mathrm{char}(k) \neq 2$.

2. Supposing f is étale at the origin 0, show that $w_0^{\mathrm{EKL}}(f) = \langle \mathrm{Jac}(f)(0) \rangle$ is an equality in $\mathrm{GW}(k)$. Show furthermore that an analogous equality holds at any k-rational point x.

As a generalization of Exercise 2.2.8(2), one may show that if f is étale at a point x, one has the following equality in $\mathrm{GW}(k)$

$$w_x^{\mathrm{EKL}}(f) = \mathrm{Tr}_{k(x)/k} \langle \mathrm{Jac}(f)(x) \rangle . \tag{2.2.9}$$

This is shown using Galois descent, as in (KW19, Lemma 33).

2.2.2 Sketch of proof for Theorem 2.2.7

Step 1: We can see that $\deg_0^{\mathbb{A}^1} f$ and $w_0^{\mathrm{EKL}}(f)$ are finitely determined in the sense that they are unchanged by changing f to $f+g$, with $g = (g_1, \ldots, g_n)$, and $g_i \in \mathfrak{m}_0^N$ for sufficiently large N, where $\mathfrak{m}_0 := (x_1, \ldots, x_n)$ denotes the maximal ideal at the origin (KW19, Lemma 17).

Step 2: By changing f to $f+g$, we may assume that f extends to a finite, flat morphism $F : \mathbb{P}_k^n \to \mathbb{P}_k^n$, where $F^{-1}(\mathbb{A}_k^n) \subseteq \mathbb{A}_k^n$ and $F|_{F^{-1}(0) \setminus \{0\}}$ is étale (KW19, Proposition 23).

Proposition 2.2.10. *(Scheja–Storch)* (SS75, §3, pp.180—182) We have that $w_0^{\mathrm{EKL}}(f)$ is a direct summand of the fiber at 0 of a family of bilinear forms over $\mathbb{A}_k^n$, which we construct below.

We will prove Proposition 2.2.10 following the construction of this family of bilinear forms.

The Scheja–Storch construction Let $F : \mathrm{Spec}(P) \to \mathrm{Spec}(A)$, where

$$P = k[x_1, \ldots, x_n]$$

$$A = k[y_1, \ldots, y_n].$$

One may show that the collection $\{t_1, \ldots, t_n\}$ is a regular sequence in $A[x_1, \ldots, x_n]$, where $t_i := y_i - F_i(x_1, \ldots, x_n)$. Then

$$B = A[x_1, \ldots, x_n] \big/ \langle t_1, \ldots, t_n \rangle$$

is a relative complete intersection, which parametrizes the fibers of F. This regular sequence determines a canonical isomorphism (SS75, Satz 3.3)

$$\theta : \mathrm{Hom}_A(B, A) \xrightarrow{\cong} B,$$

via the following procedure: we may first express

$$t_j \otimes 1 - 1 \otimes t_j = \sum_{i=1}^{n} a_{ij} \left(x_i \otimes 1 - 1 \otimes x_i \right),$$

where each a_{ij} is an element of $A[x_1, \ldots, x_n] \otimes_A A[x_1, \ldots, x_n]$. Under the projection map $A[x_1, \ldots, x_n] \otimes_A A[x_1, \ldots, x_n] \to B \otimes_A B$, we have that $\det(a_{ij})$ is mapped to some element Δ. We now consider the bijection

$$B \otimes_A B \to \mathrm{Hom}_A(\mathrm{Hom}_A(B, A), B)$$
$$b \otimes c \mapsto (\phi \mapsto \phi(b)c),$$

and define θ to be the image of Δ. We remark that a priori θ is an A-module homomorphism between $\mathrm{Hom}_A(B, A)$ and B, which both have B-module structures. It is in fact a B-module homomorphism, and is moreover an isomorphism by (SS75, Satz 3.3). Defining $\eta = \theta^{-1}(1)$, we have that η determines a bilinear form, which we denote by w

$$B \otimes_A B \to A$$

$$b \otimes c \overset{w}{\mapsto} \eta(bc).$$

Proof of Proposition 2.2.10: We note that, when $y_1 = \ldots = y_n = 0$, $\mathrm{Spec}(B)$ is the fiber of F over 0, consisting of a discrete set of points. This corresponds to a disjoint union of schemes. If b and c lie in different components, then their product is zero. This implies that the bilinear form w decomposes into an orthogonal direct sum of forms over each factor in $F^{-1}(0)$. These factors correspond to EKL forms at each point in the fiber $F^{-1}(0)$, and in particular over $0 \in F^{-1}(0)$, we recover the EKL form $w_0^{\mathrm{EKL}}(F)$.

□

The following theorem will allow us to relate the EKL forms at various points in the fiber $F^{-1}(0)$.

Theorem 2.2.11. *(Harder's Theorem)* (Lam06, VII.3.13) A family of symmetric bilinear forms over $\mathbb{A}^1_k$ is constant (respectively, has constant specialization to k-points) for characteristic not equal to 2 (resp. any k). In particular when $\mathrm{char}(k) \neq 2$, for any finite $k[t]$-module M, we have that the family of bilinear forms $M \times_{k[t]} M \to k[t]$ is pulled back from some bilinear form $N \times_k N \to k$ via the unique morphism of schemes $\mathbb{A}^1_k \to \mathrm{Spec}(k)$.

Step 3: We choose y so that $F|_{F^{-1}(y)}$ is étale. One may use the generalization of Exercise 2.2.8(2) as stated in Equation 2.2.9, combined with Proposition 2.2.4 to see that

$$\sum_{x\in F^{-1}(y)} w_x^{\mathrm{EKL}}(F) = \sum_{x\in F^{-1}(y)} \deg_x^{\mathbb{A}^1} F.$$

By Harder's theorem, we have that $\sum_{x\in F^{-1}(y)} w_x^{\mathrm{EKL}}(F) = \sum_{x\in F^{-1}(0)} w_x^{\mathrm{EKL}}(F)$, and by the local formula for degree, we see that

$$\sum_{x\in F^{-1}(y)} \deg_x^{\mathbb{A}^1} F = \deg^{\mathbb{A}^1} F = \sum_{x\in F^{-1}(0)} \deg_x^{\mathbb{A}^1} F.$$

Thus $\sum_{x\in F^{-1}(0)} w_x^{\mathrm{EKL}}(F) = \sum_{x\in F^{-1}(0)} \deg_x^{\mathbb{A}^1} F$. Since $F|_{F^{-1}(0)\smallsetminus\{0\}}$ is étale, we may iteratively apply the equality in Equation 2.2.9 to cancel terms, leaving us with the local degree and EKL form at the origin:

$$w_0^{\mathrm{EKL}}(F) = \deg_0^{\mathbb{A}^1} F.$$

Therefore by finite determinacy we recover the desired equality $w_0^{\mathrm{EKL}}(f) = \deg_0^{\mathbb{A}^1}(f)$. This concludes the proof of Theorem 2.2.7.

2.2.3 $\mathbb{A}^1$-Milnor numbers

The following section is based off of joint work by Jesse Kass and Kirsten Wickelgren (KW19, §8). A variety over a perfect field is generically smooth, although it may admit a singular locus where the dimension of the tangent space exceeds the dimension of the variety, for example a self-intersecting point on a singular elliptic curve. Singularities are generally difficult to study, although certain classes are more tractable than others. There is a particular class of singularities, called *nodes*, which are in some sense the most generic. If k is a field of characteristic not equal to 2, then a node is given by an equation $x_1^2 + \ldots + x_n^2 = 0$ over a separable algebraic closure $\overline{k}$.

Consider a point p on a hypersurface $\{f(x_1, \ldots, x_n) = 0\} \subseteq \mathbb{A}_k^n$. Fix values $a_1, \ldots, a_n$, and consider the family

$$f(x_1, \ldots, x_n) + a_1 x_1 + \ldots + a_n x_n = t,$$

parametrized over the affine t-line. This hypersurface bifurcates into nodes over $\overline{k}$. Given any hypersurface $g(x_1, \ldots, x_n)$ with a node at a k-rational point p, we define the *type* of the node as the element in $\mathrm{GW}(k)$ corresponding to the rank one form represented by the Hessian matrix at p:

$$\mathrm{type}(p) := \left\langle \frac{\partial^2 g}{\partial x_i \partial x_j}(p) \right\rangle.$$

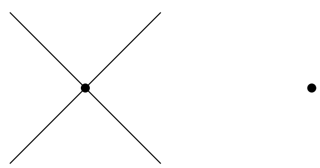
In particular, we see that:

$$\mathrm{type}(x_1^2 + a x_2^2 = 0) := \langle a \rangle$$
$$\mathrm{type}\left(\sum_{i=1}^n a_i x_i^2 = 0 \right) := \left\langle 2^n \prod_{i=1}^n a_i \right\rangle.$$

In the case where we have a node at p with $k(p) = L$, then L is separable over k (DK73, Exposé XV, Théorème 1.2.6), and we define the type of the node as the trace of the type over its residue field. In the examples above, this gives:

$$\operatorname{type}\left(\sum_{i=1}^{n} a_i x_i^2 = 0\right) := \operatorname{Tr}_{L/k}\left\langle 2^n \prod_{i=1}^{n} a_i \right\rangle.$$

Thus the type encodes the field of definition of the node, as well as its tangent direction. In the case where $k = \mathbb{R}$, we can visualize the possible $\mathbb{R}$-rational nodes in degree two as:



$x_1^2 - x_2^2 = 0$ $\qquad$ $x_1^2 + x_2^2 = 0$

(a) split $\qquad$ (b) non-split

Here we may think of *split* as corresponding to the existence of rational tangent directions, while *non-split* refers to non-rational tangent directions. Over fields that aren't $\mathbb{R}$, it is possible to have many different split nodes.

In the case where $k = \mathbb{C}$, for any $(a_1, \ldots, a_n)$ sufficiently close to 0, it is a classical result that the number of nodes in this family is a constant integer, equal to $\deg_0^{\mathrm{top}} \operatorname{grad} f =: \mu$, which is called the *Milnor number*. This admits a generalization as follows.

Theorem 2.2.12. (KW16, Corollary 45) Assume that f has a single isolated singularity at the origin. Then for a generic $(a_1, \ldots, a_n)$, we have that the sum over nodes on the hypersurface $f + a_1x_1 + \ldots + a_nx_n = t$ is

$$\sum_{\substack{\text{nodes } p \\ \text{in family}}} \text{type}(p) = \deg_0^{\mathbb{A}^1} \text{grad} f =: \mu_0^{\mathbb{A}^1} f.$$

We refer to this as the $\mathbb{A}^1$-*Milnor number*. We remark that the classical Milnor number can be recovered by taking the rank of the $\mathbb{A}^1$-Milnor number.

Example 2.2.13. Let $f(x, y) = x^3 - y^2$, over a field of characteristic not equal to 2 or 3. Let $p = (0, 0)$ be a point on the hypersurface $\{f = 0\}$. We can compute $\text{grad} f = (3x^2, -2y)$, and then we have that

$$\begin{aligned} \deg \text{grad} f &= \deg(3x^2) \cdot \deg(-2y) \\ &= \begin{pmatrix} 0 & 1/3 \\ 1/3 & 0 \end{pmatrix} \langle -2 \rangle \\ &= \mathbb{H}. \end{aligned}$$

This has rank two, so the classical Milnor number is $\mu = 2$. We can take our family to be $y^2 = x^3 + ax + t$. If $a = 0$, then we have a node at 0. In general, for $a \neq 0$, we have nodes at those t with the property that the discriminant of the curve $y^2 = x^3 + ax + t$ vanishes, that is at those t where $\Delta = -16\,(4a^3 + 27t^2) = 0$.

This has at most two solutions in t, which we may denote by $\{x^2 + u_1 y^2 = 0\}$ and $\{x^2 + u_2 y^2 = 0\}$, and we see by Theorem 2.2.12 that $\mathbb{H} = \langle u_1 \rangle + \langle u_2 \rangle$. This implies, by taking determinants, that -1 agrees with $u_1 u_2$ up to squares. This provides us with obstructions to the existence of pairs of nodes of certain types, depending on the choice of field we are working over. For example:

- Over $\mathbb{F}_5$, we see that $\langle 1 \rangle = \langle -1 \rangle$ in $\mathrm{GW}(\mathbb{F}_5)$ implying that $u_1 u_2$ is always a square. In particular, u_1 and u_2 cannot have the property that exactly one of them is a non-square, meaning that we cannot bifurcate into a split and a non-split node.

- Over $\mathbb{F}_7$ we have that $\langle 1 \rangle \neq \langle -1 \rangle$, implying $u_1 u_2$ is a non-square, so we cannot bifurcate into two split or two non-split nodes.

Exercise 2.2.14. Compute $\mu^{\mathbb{A}^1}$ for the following ADE singularities over $\mathbb{Q}$:

singularity	equation
A_n	$x^2 + y^{n+1}$
D_n	$y(x^2 + y^{n-2}) \quad (n \geq 4)$
E_6	$x^3 + y^4$
E_7	$x(x^2 + y^3)$
E_8	$x^3 + y^5$.

2.2.4 An arithmetic count of the lines on a smooth cubic surface

The following is based off of joint work of Jesse Kass and Kirsten Wickelgren (KW21). Let $f \in k[x_0, x_1, x_2, x_3]$ be a homogeneous polynomial of degree three. Consider the following surface

$$V = \{f = 0\} \subseteq \operatorname{Proj} k[x_0, x_1, x_2, x_3] = \mathbb{P}^3_k,$$

and suppose that V is smooth.

Theorem 2.2.15. *(Cayley-Salmon Theorem)* When $k = \mathbb{C}$, there are exactly 27 lines on V (Cay09).

Proof. Consider the Grassmannian $\mathrm{Gr}_{\mathbb{C}}(2,4)$, which parametrizes 2-dimensional complex subspaces $W \subseteq \mathbb{C}^{\oplus 4}$, or equivalently, lines in $\mathbb{P}^3_{\mathbb{C}}$. As the Grassmannian is a moduli space, it admits a *tautological bundle* $\mathcal{S}$ whose fiber over any point $W \in \mathrm{Gr}_{\mathbb{C}}(2,4)$ is the vector space W itself. A chosen homogeneous polynomial f of degree three defines a section σ_f of $\mathrm{Sym}^3 \mathcal{S}^*$, where

$$\sigma_f\left([W]\right) = \left. f\right|_W .$$

Thus we see that the line $\ell \subseteq \mathbb{P}^3_{\mathbb{C}}$ corresponding to $[W]$ lies on the surface V if and only if $\sigma_f[W] = 0$. One may see that σ_f has isolated zeros (EH16, Corollary 6.17), and thus we may express the Euler class of the bundle as

$$e(\mathrm{Sym}^3\mathcal{S}^*) = c_4(\mathrm{Sym}^3\mathcal{S}^*) = \sum_{\ell} \deg_{\ell}^{\mathrm{top}} \sigma_f, \tag{2.2.16}$$

where this last sum is over the zeros of σ_f. We determine $\deg_{\ell}^{\mathrm{top}} \sigma_f$ by choosing local coordinates near ℓ on $\mathrm{Gr}_{\mathbb{C}}(2,4)$ as well as a compatible trivialization for $\mathrm{Sym}^3\mathcal{S}^*$ over this coordinate patch. Then σ_f may be viewed as a function

$$\mathbb{A}^4_{\mathbb{C}} \supseteq U \xrightarrow{\sigma_f} \mathbb{A}^4_{\mathbb{C}}$$

with an isolated zero at ℓ. We can then define $\deg_{\ell}^{\mathrm{top}} \sigma_f$ as the local degree of this function. It is a fact that the smoothness of V implies that σ_f vanishes to order 1 at ℓ. Thus the Euler class counts the number of lines on V. Finally, one may compute $c_4(\mathrm{Sym}^3\mathcal{S}^*) = 27$ by applying the splitting principle and computing the cohomology of $\mathrm{Gr}_{\mathbb{C}}(2,4)$. □

In the real case, Schäfli (Sch60) and Segre (Seg42) showed that there can be 3, 7, 15, or 27 real lines on V. One of the main differences between the real and the complex case was the distinction that Segre drew between *hyperbolic* and *elliptic* lines.

Definition 2.2.17. We say that $I \in \mathrm{PGL}_2(\mathbb{R})$ is *hyperbolic* (resp. *elliptic*) if the set

$$\mathrm{Fix}(I) = \{x \in \mathbb{P}^1_{\mathbb{R}} \ : \ Ix = x\}$$

consists of two real points (resp. a complex conjugate pair of points).

To a real line $\ell \subseteq V$ we may associate an involution $I \in \mathrm{Aut}(\ell) \cong \mathrm{PSL}_2(\mathbb{R})$, where I sends $p \in \ell$ to $q \in \ell$ if $T_pV \cap V = \ell \cup Q$, for some Q satisfying $\ell \cap Q = \{p, q\}$, (that is, for any point p on a line ℓ, there is exactly one other point q having the same tangent space). We can say that ℓ is hyperbolic (resp. elliptic) whenever I is.

Alternatively, we may describe these classes of lines topologically. We think of the frame bundle as a principal SO(3)-bundle over $\mathbb{R}\mathrm{P}^3$. As SO(3) admits a double cover Spin(3), from any principal SO(3)-bundle we may obtain a principal Spin(3)-bundle. Traveling on our cubic surface along the line ℓ gives a distinguished choice of frame at every point on ℓ, that is, a loop in the frame bundle. This loop may or may not lift to the associated Spin(3)-bundle. If the loop lifts, then ℓ is hyperbolic, and if it doesn't then ℓ is elliptic.

Theorem 2.2.18. In the real case, we have the following relationship between hyperbolic and elliptic lines:

$$\#\{\text{real hyperbolic lines on } V\} - \#\{\text{real elliptic lines on } V\} = 3.$$

We refer the reader to the following sources (Seg42; BS95; HS12; OT14; FK15).

Proof sketch. Via the map $\sigma_f : \mathrm{Gr}_{\mathbb{R}}(2,4) \to \mathrm{Sym}^3\mathcal{S}^*$, we have that

$$e(\mathrm{Sym}^3\mathcal{S}^*) = \sum_{\substack{\ell \in \mathrm{Gr}_{\mathbb{R}}(2,4) \\ \sigma_f(\ell)=0}} \deg_\ell^{\mathrm{top}} \sigma_f.$$

One may also show that

$$\deg_{\ell}^{\mathrm{top}} \sigma_f = \begin{cases} 1 & \text{if } \ell \text{ is hyperbolic} \\ \\ -1 & \text{if } \ell \text{ is elliptic,} \end{cases}$$

and compute that $e(\mathrm{Sym}^3 \mathcal{S}^*) = 3$ using the Grassmannian of oriented planes. $\square$

To define a notion of hyperbolic and elliptic which holds in more generality, we introduce the *type* of a line. As before, we let $V \subseteq \mathbb{P}_k^3$ be a smooth cubic surface, and consider a closed point $\ell \in \mathrm{Gr}_k(2,4)$, with residue field $L = k(\ell)$. We can then view ℓ as a closed immersion

$$\ell \cong \mathbb{P}_L^1 \hookrightarrow \mathbb{P}_k^3 \otimes_k L.$$

Given such a line $\ell \subseteq V$, we again have an associated involution:

$$I = \begin{pmatrix} a & b \\ \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(L).$$

Since I is an involution, its fixed points satisfy $\frac{az+d}{cz+d} = z$, from which we can see they are defined over the field $L\left(\sqrt{D}\right)$, where D is the discriminant of the subscheme $\mathrm{Fix}(I) \subseteq \mathbb{P}_L^1$.

Definition 2.2.19. The *type* of a line ℓ is the element of $\mathrm{GW}(k(\ell))$ given by

$$\mathrm{type}(\ell) := \langle D \rangle = \langle ad - bc \rangle = \langle -1 \rangle \deg^{\mathbb{A}^1}(I).$$

We say a line is *hyperbolic* if $\mathrm{type}(\ell) = \langle 1 \rangle$, and *elliptic* otherwise.

Theorem 2.2.20. (KW21, Theorem 2) The number of lines on a smooth cubic surface is computed via the following weighted count

$$\sum_{\ell \subseteq V} \mathrm{Tr}_{k(\ell)/k}(\mathrm{type}(\ell)) = 15 \cdot \langle 1 \rangle + 12 \cdot \langle -1 \rangle .$$

Remark 2.2.21. We may apply the previous theorem to observe the following results:

1. If $k = \mathbb{C}$, then by taking the rank, we obtain the Cayley-Salmon Theorem (2.2.15), stating that the number of lines on a cubic surface is 27.

2. If $k = \mathbb{R}$, then $\mathrm{Tr}_{\mathbb{C}/\mathbb{R}} \langle 1 \rangle = \langle 1 \rangle \oplus \langle -1 \rangle$. Taking the signature, we recover Theorem 2.2.18, stating that the number of hyperbolic lines minus the number of elliptic lines is 3.

As a particular application, if we are working over a finite field $k = \mathbb{F}_q$, then its square classes are $\mathbb{F}_q^\times \big/ (\mathbb{F}_q^\times)^2 \cong \{1, u\}$. Thus the type of a line ℓ over $\mathbb{F}_{q^a}$ is either $\langle 1 \rangle$ or $\langle u_a \rangle$, which by Definition 2.2.19 we call hyperbolic or elliptic, respectively.

Corollary 2.2.22. (KW21, Theorem 1) For any natural number a, we have that the number of lines on V satisfies

$$\begin{aligned} &\#\{\text{elliptic lines with field of definition } \mathbb{F}_{q^{2a+1}}\} \\ &+ \#\{\text{hyperbolic lines with field of definition } \mathbb{F}_{q^{2a}}\} \equiv 0 \pmod{2}. \end{aligned}$$

In particular when all the lines in question are defined over a common field k, we have that the number of elliptic lines is even.

In order to prove Theorem 2.2.20, one considers σ_f to be a section of the bundle $\mathrm{Sym}^3\mathcal{S}^* \to \mathrm{Gr}_k(2,4)$, and computes a sum over its isolated zeros, weighted by their local index. Over the complex numbers, this is precisely Equation 2.2.16, which recovers the Euler number of the bundle. In a more general context, however, we will want to obtain an element of $\mathrm{GW}(k)$. This requires us to use an enriched notion of an *Euler class*, described below.

Digression 2.2.23. In this exposition, given a vector bundle $E \to X$ with section σ, we use the Euler class $e(E,\sigma)$ valued in $\mathrm{GW}(k)$ of (KW21, Section 4). In the literature, there are a number of other Euler classes which coincide with this definition in various settings. One may define this Euler class via Chow-Witt groups (BM00) or oriented Chow groups (Fas08) as in the work of M. Levine (Lev20). In his seminal book, Morel defines the Euler class of a bundle $E \to X$ as a cohomology class in twisted Milnor-Witt K-theory $H^n(X; \mathcal{K}_n^{\mathrm{MW}}(\det E^*))$ (Mor12), and when $\det(E^*)$ is trivial, one may relate these Euler classes up to a unit multiple via the isomorphism

$$H^n(X; \mathcal{K}_n^{\mathrm{MW}}(\det E^*)) \cong \widetilde{\mathrm{CH}}(X, \det E^*).$$

For more details, see the work of Asok and Fasel (AF16). Other versions of the Euler class in $\mathbb{A}^1$-homotopy theory occur in the work of Déglise, Jin and Khan (DJK21) and the work of Levine and Raksit (LR20). Many of these notions are equated in work of Bachmann and Wickelgren (BW21).

Definition 2.2.24. Let X be a smooth projective scheme of dimension r, and let $\mathcal{E} \to X$ be a rank r bundle. We say that $\mathcal{E}$ is *relatively oriented* if we are given an isomorphism

$$\mathrm{Hom}(\det TX, \det \mathcal{E}) \cong \mathcal{L}^{\otimes 2},$$

where $\mathcal{L}$ is a line bundle on X.

Suppose that σ is a section of a relatively oriented bundle $\mathcal{E}$ with isolated zeros, and define $Z = \{\sigma = 0\}$ to be its vanishing locus. For each $x \in Z$, we will define $\deg_x^{\mathbb{A}^1} \sigma$ as follows:

1. Choose *Nisnevich coordinates* ((KW21, Definition 17)) near $x \in Z$, that is, pick an open neighborhood $U \subseteq X$ around x, and an étale morphism $\phi : U \to \mathbb{A}_k^r$ such that $k(\phi(x)) \cong k(x)$.

2. Choose a *compatible oriented* trivialization $\mathcal{E}\big|_U$, that is, a local trivialization

$$\psi : \mathcal{E}\big|_U \to \mathcal{O}_U^{\oplus r},$$

such that the associated section $\mathrm{Hom}(\det TX, \det \mathcal{E})(U)$ is a square of a section in $\mathcal{L}(U)$. Then we have that $\psi \circ \sigma \in \mathcal{O}_U^{\oplus r}$ and there exists a $g \in (\mathfrak{m}_x^N)^{\oplus r}$, with

N sufficiently large, so that

$$\psi \circ \sigma + g \in \phi^* \mathcal{O}_{\mathbb{A}^r_k}.$$

Define $f := \psi \circ \sigma + g$, and then we have that $f : \phi(U) \to \mathbb{A}^r_k$ has an isolated zero at $\phi(x)$. Since our trivialization was compatibly oriented, this definition is independent of the choice of g.

3. Finally, we define $\deg_x^{\mathbb{A}^1} \sigma := \deg_{\phi(x)}^{\mathbb{A}^1} f \in \mathrm{GW}(k)$.

Definition 2.2.25. For a relatively oriented bundle $\mathcal{E} \to X$, and a section σ with isolated zeros, we define the *Euler class* to be

$$e(E, \sigma) := \sum_{x : \sigma(x) = 0} \deg_x^{\mathbb{A}^1} \sigma.$$

In order to conclude the proof of Theorem 2.2.20, we must identify $\deg_\ell^{\mathbb{A}^1} \sigma_f$ with $\mathrm{type}(\ell)$. Then we are able to compute $e(\mathrm{Sym}^3 \mathcal{S}^*)$ using a well-behaved choice of cubic surface, for instance the Fermat cubic. For more details, see (KW21, §5).

Remark 2.2.26. Following our definition of an Euler class for a relatively oriented bundle, we include the following closely related remarks.

1. Interesting enumerative information is still available when relative orientability fails. For an example of this in the literature, we refer the reader to the paper of Larson and Vogt (LV21) which defines relatively oriented bundles relative

to a divisor in order to compute an enriched count of bitangents to a smooth plane quartic (LV21).

2. Given a smooth projective scheme over a field, one may push forward the Euler class of its tangent bundle to obtain an Euler characteristic which is valued in $\mathrm{GW}(k)$. A particularly interesting consequence of this is an enriched version of the Riemann–Hurwitz formula, first established by M. Levine (Lev20, Theorem 12.7) and expanded upon by work of Bethea, Kass, and Wickelgren (BKW20).

Forthcoming work of Pauli investigates the related question of lines on quintic threefold (Pau22). We also refer the reader to work of M. Levine, which includes an examination of Witt-valued characteristic classes, including an Euler class of $\mathrm{Sym}^{2n-d}\mathcal{S}^*$ on $\mathrm{Gr}_k(2, n+1)$ (Lev19), and results of Bachmann and Wickelgren for symmetric bundles on arbitrary Grassmannians (BW21, Corollary 6.2). Finally, for a further investigation of enriched intersection multiplicity, we refer the reader to recent work of McKean on enriching Bézout's Theorem (McK21).

2.2.5 An arithmetic count of the lines meeting 4 lines in space

The following is based off of work by Padmavathi Srinivasan and Kirsten Wickelgren (SW21).

In enumerative geometry, one encounters the following classical question: given four complex lines in general position in $\mathbb{C}\mathrm{P}^3$, how many other complex lines meet all four? The answer is two lines, whose proof we sketch out below.

Four lines in three-space, classically Let L_1, L_2, L_3, L_4 be lines in $\mathbb{C}\mathrm{P}^3$ so that no three of them intersect at one point (we refer to this condition as *general*). Given a point $p \in L_1$, there is a unique line L_p through p which intersects both L_2 and L_3. We then examine the surface sweeped out by all such lines $Q := \bigcup_{p \in L_1} L_p$, and we claim that this is a degree two hypersurface which contains L_1, L_2, and L_3. To see this, it suffices to verify that it is the vanishing locus of a degree two homogeneous polynomial. A homogeneous polynomial of degree two, considered as an element of $H^0(\mathbb{C}\mathrm{P}^3, \mathcal{O}(2))$, will vanish on the line L_i if and only if it lies in the kernel

$$H^0(\mathbb{C}\mathrm{P}^3, \mathcal{O}(2)) \to H^0(L_i, \mathcal{O}(2)).$$

We verify that

$$\dim_k H^0(\mathbb{C}\mathrm{P}^3, \mathcal{O}(2)) = \binom{2+3}{2} = 10$$

$$\dim_k H^0(L_i, \mathcal{O}(2)) = 3,$$

therefore for $i = 1, 2, 3$ each such map has kernel of dimension ≥ 7. This implies there is a polynomial f in the common kernel of all three maps. We claim that $L_p \subseteq V(f)$ for each $p \in L_1$, and indeed since three points of L_p lie in $V(f)$, we see that $V(f)$ contains the entire line. Therefore we have containment $V(f) \supseteq Q$, and it is easy to see we must have equality. Finally by applying Bézout's Theorem, we see that $Q \cap L_4$ consists of two points, counted with multiplicity.

One might ask how to answer this question over an arbitrary field k. We recall that the Grassmannian $\mathrm{Gr}_k(2,4)$ parametrizes lines in $\mathbb{P}^3_k$ (that is, two-dimensional subspaces of $k^{\oplus 4}$), which is an appealing moduli space for this problem. We first select a basis $\{e_1, e_2, e_3, e_4\}$ of $k^{\oplus 4}$ satisfying

$$L_1 = ke_3 \oplus ke_4,$$

and we define a new line L such that

$$L = k\widetilde{e}_3 \oplus k\widetilde{e}_4,$$

where $\widetilde{e}_3$ and $\widetilde{e}_4$ are some linearly independent vectors whose definition we defer until further below. Letting ϕ_i denote the dual basis element to e_i, one may compute that

$L \cap L_1$ is nonempty if and only if

$$(\phi_1 \wedge \phi_2)(\widetilde{e}_3 \wedge \widetilde{e}_4) = 0.$$

Consider the line bundle $\det \mathcal{S}^* = \mathcal{S}^* \wedge \mathcal{S}^* \to \mathrm{Gr}_k(2,4)$, whose fiber over a point $W \in \mathrm{Gr}_k(2,4)$ is $W^* \wedge W^*$. We then have that $\phi_1 \wedge \phi_2 \in H^0(\mathrm{Gr}_k(2,4), \mathcal{S}^* \wedge \mathcal{S}^*)$ and

$$(\phi_1 \wedge \phi_2)([W]) = \phi_1|_W \wedge \phi_2|_W .$$

It is then clear that we obtain a bijection between lines intersecting L_1 and zeros of $\phi_1 \wedge \phi_2$:

$$\{L \ : \ L \cap L_1 \neq \varnothing\} = \{[W] \ : (\phi_1 \wedge \phi_2)([W]) = 0\}.$$

We may repeat this process for each line to form a section σ of $\oplus_{i=1}^4 \mathcal{S}^* \wedge \mathcal{S}^*$. Then the zeros of σ will correspond exactly to lines which meet all four of our chosen lines:

$$\{L \ : \ L \cap L_i \neq \varnothing, \ i = 1,2,3,4\} = \{[W] \in \mathrm{Gr}_k(2,4) \ : \ \sigma([W]) = 0\}.$$

In particular, if σ is a section of a relatively oriented bundle, then we may calculate an enriched count of lines meeting four lines in space, given by the Euler class

$$e\left(\oplus_{i=1}^4 \mathcal{S}^* \wedge \mathcal{S}^*, \sigma\right) = \sum_{L \ : \ L \cap L_i \neq 0} \mathrm{ind}_L \sigma. \tag{2.2.27}$$

Denote by $\mathcal{E} = \oplus_{i=1}^4 \mathcal{S}^* \wedge \mathcal{S}^*$ our rank four vector bundle over $X := \mathrm{Gr}_k(2,4)$. Since X is a smooth projective scheme of dimension four, we have that $(\det TX)^* \cong$

$\omega_X \cong \mathcal{O}(-2)^{\otimes 2}$, and $\det \mathcal{E} \cong (\otimes_{i=1}^2 \mathcal{S}^* \wedge \mathcal{S}^*)^{\otimes 2}$. Therefore $\mathrm{Hom}(\det TX, \det \mathcal{E}) \cong w_X \otimes \det \mathcal{E} \cong \mathcal{L}^{\otimes 2}$, so $\mathcal{E}$ is relatively oriented over X, and Equation 2.2.27 is a valid expression. In order to compute a local index of the section σ near a zero L, we must first parametrize Nisnevich local coordinates near L. Here we define a parametrized basis of $k^{\oplus 4}$ by

$$\widetilde{e}_1 = e_1$$

$$\widetilde{e}_2 = e_2$$

$$\widetilde{e}_3 = xe_1 + ye_2 + e_3$$

$$\widetilde{e}_4 = x'e_1 + y'e_2 + e_4.$$

We then obtain a morphism from affine space to an open cell around L:

$$\mathbb{A}^4_k = \operatorname{Spec} k[x, y, x', y'] \to U \subseteq \mathrm{Gr}_k(2, 4)$$
$$(x, y, x', y') \mapsto \operatorname{span}\{\widetilde{e}_3, \widetilde{e}_4\}.$$

Over this cell, we obtain an oriented trivialization of the bundle $\det \mathcal{S}^*$, given by $\widetilde{\phi}_3 \wedge \widetilde{\phi}_4$, where $\widetilde{\phi}_i$ denotes the dual basis element to $\widetilde{e}_i$. Under these local coordinates, we may compute the local index $\mathrm{ind}_L \sigma$ as the local $\mathbb{A}^1$-degree at the origin of the induced map $\mathbb{A}^4_k \to \mathbb{A}^4_k$. Suppose that

$$L_1 = \{\phi_1 = \phi_2 = 0\} = ke_3 \oplus ke_4.$$

Then we have that $\sigma([W]) = \left(\phi_1 \wedge \phi_2|_{[W]}, \dots\right)$. We see then that

$$\begin{aligned}(\phi_1 \wedge \phi_2)|_{k\widetilde{e}_3 \oplus k\widetilde{e}_4} &= \left(x\widetilde{\phi}_3 + y\widetilde{\phi}_4\right) \wedge \left(x'\widetilde{\phi}_3 + y'\widetilde{\phi}_4\right) \\ &= (xy' - x'y)\widetilde{\phi}_3 \wedge \widetilde{\phi}_4.\end{aligned}$$

Thus we may exhibit σ as a function

$$f = (f_1, f_2, f_3, f_4) : \mathbb{A}^4_k \to \mathbb{A}^4_k,$$

where $f_1(x, y, x', y') = xy' - x'y$. Then in the basis (x, y, x', y') we have that the Jacobian of σ has its first column as:

$$\mathrm{Jac}(f) = \det \begin{pmatrix} y' & \cdots \\ -x' & \cdots \\ -y & \cdots \\ x & \cdots \end{pmatrix}.$$

Question Is there a geometric interpretation of $\mathrm{ind}_L\sigma = \deg^{\mathbb{A}^1}_L f$?

The intersections $L \cap L_i$ for $i = 1, \dots, 4$ determine four points on $L \cong \mathbb{P}^1_{k(L)}$. Let λ_L denote the cross-ratio of these points in $k(L)^*$. Denote by P_i the plane spanned by L and L_i. We note that planes P in $\mathbb{P}^3_k$ correspond to subspaces $V \subseteq k(P)^{\oplus 4}$ where $\dim(V) = 3$. If P contains the line $L = [W]$ then it corresponds to $W \subseteq V \subseteq k(P)^{\oplus 4}$, which in turn corresponds to $k(P)$-points of $\mathrm{Proj}\left(k(L)^{\oplus 4}\big/ W\right) \cong \mathbb{P}^1_{k(L)}$. Thus we might think of the planes P_i for $i = 1, \dots, 4$ as 4 points on $\mathbb{P}^1_{k(L)}$. Let μ_L denote the cross-ratio of these points.

Theorem 2.2.28. (SW21, Theorem 1) Let L_1, L_2, L_3, L_4 be four general lines defined over k in $\mathbb{P}^3_k$. Then

$$\sum_{\{L \,:\, L\cap L_i \neq \varnothing \ \forall i\}} \mathrm{Tr}_{k(L)/k} \langle \lambda_L - \mu_L \rangle = \langle 1 \rangle + \langle -1 \rangle .$$

As a generalization, let $\pi_1, \ldots, \pi_{2n-2}$ be codimension 2 planes in $\mathbb{P}^n_k$ for n odd. Then

$$\sum_{\{L \,:\, L\cap \pi_i \neq \varnothing \ \forall i\}} \mathrm{Tr}_{k(L)/k} \det \begin{pmatrix} \cdots & c_i b_1^i & \cdots \\ & & \\ \cdots & c_i b_2^i & \cdots \end{pmatrix} = \frac{1}{2n} \binom{2n-2}{n-1} \mathbb{H},$$

where c_i are normalized coordinates for the line $\pi_i \cap L$ (defined in (SW21, Definition 10)), and $[b_1^i, b_2^i] = L \cap \pi_i \cong \mathbb{P}^1_{k(L)}$. This weighted count is expanded in forthcoming work of the author, which provides a generalized enriched count of m-planes meeting mp codimension m planes in $(m+p)$-space (Bra23).

Corollary 2.2.29. (SW21, Corollary 3) Over $\mathbb{F}_q$, we cannot have a line L over $\mathbb{F}_{q^2}$ with

$$\lambda_L - \mu_L = \begin{cases} \text{non-square} & q \equiv 3 \pmod 4 \\ & \\ \text{square} & q \equiv 1 \pmod 4. \end{cases}$$

For related results in the literature, we refer the reader to the papers of Levine and Bachmann–Wickelgren mentioned in the previous section (Lev19; BW21), as well as Wendt's work developing a Schubert calculus valued in Chow-Witt groups

(Wen20). Finally, Pauli uses Macaulay2 to compute enriched counts over a finite field of prime order and the rationals for various problems presented in these conference proceedings, including lines on a cubic surface, lines meeting four general lines in space, the EKL class, and various $\mathbb{A}^1$-Milnor numbers (Pau20).

Chapter 3

The trace of the local $\mathbb{A}^1$-degree

with R. Burklund, S. McKean, M. Montoro, M. Opie

Abstract: We prove that the local $\mathbb{A}^1$-degree of a polynomial function at an isolated zero with finite separable residue field is given by the trace of the local $\mathbb{A}^1$-degree over the residue field. This fact was originally suggested by Morel's work on motivic transfers, and by Kass and Wickelgren's work on the Scheja–Storch bilinear form. As a corollary, we generalize a result of Kass and Wickelgren relating the Scheja–Storch form and the local $\mathbb{A}^1$-degree.

3.1 Introduction

The $\mathbb{A}^1$-degree, first defined by Morel (Mor04; Mor12), provides a foundational tool for solving problems in $\mathbb{A}^1$-*enumerative geometry*.[1] In contrast to classical notions of degree, the local $\mathbb{A}^1$-degree is not integer valued: given a polynomial function $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ with isolated zero p, the local $\mathbb{A}^1$-degree of f at p, denoted by $\deg_p^{\mathbb{A}^1}(f)$, is defined to be an element of the Grothendieck–Witt group of the ground field.

Definition 3.1.1. Let k be a field. The *Grothendieck–Witt group* $\mathrm{GW}(k)$ is defined to be the group completion of the monoid of isomorphism classes of symmetric non-degenerate bilinear forms over k. The group operation is the direct sum of bilinear forms. We may also give $\mathrm{GW}(k)$ a ring structure by taking tensor products of bilinear forms for our multiplication.

The local $\mathbb{A}^1$-degree, which will be defined in Definition 3.2.11, can be related to other important invariants at rational points. The Scheja–Storch form (Definition 3.2.17) is another $\mathrm{GW}(k)$-valued invariant defined via a duality on the local complete intersection cut out by the components of a given polynomial map (see Subsection 3.2.3 for details). Kass and Wickelgren show that the isomorphism class

[1] $\mathbb{A}^1$-enumerative geometry is the application of $\mathbb{A}^1$-homotopy theory to the study of enumerative geometry over arbitrary fields. For details, see the expository paper (WW20), as well as the exposition found in (KW21; Lev20; BKW20; SW21; KW19; LV21).

of the Scheja–Storch bilinear form (SS75) is equal to the local $\mathbb{A}^1$-degree at rational points (KW19). Kass and Wickelgren also show that at points with finite separable residue field, the Scheja–Storch form is given by taking the trace of the Scheja–Storch form over the residue field (KW21, Proposition 32).

In practice, one may need to consider the local $\mathbb{A}^1$-degree at non-rational points. This is the case of interest to us. At points whose residue field is a finite extension of the ground field, Morel's work on cohomological transfer maps (Mor12) suggests the following formula: the local $\mathbb{A}^1$-degree at a non-rational point should be computed by first taking the local $\mathbb{A}^1$-degree over the residue field, and then by post-composing with a field trace. This suggestion is supported by the aforementioned results of Kass–Wickelgren on the Scheja–Storch form (KW21, Proposition 32). Our main result is to confirm this formula. We state our result precisely in Theorem 3.1.3, after introducing necessary terminology.

Definition 3.1.2. Given a separable field extension L/k of finite degree, the *trace*

$$\mathrm{Tr}_{L/k} : \mathrm{GW}(L) \to \mathrm{GW}(k)$$

is given by post-composing the field trace (which we also denote $\mathrm{Tr}_{L/k}$). That is, if $\beta : V \times V \to L$ is a representative of an isomorphism class of symmetric non-degenerate bilinear forms over L, then its trace is the isomorphism class of the following symmetric bilinear form over k

$$\mathrm{Tr}_{L/k}\beta : V \times V \to L \xrightarrow{\mathrm{Tr}_{L/k}} k.$$

When p is not a k-rational point, we can lift f to a function $f_{k(p)} : \mathbb{A}^n_{k(p)} \to \mathbb{A}^n_{k(p)}$ after fixing a choice of field embedding $k \hookrightarrow k(p)$. Moreover, we may lift p to an isolated $k(p)$-rational zero $\widetilde{p}$ of $f_{k(p)}$, and we thus obtain the local degree $\deg^{\mathbb{A}^1}_{\widetilde{p}}(f_{k(p)}) \in \mathrm{GW}(k(p))$.

We can now state our main result.

Theorem 3.1.3. Let k be a field, $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ be an endomorphism of affine space, and let $p \in \mathbb{A}^n_k$ be an isolated zero of f such that $k(p)$ is a separable extension of finite degree over k. Let $\widetilde{p}$ denote the canonical point above p. Then

$$\deg^{\mathbb{A}^1}_p(f) = \mathrm{Tr}_{k(p)/k} \deg^{\mathbb{A}^1}_{\widetilde{p}} \left(f_{k(p)}\right)$$

in $\mathrm{GW}(k)$.

As a corollary, we strengthen Kass and Wickelgren's result relating the local $\mathbb{A}^1$-degree and the Scheja–Storch form (KW19) by weakening the requirement that the point be rational.

Corollary 3.1.4. At points whose residue fields are finite separable extensions of the ground field, the local $\mathbb{A}^1$-degree coincides with the isomorphism class of the Scheja–Storch form.

In this paper we utilize the machinery of stable $\mathbb{A}^1$-homotopy theory, initially developed by Morel and Voevodsky (MV99), as well as the six functors formalism in this setting (Ayo07; CD19). We also rely heavily on results of Hoyois (Hoy14) to prove our main result. After working in the stable $\mathbb{A}^1$-homotopy category, we apply Morel's $\mathbb{A}^1$-degree to obtain the desired equality in $\mathrm{GW}(k)$.

Conventions 3.1.5. Throughout, we adopt the following conventions:

- We will use k to denote a general field. If p is a point of a k-scheme, with residue field $k(p)$ such that $k(p)/k$ a separable extension of finite degree, we call p a *finite separable point*. We may also say that p has a *finite separable residue field* in this context. We remark that all such points are closed points.

- Whenever a closed point p of a k-scheme X is chosen, we denote by ρ : $\operatorname{Spec} k(p) \to \operatorname{Spec} k$ the composite of the morphism $\operatorname{Spec} k(p) \xrightarrow{p} X$ defining the point p, and the structure map $X \to \operatorname{Spec} k$. This fixes a field embedding $k \hookrightarrow k(p)$.

- Given a scheme X over k, we denote the base change $X \times_{\operatorname{Spec} k} \operatorname{Spec} k(p)$ by $X_{k(p)}$, and given a morphism of k-schemes $f : X \to Y$, we denote its base change by $f_{k(p)} : X_{k(p)} \to Y_{k(p)}$.

- The structure map ρ allows us to define the canonical $k(p)$-rational point in

$\mathbb{A}^n_{k(p)}$ sitting above p, which we denote by $\widetilde{p}$.

3.1.1 Acknowledgements

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3.2 Preliminaries

In this section, we introduce the main notions necessary to state and prove Theorem 3.1.3. We begin in Subsection 3.2.1 by defining the local $\mathbb{A}^1$-degree. In Subsection 3.2.2, we highlight key properties of the stable motivic homotopy category in the form that we will need them. Finally, in Subsection 3.2.3, we discuss the Scheja–Storch form.

We will assume some familiarity with motivic homotopy theory. For more detail about the category of motivic spaces $\mathtt{Spc}_k^{\mathbb{A}^1}$ and the unstable motivic homotopy category $\mathcal{H}(k)$, we refer the reader to the excellent expository articles (AE17; WW20). For the construction of the stable motivic homotopy category $\mathcal{SH}(k)$, we refer the reader to (Mor04).

Notation 3.2.1. We denote by $\mathcal{SH}(k)$ the stable motivic homotopy category over the scheme $\operatorname{Spec} k$. The sphere spectrum in this category will be denoted by $\mathbf{1}_k$. We will also use $[-,-]_{\mathbb{A}^1}$ to denote $\mathbb{A}^1$-weak equivalence classes of maps between two motivic spaces, by which we mean a hom-set in the homotopy category $\mathcal{H}(k)$.

3.2.1 The local $\mathbb{A}^1$-degree

Given an endomorphism of affine space $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ with an isolated zero at a point p, we describe how to obtain an endomorphism of the sphere spectrum in the stable motivic homotopy category $\mathcal{SH}(k)$, following the exposition of (KW19, pp. 438–439). We remind the reader of Conventions 3.1.5, which we use in what follows.

Since p is an isolated zero of f, we may find an open neighborhood $U \subseteq \mathbb{A}^n_k$ for which $f^{-1}(0) \cap U = \{p\}$, that is, an open neighborhood containing no other zeros of f. Viewing $U \subseteq \mathbb{A}^n_k \subseteq \mathbb{P}^n_k$ as an open subset of projective space via a standard affine chart, we may take a Nisnevich-local pushout diagram in $\mathtt{Spc}_k^{\mathbb{A}^1}$:

$$\begin{tikzcd} U \setminus \{p\} \arrow[r] \arrow[d] & \mathbb{P}^n_k \setminus \{p\} \arrow[d] \\ U \arrow[r] \arrow[ur, phantom, "\ulcorner" near end] & \mathbb{P}^n_k. \end{tikzcd}$$

This induces an $\mathbb{A}^1$-weak equivalence on cofibers

$$\frac{U}{U \setminus \{p\}} \xrightarrow{\sim} \frac{\mathbb{P}^n_k}{\mathbb{P}^n_k \setminus \{p\}}.$$

We now appeal to the purity theorem (MV99, Theorem 2.23), a fundamental result in $\mathbb{A}^1$-homotopy, which we record for future use. While the purity theorem holds for smooth schemes over a sufficiently nice base scheme, we only need the result for smooth schemes over a field.

Theorem 3.2.2 (Morel–Voevodsky)**.** Let $Z \to X$ be a closed embedding of smooth schemes over a field k. Let $N_{X,Z}$ denote the normal bundle of Z in X. Then there is a canonical weak equivalence of motivic spaces:

$$X/(X \setminus Z) \simeq \mathrm{Th}(N_{X,Z}).$$

Returning to the situation above, we remark that projective space is endowed with a local trivialization of the tangent bundle of $\mathbb{P}^n_k$ around p, arising from the trivialization of the tangent bundle of affine space. We thus obtain canonical $\mathbb{A}^1$-weak equivalences identifying the Thom space of the trivial rank n bundle over

$\operatorname{Spec} k(p)$ with the object of study:

$$\frac{U}{U \smallsetminus \{p\}} \xrightarrow{\sim} \frac{\mathbb{P}^n_k}{\mathbb{P}^n_k \smallsetminus \{p\}} \simeq \mathrm{Th}(\mathcal{O}^n_{k(p)}) \simeq \left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \operatorname{Spec} k(p)_+.$$

We remark that U was chosen to satisfy $f(U \smallsetminus \{p\}) \subseteq \mathbb{A}^n_k \smallsetminus \{0\}$, and we can perform a completely analogous procedure to obtain $\mathbb{A}^1$-weak equivalences

$$\frac{\mathbb{A}^n_k}{\mathbb{A}^n_k \smallsetminus \{0\}} \xrightarrow{\sim} \frac{\mathbb{P}^n_k}{\mathbb{P}^n_k \smallsetminus \{0\}} \simeq \mathrm{Th}(\mathcal{O}^n_k) \simeq \mathbb{P}^n_k/\mathbb{P}^{n-1}_k.$$

Recall that in differential topology, the local degree is defined as the homotopy class of an induced map of spheres about a point. The space $\mathbb{P}^n_k/\mathbb{P}^{n-1}_k$ analogously plays the role of a sphere in $\mathtt{Spc}_k^{\mathbb{A}^1}$ when constructing the local $\mathbb{A}^1$-degree.

Definition 3.2.3. The *collapse map* is the map $c_p : \mathbb{P}^n_k/\mathbb{P}^{n-1}_k \to \mathbb{P}^n_k/(\mathbb{P}^n_k \smallsetminus \{p\})$ induced by the inclusion $\mathbb{P}^{n-1}_k \subseteq \mathbb{P}^n_k \smallsetminus \{p\}$.

Definition 3.2.4. For any $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ with an isolated zero at p, we denote by f_p the $\mathbb{A}^1$-homotopy class in the unstable motivic homotopy category assigned to the composite

$$\mathbb{P}^n_k/\mathbb{P}^{n-1}_k \xrightarrow{c_p} \frac{\mathbb{P}^n_k}{\mathbb{P}^n_k \smallsetminus \{p\}} \xleftarrow{\sim} \frac{U}{U \smallsetminus \{p\}} \xrightarrow{\overline{f}} \frac{\mathbb{A}^n_k}{\mathbb{A}^n_k \smallsetminus \{0\}} \xrightarrow{\sim} \mathbb{P}^n_k/\mathbb{P}^{n-1}_k.$$

Remark 3.2.5. Despite notational similarities, we remark that f_p and $f_{k(p)}$ are essentially unrelated. The notation f_p is consistent with that used in (KW19, Definition 11), while $f_{k(p)}$ is common notation for the base change of a morphism of schemes.

Remark 3.2.6. When p is k-rational, one can avoid the collapse map by applying the purity theorem to obtain the composite

$$\mathbb{P}^n_k/\mathbb{P}^{n-1}_k \simeq \frac{U}{U \smallsetminus \{p\}} \xrightarrow{\overline{f}} \frac{\mathbb{A}^n_k}{\mathbb{A}^n_k \smallsetminus \{0\}} \xrightarrow{\sim} \mathbb{P}^n_k/\mathbb{P}^{n-1}_k.$$

This composite yields the same element of $\left[\mathbb{P}^n_k/\mathbb{P}^{n-1}_k, \mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right]_{\mathbb{A}^1}$ as in Definition 3.2.4. Indeed, by (KW19, Lemma 10), the composite of the collapse map with the canonical $\mathbb{A}^1$-weak equivalence $\mathbb{P}^n_k/(\mathbb{P}^n_k \smallsetminus \{0\}) \xrightarrow{\sim} \mathbb{P}^n_k/\mathbb{P}^{n-1}_k$ is the class of the identity map in $\left[\mathbb{P}^n_k/\mathbb{P}^{n-1}_k, \mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right]_{\mathbb{A}^1}$.

Remark 3.2.7. The $\mathbb{A}^1$-homotopy class of f_p does not depend upon the original choice of open neighborhood $U \ni p$, provided that U contains no other zeros of f besides p. This follows immediately from our ability to provide an $\mathbb{A}^1$-weak equivalence between the cofiber $U/\left(U \smallsetminus \{p\}\right)$ and the Thom space $\mathrm{Th}(\mathcal{O}^n_k)$.

We now describe how to obtain an endomorphism of the sphere spectrum in $\mathcal{SH}(k)$ from the class f_p defined above. By (MV99, Proposition 2.17), we have a canonical $\mathbb{A}^1$-weak equivalence

$$\mathbb{P}^n_k/\mathbb{P}^{n-1}_k \simeq \left(\mathbb{P}^1_k\right)^{\wedge n}. \tag{3.2.8}$$

We recall also that $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$ as elements of the stable homotopy category. In particular by following the indexing convention of (Mor04) for motivic spheres, we see that $\Sigma^\infty \mathbb{P}^1_k = \Sigma^{2,1}\mathbf{1}_k$ in $\mathcal{SH}(k)$, where $\mathbf{1}_k$ denotes the sphere spectrum. We therefore have that

$$\Sigma^\infty \mathbb{P}^n_k/\mathbb{P}^{n-1}_k \simeq \Sigma^{2n,n}\mathbf{1}_k$$

in $\mathcal{SH}(k)$. It is immediate that, by desuspending, we obtain a canonical isomorphism

$$\mathrm{End}_{\mathcal{SH}(k)}\left(\Sigma^{2n,n}\mathbf{1}_k\right) \cong \mathrm{End}_{\mathcal{SH}(k)}(\mathbf{1}_k)$$

in the stable homotopy category. Collecting these facts together, we see that f_p determines an element in $\mathrm{End}_{\mathcal{SH}(k)}(\mathbf{1}_k)$. Abusing notation, we will refer to this endomorphism of the sphere spectrum as f_p.

Theorem 3.2.9 (Morel)**.** For any field k, there is an isomorphism

$$\deg^{\mathbb{A}^1} : \mathrm{End}_{\mathcal{SH}(k)}(\mathbf{1}_k) \cong \mathrm{GW}(k). \tag{3.2.10}$$

Morel initially required the assumption that k be perfect (Mor12), however this can be removed via work of Hoyois (Hoy15, Appendix A).

Definition 3.2.11. With notation as above, the image of f_p in $\mathrm{GW}(k)$ under $\deg^{\mathbb{A}^1}$ is the *local* $\mathbb{A}^1$*-degree* of f at p, denoted $\deg_p^{\mathbb{A}^1}(f)$.

3.2.2 Stable motivic homotopy theory

We begin by recalling a few concepts and results from stable motivic homotopy theory that will play a role in the proof of Theorem 3.1.3.

The category theory of $\mathcal{SH}(k)$ supports a six functor formalism, the general exposition of which we defer to (Hoy14, §2). Indeed, for the purposes of this paper, we need only consider this formalism in the case of functors induced by maps $\rho : \operatorname{Spec} k(p) \to \operatorname{Spec} k$, where $k(p)/k$ is a finite separable field extension. We recall that we have an adjunction

$$\rho^* : \mathcal{SH}(k) \rightleftarrows \mathcal{SH}(k(p)) : \rho_*.$$

Since ρ is separated and finite type, we also have an exceptional adjunction

$$\rho_! : \mathcal{SH}(k(p)) \rightleftarrows \mathcal{SH}(k) : \rho^!.$$

We denote by η the unit of the adjunction between the direct and inverse image functors, and by ε the counit of the exceptional adjunction. That is, we have natural transformations:

$$\eta : \mathrm{id}_{\mathcal{SH}(k)} \Rightarrow \rho_* \rho^*$$

$$\varepsilon : \rho_! \rho^! \Rightarrow \mathrm{id}_{\mathcal{SH}(k)}.$$

Remark 3.2.12. To facilitate exposition, we pause here to provide references for a few basic facts about six functors which we will make use of in this paper. Let ρ be as in Conventions 3.1.5.

1. Since ρ is smooth, ρ^* admits a left adjoint, denoted $\rho_\sharp$. As ρ is furthermore finite and étale, we have a canonical equivalence $\rho_* \simeq \rho_\sharp$ (Hoy14, p.21).

2. We have a canonical isomorphism $\rho_*\rho^*\mathbf{1}_k \simeq \rho_*\mathbf{1}_{k(p)}$. This is due to (MV99, p.112, Proposition 2.17(3)). See also (KW19, Equation 11).

3. Under our assumptions on ρ, we have canonical natural isomorphisms $\rho^! \simeq \rho^*$ and $\rho_! \simeq \rho_\sharp \simeq \rho_*$. This may be found in (Hoy14, p.21). In particular, we remark that $\rho_\sharp$ can be interpreted as a forgetful functor under the structure map ρ.

4. There is a canonical equivalence $\rho_!\mathbf{1}_{k(x)} \cong \operatorname{Spec} k(x)_+$ in $\mathcal{SH}(k)$. See (KW19, p.441).

We are now in a position to recall a description of the collapse map at the level of the stable motivic homotopy category.

Lemma 3.2.13. (Hoy14; KW19) In the stable homotopy category $\mathcal{SH}(k)$, the collapse map of Definition 3.2.3

$$c_p : \mathbb{P}^n_k/\mathbb{P}^{n-1}_k \to \mathbb{P}^n_k/(\mathbb{P}^n_k \smallsetminus \{p\}) \cong \left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \operatorname{Spec}(k(p))_+$$

is computed by applying $\mathbb{P}^n_k/\mathbb{P}^{n-1}_k \wedge (-)$ to the component of the unit η at the sphere spectrum:

$$\eta_{\mathbf{1}_k} : \mathbf{1}_k \to \rho_* \rho^* \mathbf{1}_k \cong \rho_* \mathbf{1}_{k(p)}.$$

Proof. The case $n = 1$ may be found in (Hoy14, Lemma 5.5), and the proof generalizes to higher n as in (KW19, Lemma 13). $\square$

Remark 3.2.14. We can furthermore describe $f_p \in \mathrm{End}_{\mathcal{SH}(k)}(\mathbf{1}_k)$ in the following way. Recall that f induces a map

$$\overline{f} : U/(U \smallsetminus \{p\}) \to \mathbb{A}^n_k/(\mathbb{A}^n_k \smallsetminus \{0\}).$$

As above, we have $\mathbb{A}^1$-weak equivalences

$$U/(U \smallsetminus \{p\}) \simeq \left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \mathrm{Spec}\,(k(p))_+$$

and

$$\mathbb{A}^n_k/(\mathbb{A}^n_k \smallsetminus \{0\}) \simeq \mathbb{P}^n_k/\mathbb{P}^{n-1}_k.$$

We may thus identify $\overline{f}$ with the composite $\left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \mathrm{Spec}\,(k(p))_+ \to \mathbb{P}^n_k/\mathbb{P}^{n-1}_k$ in $\mathcal{SH}(k)$. By Definition 3.2.4, we have that f_p is the composite of $\overline{f}$ and the collapse map. In $\mathcal{SH}(k)$, we can record this via the following commutative diagram:

$$\begin{tikzcd}
\left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \mathrm{Spec}\,(k(p))_+ \arrow[r, "\overline{f}"] & \mathbb{P}^n_k/\mathbb{P}^{n-1}_k \\
\mathbb{P}^n_k/\mathbb{P}^{n-1}_k \arrow[u, "\mathbb{P}^n_k/\mathbb{P}^{n-1}_k \wedge (\eta_{\mathbf{1}_k})"] \arrow[ur, bend right, "f_p"'] &
\end{tikzcd}$$

We now recall that the trace $\mathrm{Tr}_{k(p)/k} : \mathrm{GW}(k(p)) \to \mathrm{GW}(k)$ can be described purely in terms of maps in the motivic homotopy category, under the isomorphism of Theorem 3.2.9.

Definition 3.2.15. The *transfer*

$$\mathrm{Tr}_{k(p)/k} : \mathrm{End}_{\mathcal{SH}(k(p))}(\mathbf{1}_{k(p)}) \to \mathrm{End}_{\mathcal{SH}(k)}(\mathbf{1}_k)$$

is defined by sending $\omega \in \mathrm{End}_{\mathcal{SH}(k(p))}(\mathbf{1}_{k(p)})$ to the composite

$$\mathbf{1}_k \xrightarrow{\eta_{\mathbf{1}_k}} \rho_* \mathbf{1}_{k(p)} \simeq \rho_\sharp \mathbf{1}_{k(p)} \xrightarrow{\rho_\sharp \omega} \rho_\sharp \mathbf{1}_{k(p)} \simeq \rho_! \rho^! \mathbf{1}_k \xrightarrow{\varepsilon_{\mathbf{1}_k}} \mathbf{1}_k.$$

Lemma 3.2.16. (Hoy14, Proposition 5.2, Lemma 5.3) The transfer agrees with the field trace. That is, the diagram

$$\begin{array}{ccc} \mathrm{End}_{\mathcal{SH}(k(p))}\left(\mathbf{1}_{k(p)}\right) & \xrightarrow{\mathrm{Tr}_{k(p)/k}} & \mathrm{End}_{\mathcal{SH}(k)}\left(\mathbf{1}_k\right) \\ \cong\big\downarrow & & \big\downarrow\cong \\ \mathrm{GW}(k(p)) & \xrightarrow[\mathrm{Tr}_{k(p)/k}]{} & \mathrm{GW}(k) \end{array}$$

commutes, where the vertical maps are given by Morel's degree isomorphism (Equation 3.2.10), the top map is the transfer (Definition 3.2.15), and the bottom map is the trace map on the Grothendieck–Witt group of k (Definition 3.1.2).

3.2.3 The Scheja–Storch bilinear form

We give a brief description of the Scheja–Storch bilinear form (see also (SS75), (KW19), and (KW21, Section 4)). Given a polynomial map

$$f = (f_1, \ldots, f_n) : \mathbb{A}^n_k \to \mathbb{A}^n_k$$

with isolated zero p, let $\mathfrak{m}$ be the maximal ideal of $k[x_1, \ldots, x_n]$ corresponding to the point p. Consider the local algebra $Q_p = \frac{k[x_1,\ldots,x_n]_\mathfrak{m}}{(f_1,\ldots,f_n)}$. As a local complete intersection, Q_p is isomorphic to its dual $\mathrm{Hom}_k(Q_p, k)$. Scheja and Storch construct an explicit Q_p-linear isomorphism $\Theta : \mathrm{Hom}_k(Q_p, k) \to Q_p$ realizing this self-duality, which gives us a distinguished homomorphism $\eta := \Theta^{-1}(1) : Q_p \to k$.

Definition 3.2.17. Given a polynomical function $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$, the *Scheja–Storch bilinear form* $\beta_p(f) : Q_p \times Q_p \to k$ is given by $\beta_p(f)(x, y) = \eta(xy)$, where η is defined in the preceeding paragraph.

Since Q_p is commutative, the Scheja–Storch bilinear form is symmetric. By (KW19, Lemma 28) and (EL77, Proposition 3.4), the Scheja–Storch form is non-degenerate. Thus the Scheja–Storch form gives a class in $\mathrm{GW}(k)$, which we denote by $\mathrm{ind}_p(f)$.

Kass–Wickelgren show that if p is k-rational or if f is étale at p, then $\deg_p^{\mathbb{A}^1}(f) =$

$\mathrm{ind}_p(f)$ (KW19). They also show that if p is a finite separable point, then

$$\mathrm{ind}_p(f) = \mathrm{Tr}_{k(p)/k}\mathrm{ind}_{\widetilde{p}}(f_{k(p)}),$$

where $\widetilde{p}$ is the canonical $k(p)$-point above p and $f_{k(p)} : \mathbb{A}^n_{k(p)} \to \mathbb{A}^n_{k(p)}$ is the base change of f (KW21, Proposition 32). Given these two results, one would expect Theorem 3.1.3 to be true.

3.3 Main Results

We now proceed to the proof of the main theorem, as stated in Theorem 3.1.3. Our first step is to apply the machinery of Subsection 3.2.2 to frame our problem in terms of motivic homotopy theory. Recall from Conventions 3.1.5 that we have already fixed a choice of field embedding $k \hookrightarrow k(p)$. Thus, for any k-scheme X and any point $p \in X$, we have the canonical $k(p)$-rational point $\widetilde{p} \in X_{k(p)}$ sitting over p, defined via the following pullback diagram:

$$\begin{tikzcd}
\operatorname{Spec} k(p) \arrow[dr, dashed, "\widetilde{p}"] \arrow[drr, bend left, "\mathrm{id}"] \arrow[ddr, bend right, "p"'] & & \\
& X_{k(p)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Spec} k(p) \arrow[d, "\rho"] \\
& X \arrow[r] & \operatorname{Spec} k.
\end{tikzcd}$$

We write $f_{k(p)}$ and $\pi : \mathbb{A}^n_{k(p)} \to \mathbb{A}^n_k$ for the morphisms induced by base change. We then may consider the following diagram of k-schemes:

$$\begin{tikzcd}
\mathbb{A}^n_{k(p)} \arrow[r, "f_{k(p)}"] \arrow[d, "\pi"'] & \mathbb{A}^n_{k(p)} \arrow[d, "\pi"] \\
\mathbb{A}^n_k \arrow[r, "f"'] & \mathbb{A}^n_k
\end{tikzcd} \quad \text{which maps} \quad \begin{tikzcd}
\widetilde{p} \arrow[r, mapsto] \arrow[d, mapsto] & 0 \arrow[d, mapsto] \\
p \arrow[r, mapsto] & 0.
\end{tikzcd}$$

Note that the point $\widetilde{p}$ is a root of $f_{k(p)}$, so $f_{k(p)}$ has an isolated rational zero at $\widetilde{p}$. Let $U \subseteq \mathbb{A}^n_{k(p)}$ be an open neighborhood containing $\widetilde{p}$ and no other zeros of $f_{k(p)}$. As the structure map $\mathbb{A}^n_k \to \operatorname{Spec} k$ is universally open, π is an open morphism of schemes. Thus, $\pi(U)$ is an open neighborhood of p and contains no other zeros of f by construction. Taking cofibers, we obtain an induced diagram of motivic spaces

$$\begin{tikzcd}
\frac{U}{U \smallsetminus \{\widetilde{p}\}} \arrow[r, "\overline{f}_{k(p)}"] \arrow[d, "\overline{\pi}_p"'] & \frac{\mathbb{A}^n_{k(p)}}{\mathbb{A}^n_{k(p)} \smallsetminus \{0\}} \arrow[d, "\overline{\pi}_0"] \\
\frac{\pi(U)}{\pi(U) \smallsetminus \{p\}} \arrow[r, "\overline{f}"'] & \frac{\mathbb{A}^n_k}{\mathbb{A}^n_k \smallsetminus \{0\}}.
\end{tikzcd} \tag{3.3.1}$$

As discussed in Section 3.2.1, we have the following $\mathbb{A}^1$-weak equivalences:

$$\begin{aligned}
\frac{U}{U \smallsetminus \{\widetilde{p}\}} &\xrightarrow{\sim} \frac{\mathbb{P}^n_{k(p)}}{\mathbb{P}^n_{k(p)} \smallsetminus \{\widetilde{p}\}} \simeq \left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \operatorname{Spec}(k(p))_+, \\
\frac{\mathbb{A}^n_{k(p)}}{\mathbb{A}^n_{k(p)} \smallsetminus \{0\}} &\xrightarrow{\sim} \frac{\mathbb{P}^n_{k(p)}}{\mathbb{P}^n_{k(p)} \smallsetminus \{0\}} \simeq \left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \operatorname{Spec}(k(p))_+, \\
\frac{\pi(U)}{\pi(U) \smallsetminus \{p\}} &\xrightarrow{\sim} \frac{\mathbb{P}^n_k}{\mathbb{P}^n_k \smallsetminus \{p\}} \simeq \left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \operatorname{Spec}(k(p))_+, \\
\frac{\mathbb{A}^n_k}{\mathbb{A}^n_k \smallsetminus \{0\}} &\xrightarrow{\sim} \frac{\mathbb{P}^n_k}{\mathbb{P}^n_k \smallsetminus \{0\}} \simeq \mathbb{P}^n_k/\mathbb{P}^{n-1}_k.
\end{aligned}$$

Appending these $\mathbb{A}^1$-weak equivalences to Diagram 3.3.1, we obtain the following diagram in the unstable homotopy category $\mathcal{H}(k)$.

$$
\begin{tikzcd}
& \left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \operatorname{Spec}(k(p))_+ \arrow[rr, dashed] \arrow[dd, dashed] & & \left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \operatorname{Spec}(k(p))_+ \arrow[dd, dashed] \\
\frac{U}{U \setminus \{\widetilde{p}\}} \arrow[ur, "\sim"] \arrow[rr, "\overline{f}_{k(p)}"] \arrow[dd, "\overline{\pi}_p"'] & & \frac{\mathbb{A}^n_{k(p)}}{\mathbb{A}^n_{k(p)} \setminus \{0\}} \arrow[ur, "\sim"] \arrow[dd, "\overline{\pi}_0"] & \\
& \left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \operatorname{Spec}(k(p))_+ \arrow[rr, dashed] & & \mathbb{P}^n_k/\mathbb{P}^{n-1}_k \\
\frac{\pi(U)}{\pi(U) \setminus \{p\}} \arrow[ur, "\sim"] \arrow[rr, "\overline{f}"'] & & \frac{\mathbb{A}^n_k}{\mathbb{A}^n_k \setminus \{0\}} \arrow[ur, "\sim"] &
\end{tikzcd}
\tag{3.3.2}
$$

The dashed arrows above are obtained by inverting $\mathbb{A}^1$-weak equivalences.

We now turn our attention to the dashed face of the cube (3.3.2). We obtain the class f_p from Definition 3.2.4 by pre-composing with the collapse map (see Diagram 3.3.3). Working in the stable homotopy category, the top edge of the dashed face is exactly the image of $\left(f_{k(p)}\right)_{\widetilde{p}}$ under $\rho_* : \mathcal{SH}(k(p)) \to \mathcal{SH}(k)$. Taking suspension spectra, we get the following diagram in $\mathcal{SH}(k)$.

$$\begin{tikzcd}
\left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \operatorname{Spec}(k(p))_+ \arrow[r, "\rho_*(f_{k(p)})_{\widetilde{p}}"] \arrow[d, "r"'] & \left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \operatorname{Spec}(k(p))_+ \arrow[d, "g"] \\
\left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \operatorname{Spec}(k(p))_+ \arrow[r] & \mathbb{P}^n_k/\mathbb{P}^{n-1}_k \\
\mathbb{P}^n_k/\mathbb{P}^{n-1}_k \arrow[u, "\left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right)\wedge\eta_{\mathbf{1}_k}"] \arrow[ur, bend right, "f_p"'] &
\end{tikzcd} \tag{3.3.3}$$

The rest of our paper will center around the diagram above. In proving Theorem 3.1.3, we will show that r in Diagram (3.3.3) is invertible in $\mathcal{SH}(k)$, which allows us to rewrite f_p by exploiting the commutativity of this diagram.

3.3.1 The stable classes of r and g

In order to analyze Diagram (3.3.3) we state the following two lemmas, which allow us to characterize the $\mathcal{SH}(k)$-classes of r and g, respectively.

Lemma 3.3.4. Let $p \in \mathbb{A}^n_k$ be a closed point with finite separable residue field $k(p)/k$. Let $\pi : \mathbb{A}^n_{k(p)} \to \mathbb{A}^n_k$ be the projection map induced by the structure map $\rho : \operatorname{Spec} k(p) \to \operatorname{Spec} k$, and let $\widetilde{p} \in \mathbb{A}^n_{k(p)}$ be the canonical $k(p)$-rational point above p. Then for any open neighborhood U about $\widetilde{p}$ such that $U \cap \pi^{-1}(p) = \{\widetilde{p}\}$, the stable class in $\mathcal{SH}(k)$ of the map

$$\frac{U}{U \smallsetminus \{\widetilde{p}\}} \to \frac{\pi(U)}{\pi(U) \smallsetminus \{p\}} \tag{3.3.5}$$

is given by $(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k) \wedge \rho_* \mathrm{id}_{\mathbf{1}_{k(p)}}$.

Proof. The base change $\pi : \mathbb{A}^n_{k(p)} \to \mathbb{A}^n_k$ is simply Spec applied to the k-algebra homomorphism $\iota : k[x_1, ..., x_n] \hookrightarrow k(p)[x_1, ..., x_n]$. As $\iota(x_i) = x_i$, we get an induced map $T\mathbb{A}^n_{k(p)} \to \pi^* T\mathbb{A}^n_k$ which in turn induces an isomorphism $(T\mathbb{A}^n_{k(p)})_{\widetilde{p}} \xrightarrow{\sim} (\pi^* T\mathbb{A}^n_k)_{\widetilde{p}}$. The right hand side is easily seen to be $T_p\mathbb{A}^n_k \otimes_k k(p)$. As $k(p)$-vector spaces, we have isomorphisms $T_{\widetilde{p}}\mathbb{A}^n_{k(p)} \cong T_p\mathbb{A}^n_k \otimes_k k(p) \cong \mathbb{A}^n_{k(p)}$.

Next, we consider the Thom spaces $\mathrm{Th}(T_{\widetilde{p}}\mathbb{A}^n_{k(p)})$ and $\mathrm{Th}(T_p\mathbb{A}^n_k \otimes_k k(p))$. Via the purity isomorphism in Theorem 3.2.2, we have weak equivalences of k-motivic spaces

$$\frac{U}{U \smallsetminus \{\widetilde{p}\}} \simeq \mathrm{Th}\left(T_{\widetilde{p}}\mathbb{A}^n_{k(p)}\right) \xrightarrow{\sim} \left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \mathrm{Spec}\,(k(p))_+,$$
$$\frac{\pi(U)}{\pi(U) \smallsetminus \{p\}} \simeq \mathrm{Th}\left(T_p\mathbb{A}^n_k\right) \xrightarrow{\sim} \left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \mathrm{Spec}\,(k(p))_+.$$

Since base change is a left adjoint and Thom spaces are obtained by taking colimits, we deduce

$$\mathrm{Th}(T_p\mathbb{A}^n_k \otimes_k k(p)) \simeq \rho^* \mathrm{Th}(T_p\mathbb{A}^n_k)$$

as $k(p)$-motivic spaces. Moreover, the purity theorem implies that $\mathrm{Th}(T_p\mathbb{A}^n_k)$ is a $k(p)$-motivic space, so its base change to $k(p)$ is canonically identified with itself in the homotopy category of $k(p)$-motivic spaces. That is, the class in $\mathcal{SH}(k(p))$ of Equation 3.3.5 is given by the class in $\mathcal{SH}(k(p))$ of the canonical $\mathbb{A}^1$-weak equivalence

$\mathrm{Th}(T_{\widetilde{p}}\mathbb{A}^n_{k(p)}) \xrightarrow{\sim} \mathrm{Th}(T_p\mathbb{A}^n_k)$. This has the class of

$$(\mathbb{P}^n_{k(p)}/\mathbb{P}^{n-1}_{k(p)}) \wedge \mathrm{id}_{\mathbf{1}_{k(p)}}$$

in $\mathcal{SH}(k(p))$ under the identification given by the purity isomorphism. Finally, we push forward via the functor $\rho_* : \mathcal{SH}(k(p)) \to \mathcal{SH}(k)$ to get that the class of Equation 3.3.5 is $(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k) \wedge \rho_*\mathrm{id}_{\mathbf{1}_{k(p)}}$. □

Lemma 3.3.6. Let $k(p)/k$ be a finite separable field extension, let $q \in \mathbb{A}^n_k$ be any k-rational point, and let $\pi : \mathbb{A}^n_{k(p)} \to \mathbb{A}^n_k$ be the projection map induced by the structure map $\rho : \mathrm{Spec}\, k(p) \to \mathrm{Spec}\, k$. Denote by $\widetilde{q} \in \mathbb{A}^n_{k(p)}$ the canonical $k(p)$-rational point above q. Then for any open neighborhood U containing $\widetilde{q}$, the stable class in $\mathcal{SH}(k)$ of the map

$$\frac{U}{U \smallsetminus \{\widetilde{q}\}} \to \frac{\pi(U)}{\pi(U) \smallsetminus \{q\}} \tag{3.3.7}$$

is given by $(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k) \wedge \varepsilon_{\mathbf{1}_k}$.

Proof. Since q is k-rational, $\widetilde{q}$ is the unique canonical $k(p)$-rational point above q, so we may assume that $\widetilde{q}$ and q are the origins of $\mathbb{A}^n_{k(p)}$ and $\mathbb{A}^n_k$, respectively, in which case we may take $U = \mathbb{A}^n_{k(p)}$. It thus suffices to consider the class in $\mathcal{SH}(k)$ of the map of Thom spaces $\mathrm{Th}_0\mathbb{A}^n_{k(p)} \to \mathrm{Th}_0\mathbb{A}^n_k$ induced by the canonical map $\pi : \mathbb{A}^n_{k(p)} \to \mathbb{A}^n_k$. By viewing affine space as a trivial vector bundle over the origin, (Hoy14, p.9) implies that this is the desired component of the counit of the exceptional adjunction. □

Corollary 3.3.8. The map g in Diagram 3.3.3 is $\left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \varepsilon_{\mathbf{1}_k}$, and the map r is $\left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \rho_* \mathrm{id}_{\mathbf{1}_{k(p)}}$.

Remark 3.3.9. Lemmas 3.3.4 and 3.3.6 hold more generally for schemes that are locally isomorphic to affine space, as we rely only on local computations in their proofs.

3.3.2 Proof of Theorem 3.1.3

By Corollary 3.3.8, we may rewrite Diagram (3.3.3) as

$$\begin{tikzcd}
\left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \mathrm{Spec}\,(k(p))_+ \arrow[r, "\rho_*(f_{k(p)})_{\widetilde{p}}"] \arrow[d, "\left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right)\wedge\rho_*\mathrm{id}_{\mathbf{1}_{k(p)}}"', "\sim"] & \left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \mathrm{Spec}\,(k(p))_+ \arrow[d, "\left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right)\wedge\varepsilon_{\mathbf{1}_k}"] \\
\left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \mathrm{Spec}\,(k(p))_+ \arrow[r] & \mathbb{P}^n_k/\mathbb{P}^{n-1}_k \\
\mathbb{P}^n_k/\mathbb{P}^{n-1}_k \arrow[u, "\left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right)\wedge\eta_{\mathbf{1}_k}"] \arrow[ur, bend right, "f_p"'] &
\end{tikzcd} \tag{3.3.10}$$

In the stable homotopy category $\mathcal{SH}(k)$, Diagram 3.3.10 is $\mathbb{P}^n_k/\mathbb{P}^{n-1}_k \wedge (-)$ applied to the diagram

$$\begin{tikzcd}
\rho_*\mathbf{1}_{k(p)} \arrow[r, "\rho_*(f_{k(p)})_{\widetilde{p}}"] \arrow[d, "\rho_*\mathrm{id}_{\mathbf{1}_{k(p)}}"', "\sim"] & \rho_*\mathbf{1}_{k(p)} \arrow[d, "\varepsilon_{\mathbf{1}_k}"] \\
\rho_*\mathbf{1}_{k(p)} \arrow[r] & \mathbf{1}_k \\
\mathbf{1}_k \arrow[u, "\eta_{\mathbf{1}_k}"] \arrow[ur, bend right, "f_p"'] &
\end{tikzcd} \tag{3.3.11}$$

We remark that we are able to invert the weak equivalence $\rho_* \mathrm{id}_{\mathbf{1}_{k(p)}}$ of Diagram 3.3.11. Since Diagram 3.3.11 is commutative, we may express f_p as the composite

$$\mathbf{1}_k \xrightarrow{\eta_{\mathbf{1}_k}} \rho_* \rho^* \mathbf{1}_k \simeq \rho_* \mathbf{1}_{k(p)} \xrightarrow{\rho_*\left(f_{k(p)}\right)_{\widetilde{p}}} \rho_* \mathbf{1}_{k(p)} \simeq \rho_! \rho^! \mathbf{1}_k \xrightarrow{\varepsilon_{\mathbf{1}_k}} \mathbf{1}_k.$$

Recall that in the setting of Theorem 3.1.3, the morphism ρ is finite and étale, which gives a canonical isomorphism $\rho_* \simeq \rho_\sharp$ (Remark 3.2.12). Thus by Definition 3.2.15, we have $f_p = \mathrm{Tr}_{k(p)/k}\left(f_{k(p)}\right)_{\widetilde{p}}$. Applying Lemma 3.2.16, we conclude that $\deg_p^{\mathbb{A}^1}(f) = \mathrm{Tr}_{k(p)/k} \deg_{\widetilde{p}}^{\mathbb{A}^1}(f_{k(p)})$, as desired.

3.3.3 A brief proof of Corollary 3.1.4

In (KW21, Proposition 32), the authors prove that the Scheja–Storch bilinear form, denoted $\mathrm{ind}_p(f)$, is computed by the trace $\mathrm{ind}_p(f) = \mathrm{Tr}_{k(p)/k} \mathrm{ind}_{\widetilde{p}}(f_{k(p)})$. Moreover in (KW19), the authors prove that at any rational point, the Scheja–Storch form agrees with the local $\mathbb{A}^1$-degree. Combining these two results with Theorem 3.1.3, for any isolated zero p with finite separable residue field we have that

$$\mathrm{ind}_p(f) = \mathrm{Tr}_{k(p)/k} \mathrm{ind}_{\widetilde{p}}(f_{k(p)}) = \mathrm{Tr}_{k(p)/k} \deg_{\widetilde{p}}^{\mathbb{A}^1}(f_{k(p)}) = \deg_p^{\mathbb{A}^1}(f).$$

Chapter 4

Bézoutians and the $\mathbb{A}^1$-degree

with S. McKean and S. Pauli

Abstract: We prove that both the local and global $\mathbb{A}^1$-degree of an endomorphism of affine space can be computed in terms of the multivariate Bézoutian. In particular, we show that the Bézoutian bilinear form, the Scheja–Storch form, and the $\mathbb{A}^1$-degree for complete intersections are isomorphic. Our global theorem generalizes Cazanave's theorem in the univariate case, and our local theorem generalizes Kass–Wickelgren's theorem on EKL forms and the local degree. This result provides an algebraic formula for local and global degrees in motivic homotopy theory.

4.1 Introduction

Morel's $\mathbb{A}^1$-Brouwer degree (Mor06) assigns a bilinear form-valued invariant to a given endomorphism of affine space. However, Morel's construction is not explicit. In order to make computations and applications, we would like algebraic formulas for the $\mathbb{A}^1$-degree. Such formulas were constructed by Cazanave for the global $\mathbb{A}^1$-degree in dimension 1 (Caz12), Kass–Wickelgren for the local $\mathbb{A}^1$-degree at rational points and étale points (KW19), and Brazelton–Burklund–McKean–Montoro–Opie for the local $\mathbb{A}^1$-degree at separable points (BBM$^+$21). In this paper, we give a general algebraic formula for the $\mathbb{A}^1$-degree in both the global and local cases. In the global case, we remove Cazanave's dimension restriction, while in the local case, we remove previous restrictions on the residue field of the point at which the local $\mathbb{A}^1$-degree is taken.

Let k be a field, and let $f = (f_1, \ldots, f_n) : \mathbb{A}^n_k \to \mathbb{A}^n_k$ be an endomorphism of affine space with isolated zeros, so that $Q := k[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ is a complete intersection. We now recall the definition of the Bézoutian of f, as well as a special bilinear form determined by the Bézoutian. Introduce new variables $X := (X_1, \ldots, X_n)$ and $Y := (Y_1, \ldots, Y_n)$. For each $1 \leq i, j \leq n$, define the quantity

$$\Delta_{ij} := \frac{f_i(Y_1, \ldots, Y_{j-1}, X_j, \ldots, X_n) - f_i\left(Y_1, \ldots, Y_j, X_{j+1}, \ldots, X_n\right)}{X_j - Y_j}.$$

Definition 4.1.1. The *Bézoutian* of f is the image $\text{Béz}(f_1, \ldots, f_n)$ of the determinant $\det(\Delta_{ij})$ in $k[X,Y]/(f(X), f(Y))$. Given a basis $\{a_1, \ldots, a_m\}$ of Q as a k-vector space, there exist scalars $B_{i,j}$ for which

$$\text{Béz}(f_1, \ldots, f_n) = \sum_{i,j=1}^{m} B_{i,j} a_i(X) a_j(Y).$$

We define the *Bézoutian form* of f to be the class β_f in the Grothendieck–Witt ring $\text{GW}(k)$ determined by the bilinear form $Q \times Q \to k$ with Gram matrix $(B_{i,j})$.

For any isolated zero of f corresponding to a maximal ideal $\mathfrak{m}$, there is an analogous bilinear form $\beta_{f,\mathfrak{m}}$ on the local algebra $Q_\mathfrak{m}$. We refer to $\beta_{f,\mathfrak{m}}$ as the *local Bézoutian form* of f at $\mathfrak{m}$. We will demonstrate that both β_f and $\beta_{f,\mathfrak{m}}$ yield well-defined classes in $\text{GW}(k)$. Our main theorem is that the Bézoutian form of f agrees with the $\mathbb{A}^1$-degree in both the local and global contexts.

Theorem 4.1.2. Let $\text{char} k \neq 2$. Let $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ have an isolated zero at a closed point $\mathfrak{m}$. Then $\beta_{f,\mathfrak{m}}$ is isomorphic to the local $\mathbb{A}^1$-degree of f at $\mathfrak{m}$. If we further assume that all the zeros of f are isolated, then β_f is isomorphic to the global $\mathbb{A}^1$-degree of f.

Because the Bézoutian form can be explicitly computed using commutative algebraic tools, Theorem 4.1.2 provides a tractable formula for $\mathbb{A}^1$-degrees and Euler classes in motivic homotopy theory. Using the Bézoutian formula for the $\mathbb{A}^1$-degree, we are able to deduce several computational rules for the degree. We also provide a Sage implementation for calculating local and global $\mathbb{A}^1$-degrees via the Bézoutian at (BMP21a).

Remark 4.1.3. The key contribution of this article is computability. Building on the work of Kass–Wickelgren (KW19), Bachmann–Wickelgren (BW21) show that the $\mathbb{A}^1$-degree agrees with the Scheja–Storch form as elements of $\mathrm{KO}^0(k)$. In Theorem 4.5.1, we show how this immediately implies that the $\mathbb{A}^1$-degree and Scheja–Storch form determine the same element of $\mathrm{GW}(k)$. Scheja–Storch (SS75) showed that their form is a Bézoutian bilinear form (in the sense of Definition 4.3.8; see also Lemma 4.4.5 and Remark 4.4.9), which was further explored by Becker–Cardinal–Roy–Szafraniec (BCRS96). Putting these results together shows that the isomorphism class of the Bézoutian bilinear form is the $\mathbb{A}^1$-degree.

In dimension 1, Cazanave (Caz12) gives a simple formula for computing the $\mathbb{A}^1$-degree as a Bézoutian bilinear form in the global setting. However, it is not immediately clear how to adapt this to higher dimensions or the local setting. Becker–Cardinal–Roy–Szafraniec show how to compute Bézoutian bilinear forms in terms of "dualizing forms," but this method is computationally analogous to using the Eisenbud–Khimshiashvili–Levine form to compute the $\mathbb{A}^1$-degree (KW19). In the proof of Theorem 4.1.2 (found in Section 4.5), we show that our two notions of

Bézoutian bilinear forms (Definitions 4.1.1 and 4.3.8) agree up to isomorphism. Since Definition 4.1.1 is the desired generalization of Cazanave's formula, this enables us to calculate $\mathbb{A}^1$-degrees in full generality.

4.1.1 Outline

Before proving Theorem 4.1.2, we recall some classical results on Bézoutians (following (BCRS96)) in Section 4.3, as well as the work of Scheja–Storch on residue pairings (SS75) in Section 4.4. We then discuss a local decomposition procedure for the Scheja–Storch form and show that the global Scheja–Storch form is isomorphic to the Bézoutian form in Section 4.4.1. In Section 4.5, we complete the proof of Theorem 4.1.2 by applying the work of Kass–Wickelgren (KW19) and Bachmann–Wickelgren (BW21) on the local $\mathbb{A}^1$-degree and the Scheja–Storch form. Using Theorem 4.1.2, we give an algorithm for computing the local and global $\mathbb{A}^1$-degree at the end of Section 4.5.1, available at (BMP21a). In Section 4.6, we establish some basic properties for computing degrees. In Section 4.7, we provide a step-by-step illustration of our ideas by working through some explicit examples. Finally, we implement our code to compute some examples of $\mathbb{A}^1$-Euler characteristics of Grassmannians in Section 4.8. We check our computations by proving a general formula for the $\mathbb{A}^1$-Euler characteristic of a Grassmannian in Theorem 4.8.4. The $\mathbb{A}^1$-Euler characteristic of Grassmannians is essentially a folklore result that follows from the work of Hoyois, Levine, and Bachmann–Wickelgren.

4.1.2 Background

Let $\mathrm{GW}(k)$ denote the Grothendieck–Witt group of isomorphism classes of symmetric, non-degenerate bilinear forms over a field k. Morel's $\mathbb{A}^1$-Brouwer degree (Mor06, Corollary 1.24)

$$\deg : [\mathbb{P}^n_k/\mathbb{P}^{n-1}_k, \mathbb{P}^n_k/\mathbb{P}^{n-1}_k]_{\mathbb{A}^1} \to \mathrm{GW}(k),$$

which is a group isomorphism (in fact, a ring isomorphism (Mor04, Lemma 6.3.8)) for $n \geq 2$, demonstrates that bilinear forms play a critical role in motivic homotopy theory. However, Morel's $\mathbb{A}^1$-degree is non-constructive. Kass and Wickelgren addressed this problem by expressing the $\mathbb{A}^1$-degree as a sum of local degrees (KW21, Lemma 19) and providing an explicit formula (building on the work of Eisenbud–Levine (EL77) and Khimshiashvili (Him77)) for the local $\mathbb{A}^1$-degree (KW19) at rational points and étale points. This explicit formula can also be used to compute the local $\mathbb{A}^1$-degree at points with separable residue field by (BBM$^+$21). Together, these results allow one to compute the global $\mathbb{A}^1$-degree of a morphism $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ with only isolated zeros by computing the local $\mathbb{A}^1$-degrees of f over its zero locus, so long as the residue field of each point in the zero locus is separable over the base field. In the local case, Theorem 4.1.2 gives a commutative algebraic formula for the local $\mathbb{A}^1$-degree at any closed point.

Cazanave showed that the Bézoutian gives a formula for the global $\mathbb{A}^1$-degree of any endomorphism of $\mathbb{P}^1_k$ (Caz12). An advantage to Cazanave's formula is that one does not need to determine the zero locus or other local information about f. We extend Cazanave's formula for morphisms $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ with isolated zeros. The work of Scheja–Storch on global complete intersections (SS75) is central to both (KW19) and our result. We also rely on the work of Becker–Cardinal–Roy–Szafraniec (BCRS96), who describe a procedure for recovering the global version of the Scheja–Storch form.

Theorem 4.1.2 has applications wherever Morel's $\mathbb{A}^1$-degree is used. One particularly successful application of the $\mathbb{A}^1$-degree has been the $\mathbb{A}^1$-enumerative geometry program. The goal of this program is to enrich enumerative problems over arbitrary fields by producing $\mathrm{GW}(k)$-valued enumerative equations and interpreting them geometrically over various fields. Notable results in this direction include Srinivasan and Wickelgren's count of lines meeting four lines in three-space (SW21), Larson and Vogt's count of bitangents to a smooth plane quartic (LV21), and Bethea, Kass, and Wickelgren's enriched Riemann–Hurwitz formula (BKW20). See (McK21; Pau22) for other related works. For a more detailed account of recent developments in $\mathbb{A}^1$-

enumerative geometry, see (Bra21; PW21).

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4.2 Notation and conventions

In this section, we fix some standard terminology and notation. Let k denote an arbitrary field. We will always use $f = (f_1, \ldots, f_n) : \mathbb{A}^n_k \to \mathbb{A}^n_k$ to denote an endomorphism of affine space, assumed to have isolated zeros when we work with it in the global context. We denote by Q the global algebra associated to this endomorphism

$$Q := \frac{k[x_1, \ldots, x_n]}{(f_1, \ldots, f_n)}.$$

The maximal ideals of Q correspond to the maximal ideals of $k[x_1, \ldots, x_n]$ on which f vanishes. For any maximal ideal $\mathfrak{m}$ of $k[x_1, \ldots, x_n]$ on which f vanishes, we denote by $Q_{\mathfrak{m}}$ the local algebra

$$Q_{\mathfrak{m}} := \frac{k[x_1, \ldots, x_n]_{\mathfrak{m}}}{(f_1, \ldots, f_n)}.$$

If $\lambda : V \to k$ is a k-linear form on any k-algebra, we will denote by Φ_λ the associated bilinear form given by

$$\begin{aligned} \Phi_\lambda : V \times V &\to k \\ (a, b) &\mapsto \lambda(ab). \end{aligned}$$

Definition 4.2.1. We say that λ is a *dualizing linear form* if Φ_λ is non-degenerate as a symmetric bilinear form (BCRS96, 2.1). If λ is dualizing, then we say that two vector space bases $\{a_i\}$ and $\{b_i\}$ of V are *dual with respect to* λ if

$$\lambda(a_i b_j) = \delta_{ij},$$

where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$. We show in Remark 4.3.6 that if $\{a_i\}$ and $\{b_i\}$ are dual with respect to λ, then λ is a dualizing linear form.

More notation will be introduced as we provide an overview of Bézoutians and the Scheja–Storch bilinear form. We will borrow and clarify notation from both (SS75) and (BCRS96).

4.3 Bézoutians

We first provide an overview of the construction of the *Bézoutian*, following (BCRS96). Given one of our n polynomials f_i, we introduce two sets of auxiliary indeterminants and study how f_i changes when we incrementally exchange one set of indeterminants for the other. Explicitly, consider variables $X := (X_1, \ldots, X_n)$ and $Y := (Y_1, \ldots, Y_n)$. For any $1 \leq i, j \leq n$, we denote by Δ_{ij} the quantity

$$\Delta_{ij} := \frac{f_i(Y_1, \ldots, Y_{j-1}, X_j, \ldots, X_n) - f_i\,(Y_1, \ldots, Y_j, X_{j+1}, \ldots, X_n)}{X_j - Y_j}.$$

Note that Δ_{ij} is a multivariate polynomial. Indeed, $f_i(Y_1, \ldots, Y_{j-1}, X_j, \ldots, X_n)$ and $f_i(Y_1, \ldots, Y_j, X_{j+1}, \ldots, X_n)$ differ only in the terms in which X_j or Y_j appear, so we can expand the difference

$$f_i(Y_1, \ldots, Y_{j-1}, X_j, \ldots, X_n) - f_i\,(Y_1, \ldots, Y_j, X_{j+1}, \ldots, X_n) = \sum_{\ell \geq 1} g_\ell \cdot (X_j - Y_j)^\ell,$$

where $g_\ell \in k[Y_1, \ldots, Y_{j-1}, X_{j+1}, \ldots, X_n]$. In this notation, $\Delta_{ij} = \sum_{\ell \geq 1} g_\ell \cdot (X_j - Y_j)^{\ell-1}$.

We view Δ_{ij} as living in the tensor product ring $Q \otimes_k Q$, under the isomorphism

$$\varepsilon : \frac{k[X,Y]}{(f(X), f(Y))} \xrightarrow{\cong} Q \otimes_k Q,$$

given by sending X_i to $x_i \otimes 1$, and Y_i to $1 \otimes x_i$.

Definition 4.3.1. We define the *Bézoutian* of the polynomials $f_1, \ldots, f_n$ to be the image $\text{Béz}(f_1, \ldots, f_n)$ of the determinant $\det(\Delta_{ij})$ in $Q \otimes_k Q$.

Example 4.3.2. Let $(f_1, f_2, f_3) = (x_1^2, x_2^2, x_3^2)$. Then we have that

$$\begin{aligned}
\text{Béz}(f_1, f_2, f_3) &= \varepsilon \left(\det \begin{pmatrix} X_1 + Y_1 & 0 & 0 \\ 0 & X_2 + Y_2 & 0 \\ 0 & 0 & X_3 + Y_3 \end{pmatrix} \right) \\
&= \varepsilon \left((X_1 + Y_1)(X_2 + Y_2)(X_3 + Y_3) \right) \\
&= x_1x_2x_3 \otimes 1 + x_1x_2 \otimes x_3 + x_1x_3 \otimes x_2 + x_2x_3 \otimes x_1 \\
&\quad + x_1 \otimes x_2x_3 + x_2 \otimes x_1x_3 + x_3 \otimes x_1x_2 + 1 \otimes x_1x_2x_3.
\end{aligned}$$

There is a natural multiplication map $\delta : Q \otimes_k Q \to Q$, defined by $\delta(a \otimes b) = ab$, that sends the Bézoutian of f to the image of the Jacobian of f in Q.

Proposition 4.3.3. Let $\text{Jac}(f_1, \ldots, f_n)$ be the image of the Jacobian determinant $\det(\frac{\partial f_i}{\partial x_j})$ in Q. Then

$$\delta\left(\text{Béz}(f_1, \ldots, f_n)\right) = \text{Jac}(f_1, \ldots, f_n) \in Q.$$

Proof. Note that $(\delta\circ\varepsilon)(a(X,Y)) = a(x,x)$ and $\delta\circ\varepsilon$ is an algebra homomorphism. In particular, $\delta\circ\varepsilon$ preserves the multiplication and addition occurring in the determinant which defines $\mathrm{Béz}(f_1,\ldots,f_n)$. Therefore it suffices for us to verify that

$$(\delta\circ\varepsilon)\left(\Delta_{ij}\right) = \frac{\partial f_i}{\partial x_j}.$$

Recall that

$$\Delta_{ij} = \frac{f_i(Y_1,\ldots,Y_{j-1},X_j,\ldots,X_n) - f_i(Y_1,\ldots,Y_j,X_{j+1},\ldots,X_n)}{X_j - Y_j}.$$

Taking the x_j-Taylor expansion of $f(x_1,\ldots,x_n)$ about Y_j gives us

$$f_i(x_1,\ldots,x_n) = f_i(x_1,\ldots,Y_j,\ldots,x_n) + \sum_{\ell\geq 1} \frac{\partial^\ell f_i}{\partial x_j^\ell}\cdot(x_j - Y_j)^\ell.$$

We now subtract $f_i(x_1,\ldots,Y_j,\ldots,x_n)$ from both sides, evaluate $x_j \mapsto X_j$, and divide by $X_j - Y_j$ to deduce

$$\frac{f_i(x_1,\ldots,X_j,\ldots,x_n) - f_i(x_1,\ldots,Y_j,\ldots,x_n)}{(X_j - Y_j)} = \frac{\partial f_i}{\partial x_j} + \sum_{\ell\geq 2}\frac{\partial^\ell f_i}{\partial x_j^\ell}\cdot(X_j - Y_j)^{\ell-1}.$$

Finally, evaluating $X_j \mapsto x_j$ and $Y_j \mapsto x_j$ gives us $(\delta\circ\varepsilon)(\Delta_{ij}) = \frac{\partial f_i}{\partial x_j}$, as desired. □

Lemma 4.3.4. Let $a_1,\ldots,a_m$ be any vector space basis for Q, and write the Bézoutian as

$$\mathrm{Béz}(f_1,\ldots,f_n) = \sum_{i=1}^m a_i \otimes b_i$$

for some $b_1,\ldots,b_n \in Q$. Then $\{b_i\}_{i=1}^m$ is a basis for Q .

Proof. This is (BCRS96, 2.10(iii)). □

This allows us to associate to the Bézoutian a pair of vector space bases for Q. Given any such pair of bases, we will construct a unique linear form for which the bases are dual. Before doing so, we establish some equivalent conditions for the duality of a linear form given a pair of bases.

Proposition 4.3.5. Let $\{a_i\}$ and $\{b_i\}$ be a pair of bases for B. Consider the induced k-linear isomorphism

$$\begin{aligned}\Theta : \operatorname{Hom}_k(Q, k) &\to Q \\ \varphi &\mapsto \sum_i \varphi(a_i)b_i.\end{aligned}$$

Given a linear form $\lambda : Q \to k$, the following are equivalent:

1. We have that $\Theta(\lambda) = \sum_i \lambda(a_i)b_i = 1$.

2. For any $a \in Q$, we have $a = \sum_i \lambda(aa_i)b_i$.

3. We have that $\{a_i\}$ and $\{b_i\}$ are dual with respect to λ.

Proof. Note that (2) implies (1) by setting $a = 1$. Next, we remark that Θ is a Q-module isomorphism by (SS75, 3.3 Satz), where the Q-module structure on $\operatorname{Hom}_k(Q, k)$ is given by $a \cdot \varphi = \varphi(a \cdot -)$. This allows us to conclude that $a \cdot \Theta(\lambda) =$

$\Theta(a \cdot \lambda)$ for any linear form λ. In particular, we have

$$a \sum_i \lambda(a_i) b_i = \sum_i \lambda(a a_i) b_i.$$

It follows from this identity that (1) implies (2). Now suppose that (2) holds. By setting $a = b_j$ for some j, we have

$$\sum_i \lambda(a_i b_j) b_i = b_j.$$

Since $\{b_i\}$ is a basis, it follows that $\lambda(a_i b_j) = \delta_{ij}$. Thus the bases $\{a_i\}$ and $\{b_i\}$ are dual with respect to λ. Finally, suppose that (3) holds, so that $\lambda(a_i b_j) = \delta_{ij}$. For any $a \in Q$, write a as $a := \sum_j c_j b_j$ for some scalars c_j. Then

$$\begin{aligned}\sum_i \lambda(a a_i) b_i = \sum_i \lambda\left(a_i \sum_j c_j b_j\right) b_i &= \sum_i \left(\sum_j c_j \lambda(a_i b_j)\right) b_i \\ &= \sum_i c_i b_i = a.\end{aligned}$$

Thus (3) implies (2). $\square$

Remark 4.3.6. If $\{a_i\}$ and $\{b_i\}$ are dual with respect to λ, then λ is a dualizing form. Indeed, suppose there exists $x \in Q$ such that $\Phi_\lambda(x, y) = 0$ for all $y \in Q$. Write $x = \sum_i x_i a_i$ with $x_i \in k$. Then

$$\begin{aligned}0 = \lambda(x b_j) &= \lambda\left(\sum_i x_i a_i b_j\right) \\ &= \sum_i x_i \lambda(a_i b_j) = x_j\end{aligned}$$

for all j, so $x = 0$.

Corollary 4.3.7. Let $\{a_i\}$ and $\{b_i\}$ be two k-vector space bases for Q. Then there exists a unique dualizing linear form $\lambda : Q \to k$ such that $\{a_i\}$ and $\{b_i\}$ are dual with respect to λ.

Proof. As Θ is a k-algebra isomorphism, it admits a unique preimage of 1. Thus, given any pair of bases $\{a_i\}$ and $\{b_i\}$ of Q, there is a unique dualizing linear form with respect to which $\{a_i\}$ and $\{b_i\}$ are dual. $\square$

Definition 4.3.8. We call Φ_λ a *Bézoutian bilinear form* if $\lambda : Q \to k$ is a dualizing linear form such that

$$\text{Béz}(f_1, \ldots, f_n) = \sum_{i=1}^{m} a_i \otimes b_i,$$

where $\{a_i\}$ and $\{b_i\}$ are dual bases with respect to λ.

A priori this is different than the Bézoutian form detailed in Definition 4.1.1, although we will prove that they define the same class in $\mathrm{GW}(k)$ in Section 4.5.1.

Proposition 4.3.9. Given a function $f : \mathbb{A}_k^n \to \mathbb{A}_k^n$ with isolated zeros, its Bézoutian bilinear form is a well-defined class in $\mathrm{GW}(k)$.

Proof. Let Φ_λ be a Bézoutian bilinear form for f. Recall that $\Phi_\lambda : Q \times Q \to k$ is defined by $\Phi_\lambda(a, b) = \lambda(ab)$. Since λ is a dualizing linear form, Φ_λ is non-degenerate and as Q is commutative, Φ_λ is symmetric. Lemma 4.3.4 implies that given a basis

$a_1, \ldots, a_m$ for Q, we can write

$$\mathrm{Béz}(f_1, \ldots, f_n) = \sum_{i=1}^{m} a_i \otimes b_i,$$

and obtain a second basis $b_1, \ldots, b_m$ for Q. By Corollary 4.3.7, there is a dualizing linear form for the two bases $\{a_i\}_{i=1}^m$ and $\{b_i\}_{i=1}^m$. It remains to show that if

$$\mathrm{Béz}(f_1, \ldots, f_n) = \sum_{i=1}^{m} a_i \otimes b_i = \sum_{i=1}^{m} a_i' \otimes b_i',$$

for some bases $\{a_i\}, \{b_i\}$ dual with respect to λ and $\{a_i'\}, \{b_i'\}$ dual with respect to λ', then Φ_λ and $\Phi_{\lambda'}$ are isomorphic. We will in fact show that $\lambda = \lambda'$, so that $\Phi_\lambda = \Phi_{\lambda'}$. Write $a_i = \sum_s \alpha_{is} a_s'$ and $b_i = \sum_s \beta_{is} b_s'$. Then

$$\begin{aligned} \sum_{i=1}^{m} a_i' \otimes b_i' = \sum_{i=1}^{m} a_i \otimes b_i &= \sum_i \left(\sum_s \alpha_{is} a_s' \right) \otimes \left(\sum_t \beta_{it} b_t' \right) \\ &= \sum_{s,t} \left(\sum_i \alpha_{is} \beta_{it} \right) a_s' \otimes b_t'. \end{aligned}$$

Since $\{a_s' \otimes b_t'\}$ is a basis for $Q \otimes_k Q$, we conclude that $\sum_i \alpha_{is}\beta_{it} = \delta_{st}$. In particular, $(\alpha_{ij})^{-1} = (\beta_{ij})^T$, so $(\beta_{ij})(\alpha_{ij})^T$ is the identity matrix. Thus $\sum_j \alpha_{sj}\beta_{tj} = \delta_{st}$.

Now given $g = \sum_i c_i a_i = \sum_i c_i' a_i' \in Q$ and $1 = \sum_i d_i b_i = \sum_i d_i' b_i'$, we have that

$$\lambda(g) = \lambda \left(\sum_i (c_i a_i) \cdot \sum_j d_j b_j \right) = \sum_{i,j} c_i d_j \lambda(a_i b_j) = \sum_i c_i d_i.$$

Similarly, we have $\lambda'(g) = \sum_i c_i' d_i'$. By our change of bases, we have $c_j' = \sum_i c_j \alpha_{ij}$

and $d'_j = \sum_i d_i \beta_{ij}$. Thus

$$\begin{aligned}\lambda'(g) &= \sum_j c'_j d'_j = \sum_j \left(\sum_s c_s \alpha_{sj} \right) \left(\sum_t d_t \beta_{tj} \right) \\ &= \sum_{s,t} c_s d_t \left(\sum_j \alpha_{sj} \beta_{tj} \right) = \sum_s c_s d_s = \lambda(g).\end{aligned}$$

Therefore $\lambda = \lambda'$, as desired. $\square$

Example 4.3.10. Continuing Example 4.3.2, let $f = (x_1^2, x_2^2, x_3^2)$, so that

$$\begin{aligned}\varepsilon^{-1}(\text{Béz}(f_1, f_2, f_3)) &= (X_1 + Y_1)(X_2 + Y_2)(X_3 + Y_3) \\ &= X_1X_2X_3 + X_1X_2Y_3 + X_1Y_2X_3 + X_1Y_2Y_3 \\ &\quad + Y_1X_2X_3 + Y_1X_2Y_3 + Y_1Y_2X_3 + Y_1Y_2Y_3.\end{aligned}$$

We give two bases for $k[Z_1, Z_2, Z_3]/(Z_1^2, Z_2^2, Z_3^2)$ in the following table, where we replace Z by either X or Y. We pair off these bases in a convenient way.

i	a_i	b_i
1	1	$Y_1Y_2Y_3$
2	X_1	Y_2Y_3
3	X_2	Y_1Y_3
4	X_3	Y_1Y_2
5	X_1X_2	Y_3
6	X_1X_3	Y_2
7	X_2X_3	Y_1
8	$X_1X_2X_3$	1

The Bézoutian we computed is in the desired form $\sum_{i=1}^{8} a_i \otimes b_i$, so we now need to compute the dualizing linear form λ for $\{a_i\}$ and $\{b_i\}$. Since $1 = 1 \cdot b_8 + \sum_{i=1}^{7} 0 \cdot b_i$, we define λ by $\lambda(a_i) = 0$ for $1 \leq i \leq 7$ and $\lambda(a_8) = \lambda(X_1X_2X_3) = 1$. Now let $g \in k[X_1, X_2, X_3]/(X_1^2, X_2^2, X_3^2)$ be arbitrary. We can write g as

$$g = c_1 + c_2X_1 + c_3X_2 + c_4X_3 + c_5X_1X_2 + c_6X_1X_3 + c_7X_2X_3 + c_8X_1X_2X_3.$$

Then λ is the dualizing linear form sending

$$\lambda : \frac{k[X_1, X_2, X_3]}{(X_1^2, X_2^2, X_3^2)} \to k$$
$$g \mapsto c_8.$$

Finally we can compute the Gram matrix of Φ_λ in the basis $\{a_i\}$. Note that a_ia_j is a scalar multiple of $X_1X_2X_3$ if and only if $i+j-1=8$. Thus the Gram matrix is

$$\Phi_\lambda = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cong \bigoplus_{i=1}^{4} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

4.4 The Scheja–Storch bilinear form

Associated to any polynomial with an isolated zero, Eisenbud and Levine (EL77) and Khimshiashvili (Him77) used the Scheja–Storch construction (SS75) to produce a bilinear form on the local algebra $Q_\mathfrak{m}$. Kass and Wickelgren proved that this Eisenbud–Khimshiashvili–Levine bilinear form computes the local $\mathbb{A}^1$-degree (KW19). The machinery of Scheja and Storch works in great generality; in particular, one may produce a Scheja–Storch bilinear form on the global algebra Q as well as the local algebras $Q_\mathfrak{m}$. We will provide a brief account of the Scheja–Storch construction before comparing it with the Bézoutian.

In (SS75), $k\langle X\rangle := k\left\langle X_1, \ldots, X_n\right\rangle$ denotes either a polynomial ring $k\left[X_1, \ldots, X_n\right]$

or a power series ring $k[[X_1, \ldots, X_n]]$. We will also use this notation, although we will focus on the situation where $k\langle X\rangle$ is a polynomial ring. Let $\rho : k\langle X\rangle \to Q$ denote the map obtained by quotienting out by the ideal $(f_1, \ldots, f_n)$, let $\mu_1 : k\langle X\rangle \otimes_k k\langle X\rangle \to k\langle X\rangle$ denote the multiplication map, and let $\mu : Q \otimes_k Q \to Q$ denote the multiplication map on the global algebra, fitting into a commutative diagram

$$\begin{tikzcd}
k\langle X\rangle \otimes_k k\langle X\rangle \arrow[r, "\mu_1"] \arrow[d, "\rho\otimes\rho"'] & k\langle X\rangle \arrow[d, "\rho"] \\
Q \otimes_k Q \arrow[r, "\mu"'] & Q.
\end{tikzcd}$$

We remark that $f_j \otimes 1 - 1 \otimes f_j$ lies in $\ker(\mu_1)$, and that $\ker(\mu_1)$ is generated by elements of the form $X_i \otimes 1 - 1 \otimes X_i$. Thus for any j, there are elements $a_{ij} \in k\langle X\rangle \otimes_k k\langle X\rangle$ such that

$$f_j \otimes 1 - 1 \otimes f_j = \sum_{i=1}^{n} a_{ij} \left(X_i \otimes 1 - 1 \otimes X_i\right). \tag{4.4.1}$$

We denote by Δ the following distinguished element in the tensor algebra $Q \otimes_k Q$

$$\Delta := (\rho \otimes \rho)\left(\det(a_{ij})\right),$$

which corresponds to the Bézoutian which we will later demonstrate. It is true that Δ is independent of the choice of a_{ij}, as shown by Scheja and Storch (SS75, 3.1 Satz). We now define an important isomorphism χ of k-algebras used in the Scheja–Storch construction. However, we will phrase this more categorically than in (SS75), as it will benefit us later.

Proposition 4.4.2. Consider two endofunctors $F, G : \mathtt{Alg}_k^{\mathrm{fg}} \to \mathtt{Alg}_k^{\mathrm{fg}}$ on the category of finitely generated k-algebras, where $F(A) = A \otimes_k A$ and $G(A) = \mathrm{Hom}_k\left(\mathrm{Hom}_k\left(A, k\right), A\right)$. Then there is a natural isomorphism $\chi : F \to G$ whose component at a k-algebra A is

$$\chi_A : A \otimes_k A \to \mathrm{Hom}_k\left(\mathrm{Hom}_k\left(A, k\right), A\right)$$
$$b \otimes c \mapsto \left[\varphi \mapsto \varphi(b)c\right].$$

Proof. This canonical isomorphism is given in (SS75, p.181), so it will suffice for us to verify naturality. Let $g : A \to B$ be any morphism of k-algebras. Consider the induced maps $g \otimes g : A \otimes_k A \to B \otimes_k B$ and

$$g_* : \mathrm{Hom}_k\left(\mathrm{Hom}_k(A, k), A\right) \to \mathrm{Hom}_k\left(\mathrm{Hom}_k\left(B, k\right), B\right)$$
$$\psi \mapsto \left[\varepsilon \mapsto g \circ \psi(\varepsilon \circ g)\right].$$

It remains to show that the following diagram commutes.

$$\begin{tikzcd}
A \otimes_k A \arrow[r, "\chi_A"] \arrow[d, "g\otimes g"'] & \mathrm{Hom}_k\left(\mathrm{Hom}_k\left(A, k\right), A\right) \arrow[d, "g_*"] \\
B \otimes_k B \arrow[r, "\chi_B"'] & \mathrm{Hom}_k\left(\mathrm{Hom}_k\left(B, k\right), B\right)
\end{tikzcd}$$

To see this, we compute $g_* \circ \chi_A = [b \otimes c \mapsto [\varepsilon \mapsto g((\varepsilon \circ g)(b) \cdot c)]]$. Note that $\varepsilon \circ g : B \to k$, so $(\varepsilon \circ g)(b) \in k$. Since g is k-linear, we have $g((\varepsilon \circ g)(b) \cdot c) = \varepsilon(g(b)) \cdot g(c)$. Next, we compute $\chi_B \circ (g \otimes g) = [b \otimes c \mapsto [\varepsilon \mapsto \varepsilon(g(b)) \cdot g(c)]]$. Thus $g_* \circ \chi_A = \chi_B \circ (g \otimes g)$, so the diagram commutes. $\square$

We now let $\Theta := \chi_Q(\Delta)$ denote the image of Δ under the component of this natural isomorphism at the global algebra Q. We have that Θ is a k-linear map $\Theta : \mathrm{Hom}_k(Q, k) \to Q$. Letting η denote $\Theta^{-1}(1)$, we obtain a well-defined linear form $\eta : Q \to k$ by (SS75, 3.3 Satz).

Definition 4.4.3. We refer to $\Phi_\eta : Q \times Q \to k$ as the *global Scheja–Storch bilinear form.*

The Bézoutian gives us an explicit formula for Δ. As a result, the global Scheja–Storch form agrees with the Bézoutian form.

Proposition 4.4.4. In $Q \otimes_k Q$, we have $\Delta = \mathrm{Béz}(f_1, \ldots, f_n)$.

Proof. We first compute

$$\begin{aligned}
\sum_{i=1}^{n} \Delta_{ji}\,(X_i - Y_i) &= \sum_{i=1}^{n} \frac{f_j(Y_1, \ldots, Y_{i-1}, X_i, \ldots, X_n) - f_j(Y_1, \ldots, Y_i, X_{i+1}, \ldots, X_n)}{(X_i - Y_i)} \cdot (X_i - Y_i) \\
&= \sum_{i=1}^{n} f_j(Y_1, \ldots, Y_{i-1}, X_i, \ldots, X_n) - f_j(Y_1, \ldots, Y_i, X_{i+1}, \ldots, X_n) \\
&= f_j(X_1, \ldots, X_n) - f_j(Y_1, \ldots, Y_n).
\end{aligned}$$

Let $\varphi : k\langle X\rangle \otimes_k k\langle X\rangle \xrightarrow{\sim} k\langle X, Y\rangle$ be the ring isomorphism given by $\varphi(b \otimes c) = b(X)c(Y)$. Note that $\varphi(x_i \otimes 1) = X_i$ and $\varphi(1 \otimes x_i) = Y_i$, so the inverse of φ is characterized by $\varphi^{-1}(X_i) = x_i \otimes 1$ and $\varphi^{-1}(Y_i) = 1 \otimes x_i$. It follows that

$$f_j \otimes 1 - 1 \otimes f_j = \varphi^{-1}(f_j(X) - f_j(Y)) = \sum_{i=1}^{n} \varphi^{-1}(\Delta_{ji}(X_i - Y_i))$$
$$= \sum_{i=1}^{n} \varphi^{-1}(\Delta_{ji})(x_i \otimes 1 - 1 \otimes x_i).$$

We may thus set $a_{ij} = \varphi^{-1}(\Delta_{ji})$, and (SS75, 3.1 Satz) implies that $\Delta = (\rho \otimes \rho)(\det(a_{ij}))$. On the other hand, $(\rho \otimes \rho)(\varphi^{-1}(\det(\Delta_{ji}))) = \text{Béz}(f_1, \ldots, f_n)$ by Definition 4.3.1. □

Lemma 4.4.5. The Bézoutian bilinear form and the global Scheja–Storch bilinear form are identical.

Proof. We showed in Proposition 4.4.4 that Δ is the Bézoutian in $Q \otimes_k Q$. We now show that the associated forms are identical. Pick bases $\{a_i\}$ and $\{b_i\}$ of Q such that

$$\Delta = \text{Béz}(f_1, \ldots, f_n) = \sum_{i=1}^{m} a_i \otimes b_i.$$

Since the natural isomorphism χ has k-linear components, Δ is mapped to

$$\Theta := \chi_Q(\Delta) = \left[\varphi \mapsto \sum_{i=1}^{m} \varphi(a_i) b_i\right].$$

Thus $\eta := \Theta^{-1}(1)$ is the linear form $\eta : Q \to k$ satisfying $\sum_{i=1}^{m} \eta(a_i) b_i = 1$. By Proposition 4.3.5, this implies that η is the form for which $\{a_i\}$ and $\{b_i\}$ are dual bases. As in Definition 4.3.8, this tells us that η is the linear form producing the Bézoutian bilinear form. □

4.4.1 Local decomposition

While our discussion of the Scheja–Storch form in the previous section was global, it is perfectly valid to localize at a maximal ideal and repeat the story again (SS75, p.180–181). The fact that Q is an Artinian ring then gives a convenient way to relate the global version of η to the local version of η. This local decomposition has been utilized previously, for example in (KW19).

Let $\mathfrak{m}$ be a maximal ideal in $k[x_1, \ldots, x_n]$ at which the morphism $f = (f_1, \ldots, f_n)$ has an isolated root. Letting $\rho_\mathfrak{m}$ denote the quotient map $k\left\langle X \right\rangle_\mathfrak{m} \to Q_\mathfrak{m}$, we have a commutative diagram

$$\begin{tikzcd} k\left\langle X \right\rangle_\mathfrak{m} \otimes_k k\left\langle X \right\rangle_\mathfrak{m} \arrow[r, "\mu_1'"] \arrow[d, "\rho_\mathfrak{m}\otimes\rho_\mathfrak{m}"'] & k\left\langle X \right\rangle_\mathfrak{m} \arrow[d, "\rho_\mathfrak{m}"] \\ Q_\mathfrak{m} \otimes_k Q_\mathfrak{m} \arrow[r, "\mu'"'] & Q_\mathfrak{m}. \end{tikzcd}$$

In $k\langle X\rangle_\mathfrak{m} \otimes_k k\langle X\rangle_\mathfrak{m}$, we can again write

$$f_j \otimes 1 - 1 \otimes f_j = \sum_{i=1}^{n} \widetilde{a}_{ij}(X_i \otimes 1 - 1 \otimes X_i)$$

to obtain the local Bézoutian $\Delta_\mathfrak{m} := (\rho_\mathfrak{m} \otimes \rho_\mathfrak{m})(\det(\widetilde{a}_{ij})) \in Q_\mathfrak{m} \otimes_k Q_\mathfrak{m}$. Let $\lambda_\mathfrak{m} : Q \to Q_\mathfrak{m}$ be the localization map. From (SS75, p.181) we have $(\lambda_\mathfrak{m} \otimes \lambda_\mathfrak{m})(\Delta) = \Delta_\mathfrak{m}$. Via the natural isomorphism χ in Proposition 4.4.2, we have a commutative diagram of the form

$$\begin{tikzcd}
Q \otimes_k Q \arrow[r, "\chi_Q"] \arrow[d, "\lambda_\mathfrak{m} \otimes \lambda_\mathfrak{m}"'] & \operatorname{Hom}_k\left(\operatorname{Hom}_k(Q,k),Q\right) \arrow[d, "\lambda_{\mathfrak{m}*}"] \\
Q_\mathfrak{m} \otimes_k Q_\mathfrak{m} \arrow[r, "\chi_{Q_\mathfrak{m}}"'] & \operatorname{Hom}_k\left(\operatorname{Hom}_k(Q_\mathfrak{m},k),Q_\mathfrak{m}\right).
\end{tikzcd}$$

Tracing Δ through this diagram, we see that

$$\begin{tikzcd}
\Delta \arrow[r, mapsto] \arrow[d, mapsto] & \Theta \arrow[d, mapsto] \\
\Delta_\mathfrak{m} \arrow[r, mapsto] & \Theta_\mathfrak{m},
\end{tikzcd}$$

where $\Theta_\mathfrak{m} = \chi_{Q_\mathfrak{m}}(\Delta_\mathfrak{m})$. Unwinding $\Theta_\mathfrak{m} = \lambda_{\mathfrak{m}*}(\Theta)$, we find that $\Theta_\mathfrak{m}$ is the map

$$\begin{aligned}
\Theta_\mathfrak{m} : \operatorname{Hom}_k(Q_\mathfrak{m},k) &\to Q_\mathfrak{m} \\
\psi &\mapsto \lambda_\mathfrak{m} \circ \Theta\left(\psi \circ \lambda_\mathfrak{m}\right).
\end{aligned}$$

Recall that as Q is a zero-dimensional Noetherian commutative k-algebra, the localization maps induce a k-algebra isomorphism[1]

$$(\lambda_\mathfrak{m})_\mathfrak{m} : Q \xrightarrow{\sim} \prod_\mathfrak{m} Q_\mathfrak{m}.$$

This is reflected by an internal decomposition of Q in terms of orthogonal idempotents (BCRS96, 2.13), which we now describe (see also (Sta21, Lemma 00JA)). By the Chinese remainder theorem, we may pick a collection of pairwise orthogonal

[1] Q is Artinian by (Sta21, Lemma 00KH), so the claimed isomorphism exists by (Sta21, Lemma 00JA).

idempotents $\{e_\mathfrak{m}\}_\mathfrak{m}$ such that $\sum_\mathfrak{m} e_\mathfrak{m} = 1$. The internal decomposition of Q is then

$$Q = \bigoplus_{\mathfrak{m}} Q \cdot e_\mathfrak{m},$$

and the localization maps restrict to isomorphisms $\lambda_\mathfrak{m}|_{Q\cdot e_\mathfrak{m}} : Q \cdot e_\mathfrak{m} \xrightarrow{\sim} Q_\mathfrak{m}$ with $\lambda_\mathfrak{m}(e_\mathfrak{m}) = 1$. Moreover, $\lambda_\mathfrak{m}(Q \cdot e_\mathfrak{n}) = 0$ for any $\mathfrak{n} \neq \mathfrak{m}$.

Proposition 4.4.6. Suppose $\ell : Q \to k$ is a linear form which factors through the localization $\lambda_\mathfrak{m} : Q \to Q_\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$. Then $\Theta(\ell)$ lies in $Q \cdot e_\mathfrak{m}$.

Proof. Recall that $\lambda_\mathfrak{m}|_{Q\cdot e_\mathfrak{n}} = 0$ for $\mathfrak{n} \neq \mathfrak{m}$. Since $e_\mathfrak{m} \cdot e_\mathfrak{n} = 0$ for $\mathfrak{n} \neq \mathfrak{m}$ and $e_\mathfrak{m}$ is idempotent, the localization $\lambda_\mathfrak{m} : Q \to Q_\mathfrak{m}$ can be written as the following composition:

$$\lambda_\mathfrak{m} : Q \xrightarrow{-\cdot e_\mathfrak{m}} Q \xrightarrow{\lambda_\mathfrak{m}} Q_\mathfrak{m}.$$

Since ℓ factors through the localization, it can be written as a composite

$$\ell : Q \xrightarrow{-\cdot e_\mathfrak{m}} Q \xrightarrow{\lambda_\mathfrak{m}} Q_\mathfrak{m} \xrightarrow{\ell_\mathfrak{m}} k.$$

Thus $\Theta(\ell) = \Theta(\ell_\mathfrak{m} \circ \lambda_\mathfrak{m} \circ (e_\mathfrak{m} \cdot -))$. Scheja–Storch proved that Θ respects the Q-module structure on $\mathrm{Hom}_k(Q, k)$ given by $a \cdot \sigma = \sigma(a \cdot -)$ (SS75, 3.3 Satz). That is, $\Theta(\sigma(a \cdot -)) = \Theta(a \cdot \sigma) = a\Theta(\sigma)$ for any $a \in Q$ and $\sigma \in \mathrm{Hom}_k(Q, k)$. Thus

$$\Theta(\ell) = e_\mathfrak{m} \cdot \Theta\left(\ell_\mathfrak{m} \circ \lambda_\mathfrak{m}\right),$$

so $\Theta(\ell) \in Q \cdot e_\mathfrak{m}$. $\square$

Returning to the Scheja–Storch form, we have the following commutative diagram relating $\Theta_\mathfrak{m}$ and Θ:

$$\begin{tikzcd} \mathrm{Hom}_k(Q,k) \arrow[r, "\Theta"] & Q \arrow[d, "\lambda_\mathfrak{m}"] \\ \mathrm{Hom}_k(Q_\mathfrak{m},k) \arrow[u, "-\circ\lambda_\mathfrak{m}"] \arrow[r, "\Theta_\mathfrak{m}"'] & Q_\mathfrak{m}. \end{tikzcd}$$

This coherence between Θ and $\Theta_\mathfrak{m}$ allows us to relate the local linear forms $\eta_\mathfrak{m} := \Theta_\mathfrak{m}^{-1}(1)$ to the global linear form $\eta := \Theta^{-1}(1)$ in the following way.

Proposition 4.4.7. For each maximal ideal $\mathfrak{m}$ of Q, let $\eta_\mathfrak{m} := \Theta_\mathfrak{m}^{-1}(1) : Q_\mathfrak{m} \to k$, and let $\eta := \Theta^{-1}(1) : Q \to k$. Then $\eta = \sum_\mathfrak{m} \eta_\mathfrak{m} \circ \lambda_\mathfrak{m}$.

Proof. It suffices to show that $\Theta(\sum_\mathfrak{m} \eta_\mathfrak{m} \circ \lambda_\mathfrak{m}) = 1$. Since $\eta_\mathfrak{m} = \Theta_\mathfrak{m}^{-1}(1)$ by definition, we have $1 = \Theta_\mathfrak{m}(\eta_\mathfrak{m}) := \lambda_\mathfrak{m}(\Theta(\eta_\mathfrak{m} \circ \lambda_\mathfrak{m}))$. By Proposition 4.4.6, we have $\Theta(\eta_\mathfrak{m} \circ \lambda_\mathfrak{m}) \in Q \cdot e_\mathfrak{m}$. Since $\lambda_\mathfrak{m}(\Theta(\eta_\mathfrak{m} \circ \lambda_\mathfrak{m})) = 1$ and $\lambda_\mathfrak{m}|_{Q\cdot e_\mathfrak{m}}$ is an isomorphism sending $e_\mathfrak{m}$ to 1, it follows that $\Theta(\eta_\mathfrak{m} \circ \lambda_\mathfrak{m}) = e_\mathfrak{m}$. Finally, since Θ is k-linear, we have

$$\begin{aligned} \Theta\left(\sum_\mathfrak{m} \eta_\mathfrak{m} \circ \lambda_\mathfrak{m}\right) &= \sum_\mathfrak{m} \Theta(\eta_\mathfrak{m} \circ \lambda_\mathfrak{m}) \\ &= \sum_\mathfrak{m} e_\mathfrak{m} = 1. \end{aligned}$$ □

Using this local decomposition procedure for the linear forms $\eta_\mathfrak{m}$ and η, we obtain a local decomposition for Scheja–Storch bilinear forms.

Lemma 4.4.8. *(Local decomposition of Scheja–Storch forms)* Let η and $\eta_\mathfrak{m}$ be as in Proposition 4.4.7. Then $\Phi_\eta = \bigoplus_\mathfrak{m} \Phi_{\eta_\mathfrak{m}}$. In particular, the global Scheja–Storch form is a sum over local Scheja–Storch forms

$$\mathrm{SS}(f) = \sum_{\mathfrak{m}} \mathrm{SS}_{\mathfrak{m}}(f).$$

Proof. For each maximal ideal $\mathfrak{m}$, let $\{w_{\mathfrak{m},i}\}_i$ be a k-vector space basis for $Q_{\mathfrak{m}}$. Let $\{v_{\mathfrak{m},i}\}_{\mathfrak{m},i}$ (ranging over all i and all maximal ideals) be a basis of Q such that $\lambda_{\mathfrak{m}}(v_{\mathfrak{m},i}) = w_{\mathfrak{m},i}$ for each i and $\mathfrak{m}$, and $\lambda_{\mathfrak{m}}(v_{\mathfrak{n},i}) = 0$ for $\mathfrak{m} \neq \mathfrak{n}$. We now compare the Gram matrix for $\eta : Q \to k$ and the Gram matrices for $\eta_{\mathfrak{m}} : Q_{\mathfrak{m}} \to k$ in these bases. Via the internal decomposition consisting of pairwise orthogonal idempotents, we have $v_{\mathfrak{m},i} \cdot v_{\mathfrak{n},j} = 0$ if $\mathfrak{m} \neq \mathfrak{n}$. Thus

$$\eta(v_{\mathfrak{m},i} \cdot v_{\mathfrak{n},j}) = 0,$$

so the Gram matrix for Φ_η will be a block sum indexed over the maximal ideals. If $\mathfrak{m} = \mathfrak{n}$, then Proposition 4.4.7 implies

$$\begin{aligned} \eta(v_{\mathfrak{m},i} \cdot v_{\mathfrak{m},j}) &= \sum_{\mathfrak{n}} \eta_{\mathfrak{n}}\left(\lambda_{\mathfrak{n}}\left(v_{\mathfrak{m},i} \cdot v_{\mathfrak{m},j}\right)\right) = \eta_{\mathfrak{m}}(\lambda_{\mathfrak{m}}\left(v_{\mathfrak{m},i} \cdot v_{\mathfrak{m},j}\right)) \\ &= \eta_{\mathfrak{m}}\left(w_{\mathfrak{m},i} \cdot w_{\mathfrak{m},j}\right). \end{aligned}$$

Thus the Gram matrices of Φ_η and $\bigoplus_{\mathfrak{m}} \Phi_{\eta_{\mathfrak{m}}}$ are equal, so $\Phi_\eta = \bigoplus_{\mathfrak{m}} \Phi_{\eta_{\mathfrak{m}}}$. □

Remark 4.4.9. The local Scheja–Storch bilinear form is given by $\Phi_{\eta_{\mathfrak{m}}} : Q_{\mathfrak{m}} \times Q_{\mathfrak{m}} \to k$. Given a basis $\{a_1, \ldots, a_m\}$ of $Q_{\mathfrak{m}}$, we may write $\Delta_{\mathfrak{m}} = \sum a_i \otimes b_i$ and define the local Bézoutian bilinear form as a suitable dualizing form. Replacing Q, Δ, Θ, and η with

$Q_{\mathfrak{m}}$, $\Delta_{\mathfrak{m}}$, $\Theta_{\mathfrak{m}}$, and $\eta_{\mathfrak{m}}$, the results of Sections 4.3 and 4.4 also hold for local Bézoutians and the local Scheja–Storch form. In particular, the local analog of Lemma 4.4.5 implies that the local Scheja–Storch form is equal to the local Bézoutian form.

4.5 Proof of Theorem 4.1.2

We now relate the Scheja–Storch form to the $\mathbb{A}^1$-degree. The following theorem was first proven in the case where p is a rational zero by Kass and Wickelgren (KW19), and then in the case where p has finite separable residue field over the ground field in (BBM$^+$21, Corollary 1.4). Recent work of Bachmann and Wickelgren (BW21) gives a general result about the relation between local $\mathbb{A}^1$-degrees and Scheja–Storch forms.

Theorem 4.5.1. Let $\operatorname{char} k \neq 2$. Let $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ be an endomorphism of affine space with an isolated zero at a closed point p. Then we have that the local $\mathbb{A}^1$-degree of f at p and the Scheja–Storch form of f at p coincide as elements of $\mathrm{GW}(k)$:

$$\deg_p^{\mathbb{A}^1}(f) = \mathrm{SS}_p(f).$$

Proof. We may rewrite f as a section of the trivial rank n bundle over affine space $\mathcal{O}^n_{\mathbb{A}^n_k} \to \mathbb{A}^n_k$. Under the hypothesis that p is isolated, we may find a neighborhood $X \subseteq \mathbb{A}^n_k$ of p where the section f is non-degenerate (meaning it is cut out by a regular sequence). By (BW21, Corollary 8.2), the local index of f at p with the trivial orientation, corresponding to the representable Hermitian K-theory spectrum KO, agrees with the local Scheja–Storch form as elements of $\mathrm{KO}^0(k)$:

$$\mathrm{ind}_p(f, \rho_{\mathrm{triv}}, \mathrm{KO}) = \mathrm{SS}_p(f). \tag{4.5.2}$$

Let $\mathbb{S}$ denote the sphere spectrum in the stable motivic homotopy category $\mathcal{SH}(k)$. It is a well-known fact that Hermitian K-theory receives a map from the sphere spectrum, inducing an isomorphism $\pi_0(\mathbb{S}) \xrightarrow{\sim} \pi_0(\mathrm{KO})$ if $\mathrm{char} k \neq 2$ (see for example (Hor05, 6.9) for more detail); this is the only place where we use the assumption that $\mathrm{char} k \neq 2$. Combining this with the fact that that $\pi_0(\mathbb{S}) = \mathrm{GW}(k)$ under Morel's degree isomorphism, we observe that Equation 4.5.2 is really an equality in $\mathrm{GW}(k)$. By (BW21, Theorem 7.6, Example 7.7), the local index associated to the representable theory agrees with the local $\mathbb{A}^1$-degree:

$$\mathrm{ind}_p(f, \rho_{\mathrm{triv}}, \mathrm{KO}) = \deg_p^{\mathbb{A}^1}(f).$$

Combining these equalities gives the desired equality in $\mathrm{GW}(k)$. $\square$

Remark 4.5.3. Bachmann and Wickelgren in fact show that $\deg_Z^{\mathbb{A}^1}(f) = \mathrm{SS}_Z(f)$ for any isolated zero locus Z of f (BW21, Corollary 8.2). This gives an alternate viewpoint on the local decomposition described in Lemma 4.4.8

Corollary 4.5.4. Let $\operatorname{char} k \neq 2$. The local Bézoutian bilinear form is the local $\mathbb{A}^1$-degree.

Proof. As discussed in Remark 4.4.9, we can modify Lemma 4.4.5 to the local case by replacing Q, Δ, Θ, and η with $Q_{\mathfrak{m}}$, $\Delta_{\mathfrak{m}}$, $\Theta_{\mathfrak{m}}$, and $\eta_{\mathfrak{m}}$. The local Bézoutian form is thus equal to the local Scheja–Storch form, which is equal to the local $\mathbb{A}^1$-degree by Theorem 4.5.1. □

In contrast to previous techniques for computing the local $\mathbb{A}^1$-degree at rational or separable points, Corollary 4.5.4 gives an algebraic formula for the local $\mathbb{A}^1$-degree at any closed point.

As a result of the local decomposition of Scheja–Storch forms, the Bézoutian form agrees with the $\mathbb{A}^1$-degree globally as well.

Corollary 4.5.5. Let $\operatorname{char} k \neq 2$. The Bézoutian bilinear form is the global $\mathbb{A}^1$-degree.

Proof. Let Φ_η denote the Bézoutian bilinear form, which is equal to the global Scheja–Storch bilinear form by Lemma 4.4.5. By Lemma 4.4.8, the global Scheja–Storch form decomposes as a block sum of local Scheja–Storch forms. By Theorem 4.5.1, the local Scheja–Storch bilinear form agrees with the local $\mathbb{A}^1$-degree. Finally, we have that the sum of local $\mathbb{A}^1$-degrees is the global $\mathbb{A}^1$-degree. Putting this all together, we have

$$\Phi_\eta = \mathrm{SS}(f) = \sum_{\mathfrak{m}} \mathrm{SS}_{\mathfrak{m}}(f) = \sum_{\mathfrak{m}} \deg_{\mathfrak{m}}^{\mathbb{A}^1}(f) = \deg^{\mathbb{A}^1}(f). \tag{4.5.6}$$

□

Remark 4.5.7. It is not known if GW is represented by KO over fields of characteristic 2, which is the source of our assumption that $\mathrm{char} k \neq 2$. If this problem is resolved, one can remove any characteristic restrictions from our results. Alternately, Lemma 4.4.8 implies Corollaries 4.5.4 and 4.5.5 if all roots of f satisfy $\deg_p^{\mathbb{A}^1}(f) = \mathrm{SS}_p(f)$. By (KW19), (BBM⁺21), and (KW21, Proposition 34), Corollaries 4.5.4 and 4.5.5 are true in any characteristic if all roots of f are rational, étale, or separable.

4.5.1 Computing the Bézoutian bilinear form

We now prove Theorem 4.1.2 by describing a method for computing the class in $\mathrm{GW}(k)$ of the Bézoutian bilinear form in terms of the Bézoutian.

Proof of Theorem 4.1.2. Let R denote either a global algebra Q or a local algebra $Q_{\mathfrak{m}}$. Let $\{\alpha_i\}$ be any basis for R, and express

$$\text{Béz}(f_1, \ldots, f_n) = \sum_{i,j} B_{i,j} \alpha_i \otimes \alpha_j.$$

Rewriting this, we have

$$\text{Béz}(f_1, \ldots, f_n) = \sum_i \alpha_i \otimes \left(\sum_j B_{i,j}\alpha_j \right).$$

Let $\beta_i := \sum_j B_{i,j}\alpha_j$, so that $\{\alpha_i\}$ and $\{\beta_i\}$ are dual bases. Then for any linear form $\lambda : R \to k$ for which $\{\alpha_i\}$ and $\{\beta_i\}$ are dual, we will have that Φ_λ agrees with the global or local $\mathbb{A}^1$-degree (depending on our choice of R) by Corollaries 4.5.4 and 4.5.5. Let λ be such a form. The product of α_i and β_j is given by

$$\alpha_i\beta_j = \alpha_i \cdot \sum_s B_{j,s}\alpha_s.$$

Applying λ to each side, we get an indicator function

$$\delta_{ij} = \lambda(\alpha_i\beta_j) = \lambda\left(\alpha_i \sum_s B_{j,s}\alpha_s \right) = \sum_s B_{j,s}\lambda\left(\alpha_i\alpha_s\right).$$

Varying over all i, j, s, this equation above tells us that the identity matrix is equal to the product of the matrix $(B_{j,s})$ and the matrix $(\lambda(\alpha_i\alpha_s)) = (\lambda(\alpha_s\alpha_i))$. Explicitly, we have that

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} B_{1,1} & B_{1,2} & \cdots & B_{1,m} \\ B_{2,1} & B_{2,2} & \cdots & B_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m,1} & B_{m,2} & \cdots & B_{m,m} \end{pmatrix} \begin{pmatrix} \lambda(\alpha_1^2) & \lambda(\alpha_1\alpha_2) & \cdots & \lambda(\alpha_1\alpha_m) \\ \lambda(\alpha_2\alpha_1) & \lambda(\alpha_2^2) & \cdots & \lambda(\alpha_2\alpha_m) \\ \vdots & \vdots & \ddots & \vdots \\ \lambda(\alpha_m\alpha_1) & \lambda(\alpha_m\alpha_2) & \cdots & \lambda(\alpha_m^2) \end{pmatrix}.$$

Thus the Gram matrix for Φ_λ in the basis $\{\alpha_i\}$ is $(B_{i,j})^{-1}$. We conclude by proving that $(B_{i,j})$ and $(B_{i,j})^{-1}$ represent the same element of $\text{GW}(k)$. Since any symmetric bilinear form can be diagonalized, there is an invertible $m \times m$ matrix S such that $S^T \cdot (B_{i,j}) \cdot S$ is diagonal. Since $(S^T \cdot (B_{i,j}) \cdot S) \cdot (S^{-1} \cdot (B_{i,j})^{-1} \cdot (S^{-1})^T)$ is equal to the identity matrix, it follows that $S^{-1} \cdot (\lambda(\alpha_i \alpha_j)) \cdot (S^{-1})^T$ is diagonal with entries inverse to the diagonal entries of $S^T \cdot (B_{i,j}) \cdot S$. Applying the equality $\langle a \rangle = \langle 1/a \rangle$ along the diagonals, it follows that $(B_{i,j})^{-1}$ and $(B_{i,j})$ define the same element in $\mathrm{GW}(k)$. Theorem 4.1.2 now follows from Corollaries 4.5.4 and 4.5.5. $\square$

The following tables describe algorithms for computing the global and local $\mathbb{A}^1$-degrees in terms of the Bézoutian bilinear form. A Sage implementation of these algorithms is available at (BMP21a).

Computing the global $\mathbb{A}^1$-degree via the Bézoutian:

1. Compute the Δ_{ij} and the image of their determinant $\mathrm{Béz}(f) = \det(\Delta_{ij})$ in $k[X,Y]/(f(X), f(Y))$.

2. Pick a k-vector space basis $a_1, \ldots, a_m$ of $Q = k[X_1, \ldots, X_n]/(f_1, \ldots, f_n)$. Find $B_{i,j} \in k$ such that
$$\mathrm{Béz}(f) = \sum_{i=1}^{m} B_{i,j} a_i(X) a_j(Y).$$

3. The matrix $B = (B_{i,j})$ represents $\deg^{\mathbb{A}^1}(f)$. Diagonalize B to write its class in $\mathrm{GW}(k)$.

Computing the local $\mathbb{A}^1$-degree via the Bézoutian:

1. Compute the Δ_{ij} and the image of their determinant $\text{Béz}(f) = \det(\Delta_{ij})$ in $k[X,Y]/(f(X), f(Y))$.

2. Pick a k-vector space basis $a_1, \ldots, a_m$ of $Q_\mathfrak{m} = k[X_1, \ldots, X_n]_\mathfrak{m}/(f_1, \ldots, f_n)$. Find $B_{i,j} \in k$ such that
$$\text{Béz}(f) = \sum_{i=1}^{m} B_{i,j} a_i(X) a_j(Y).$$

3. The matrix $B = (B_{i,j})$ represents $\deg_\mathfrak{m}^{\mathbb{A}^1}(f)$. Diagonalize B to write its class in $\text{GW}(k)$.

4.6 Calculation rules

Using the Bézoutian characterization of the $\mathbb{A}^1$-degree, we are able to establish various calculation rules for local and global $\mathbb{A}^1$-degrees. See (KST21; QSW22) for related results in the local case.

Our ultimate goal in this section is the product rule for the $\mathbb{A}^1$-degree (see Proposition 4.6.5), which was already known by the work of Morel. See the paragraph preceding Proposition 4.6.5 for a more detailed discussion.

Proposition 4.6.1. Suppose that $f = (f_1, \ldots, f_n)$ and $g = (g_1, \ldots, g_n)$ are endomorphisms of affine space that generate the same ideal

$$I = (f_1, \ldots, f_n) = (g_1, \ldots, g_n) \triangleleft k[x_1, \ldots, x_n].$$

If $\text{Béz}(f) = \text{Béz}(g)$ in $k[X, Y]$, then $\deg^{\mathbb{A}^1}(f) = \deg^{\mathbb{A}^1}(g)$, and $\deg_p^{\mathbb{A}^1}(f) = \deg_p^{\mathbb{A}^1}(g)$ for all p.

Proof. We may choose the same basis for $Q = k[x_1, \ldots, x_n]/I$ (or Q_p in the local case) in our computation for the degrees of f and g. The Bézoutians $\text{Béz}(f) = \text{Béz}(g)$ will have the same coefficients in this basis, so their Gram matrices will coincide. $\square$

The following result is the global analogue of (QSW22, Lemma 14).

Lemma 4.6.2. Let $f = (f_1, \ldots, f_n) : \mathbb{A}^n_k \to \mathbb{A}^n_k$ be an endomorphism of $\mathbb{A}^n_k$ with only isolated zeros. Let $A \in k^{n \times n}$ be an invertible matrix. Then

$$\langle \det A \rangle \cdot \deg^{\mathbb{A}^1}(f) = \deg^{\mathbb{A}^1}(A \circ f)$$

as elements of $\text{GW}(k)$.

Proof. Write $A = (a_{ij})$ and

$$\Delta^g_{ij} = \frac{g_i(X_1, \ldots, X_j, Y_{j+1}, \ldots, Y_n) - g_i(X_1, \ldots, X_{j-1}, Y_j \ldots, Y_n)}{X_j - Y_j},$$

where g is either f or $A \circ f$. Then $\Delta^{A\circ f}_{ij} = \sum_{l=1}^n a_{il} \Delta^f_{lj}$, and thus $(\Delta^{A\circ f}_{ij}) = A \cdot (\Delta^f_{ij})$ as matrices over $k[X, Y]$. The ideals generated by $A \circ (f_1, \ldots, f_n)$ and $(f_1, \ldots, f_n)$

are equal, and the images in $Q \otimes_k Q$ of $\det(\Delta_{ij}^{A\circ f})$ and $\det A \cdot \det(\Delta f_{ij})$ are equal. Thus the Gram matrix of the Bézoutian bilinear form for $A \circ f$ is $\det A$ times the Gram matrix of the Bézoutian bilinear form for f. Proposition 4.6.1 then proves the claim. $\square$

Example 4.6.3. We may apply Lemma 4.6.2 in the case where A is a permutation matrix associated to some permutation $\sigma \in \Sigma_n$. Letting $f_\sigma := \big(f_{\sigma(1)}, \ldots, f_{\sigma(n)}\big)$, we observe that

$$\deg_p^{\mathbb{A}^1}(f_\sigma) = \langle \operatorname{sgn}(\sigma) \rangle \cdot \deg_p^{\mathbb{A}^1}(f)$$

at any isolated zero p of f, and an analogous statement is true for global degrees as well.

Next, we prove a lemma inspired by (KST21, Lemma 12).

Lemma 4.6.4. Let $f, g : \mathbb{A}_k^n \to \mathbb{A}_k^n$ be two endomorphisms of $\mathbb{A}_k^n$. Assume that f and g are quasi-finite. Let $L \in M_n(k)$ be an invertible $n \times n$ matrix, which defines a morphism $L : \mathbb{A}_k^n \to \mathbb{A}_k^n$ given by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n) \cdot L^T$. Let I_n denote the $n \times n$ identity matrix, and assume that $\det(I_n + t(L - I_n)) \in k[t]$ is in fact an element of $k^\times$. Then $\deg^{\mathbb{A}^1}(f \circ g) = \deg^{\mathbb{A}^1}(f \circ L \circ g)$.

Proof. Quasi-finite morphisms have isolated zero loci by (Sta21, Definition 01TD (3)). The composition of quasi-finite morphisms is again quasi-finite (Sta21, Lemma

01TL), so $f \circ g$ has isolated zero locus.

Next, we show that L is also quasi-finite. We will actually prove a stronger statement. Let $A_t \in M_n(k[t])$ be an invertible $n \times n$ matrix, which implies that $\det A_t \in k[t]^\times = k^\times$. This matrix determines a family of morphisms $A_t : \mathbb{A}^n_k \times \mathbb{A}^1_k \to \mathbb{A}^n_k$ by $(x_1, \ldots, x_n, t) \mapsto (x_1, \ldots, x_n) \cdot A_t^T$. Given $t_0 \in \mathbb{A}^1_k$, the morphism A_{t_0} has Jacobian determinant $\det(\frac{\partial (A_{t_0})_i}{\partial x_j}) = \det A_t$, which is a unit. In particular, A_{t_0} is unramified for each $t_0 \in \mathbb{A}^1_k$. Thus A_{t_0} is locally quasi-finite (Sta21, Lemma 02VF). Since $\mathbb{A}^n_k$ is Noetherian, $A_{t_0} : \mathbb{A}^n_k \to \mathbb{A}^n_k$ is quasi-compact. Quasi-compact and locally quasi-finite morphisms are quasi-finite (Sta21, Lemma 01TJ), so we conclude that A_{t_0} is quasi-finite for each $t_0 \in \mathbb{A}^1_k$.

Just as in (KST21, Lemma 12), we now define $L_t = I_n + t \cdot (L - I_n)$. Our assumption on $\det(I_n + t(L - I_n))$ implies that L_t is invertible. Thus L_t is quasi-finite, so $f \circ L_t \circ g$ is quasi-finite and hence only has isolated zeros for all t. Set

$$\widetilde{Q} = \frac{k[t][x_1, \ldots, x_n]}{(f \circ L_t \circ g)}.$$

Then (SS75, p. 182) gives us a Scheja–Storch form $\widetilde{\eta} : \widetilde{Q} \to k[t]$ such that the bilinear form $\Phi_{\widetilde{\eta}} : \widetilde{Q} \times \widetilde{Q} \to k[t]$ is symmetric and non-degenerate. By Harder's theorem (KW19, Lemma 30), the stable isomorphism class of $\Phi_{\widetilde{\eta}} \otimes_k k(t_0) \in \mathrm{GW}(k)$ is independent of $t_0 \in \mathbb{A}^1_k(k)$. In particular, the Scheja–Storch bilinear forms of $f \circ L_0 \circ g = f \circ g$ and $f \circ L_1 \circ g = f \circ L \circ g$ are isomorphic. $\square$

The following product rule is a consequence of Morel's proof that the $\mathbb{A}^1$-degree is a ring isomorphism (Mor04, Lemma 6.3.8). We give a more hands-on proof of this product rule. See (KST21, Theorem 13) and (QSW22, Theorem 26) for an analogous proof of the product rule for local degrees at rational points.

Proposition 4.6.5 (Product rule)**.** Let $f, g : \mathbb{A}^n_k \to \mathbb{A}^n_k$ be two quasi-finite endomorphisms of $\mathbb{A}^n_k$. Then $\deg^{\mathbb{A}^1}(f \circ g) = \deg^{\mathbb{A}^1}(f) \cdot \deg^{\mathbb{A}^1}(g)$.

Proof. We follow the proofs of (KST21, Theorem 13) and (QSW22, Theorem 26). The general idea is to mimic the Eckmann–Hilton argument (EH62). Let $x := (x_1, \ldots, x_n)$ and $y := (y_1, \ldots, y_n)$. Define $\widetilde{f}, \widetilde{g} : \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{A}^n \times \mathbb{A}^n$ by $\widetilde{f}(x, y) = (f(x), y)$ and $\widetilde{g}(x, y) = (g(x), y)$, and note that $\widetilde{f}$ and $\widetilde{g}$ are both quasi-finite because f and g are quasi-finite. Since $(f \circ g, y)$ and $\widetilde{f} \circ \widetilde{g}$ define the same ideal in $k[x, y]$ and have the same Bézoutian, we have $\deg^{\mathbb{A}^1}(f \circ g) = \deg^{\mathbb{A}^1}(\widetilde{f} \circ \widetilde{g})$ by Proposition 4.6.1.

Let $g \times f : \mathbb{A}^n_k \times \mathbb{A}^n_k \to \mathbb{A}^n_k \times \mathbb{A}^n_k$ be given by $(g \times f)(x, y) = (g(x), f(y))$. Using Lemma 4.6.4 repeatedly, we will show that $\deg^{\mathbb{A}^1}(\widetilde{f} \circ \widetilde{g}) = \deg^{\mathbb{A}^1}(g \times f)$. Let I_n be the $n \times n$ identity matrix, and let

$$L_1 = \begin{pmatrix} I_n & 0 \\ -I_n & I_n \end{pmatrix}, \quad L_2 = \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

By construction, $\det(I_{2n} + t(L_1 - I_{2n})) = \det(I_{2n} + t(L_2 - I_{2n})) = 1$, so Lemma 4.6.4

implies that

$$\begin{aligned}
\deg^{\mathbb{A}^1}(\widetilde{f}\circ\widetilde{g}) &= \deg^{\mathbb{A}^1}(\widetilde{f}\circ L_1\circ\widetilde{g}) \\
&= \deg^{\mathbb{A}^1}(\widetilde{f}\circ L_2\circ(L_1\circ\widetilde{g})) \\
&= \deg^{\mathbb{A}^1}(\widetilde{f}\circ L_1\circ(L_2\circ L_1\circ\widetilde{g})).
\end{aligned}$$

One can check that $A\circ\widetilde{f}\circ L_1\circ L_2\circ L_1\circ\widetilde{g} = g\times f$. By Lemma 4.6.2, we have

$$\begin{aligned}
\langle\det A\rangle\cdot\deg^{\mathbb{A}^1}(\widetilde{f}\circ\widetilde{g}) &= \langle\det A\rangle\cdot\deg^{\mathbb{A}^1}(\widetilde{f}\circ L_1\circ L_2\circ L_1\circ\widetilde{g}) \\
&= \deg^{\mathbb{A}^1}(g\times f).
\end{aligned}$$

Since $\det A = 1$, it just remains to show that $\deg^{\mathbb{A}^1}(g\times f) = \deg^{\mathbb{A}^1}(g)\cdot\deg^{\mathbb{A}^1}(f)$. Let $a_1,\ldots,a_m$ be a basis for $\frac{k[x_1,\ldots,x_n]}{(g_1,\ldots,g_n)}$ and $a_1',\ldots,a_{m'}'$ be a basis for $\frac{k[y_1,\ldots,y_n]}{(f_1,\ldots,f_n)}$. Write $\text{Béz}(g) = \sum_{i,j=1}^{m} B_{ij}a_i\otimes a_j$ and $\text{Béz}(f) = \sum_{i,j=1}^{m'} B_{ij}'a_i'\otimes a_j'$. By Theorem 4.1.2, (B_{ij}) and (B_{ij}') are the Gram matrices for $\deg^{\mathbb{A}^1}(g)$ and $\deg^{\mathbb{A}^1}(f)$, respectively. Next, we have $\text{Béz}(g\times f) = \text{Béz}(g)\cdot\text{Béz}(f)$, since

$$(\Delta_{ij}^{g\times f}) = \begin{pmatrix} (\Delta_{ij}^{g}) & 0 \\ 0 & (\Delta_{ij}^{f}) \end{pmatrix}.$$

Note that $\{a_i(x)a_{i'}'(y)\}_{i,i'=1}^{m,m'}$ is a basis of $\frac{k[x_1,\ldots,x_n,y_1,\ldots,y_n]}{(g_1(x),\ldots,g_n(x),f_1(y),\ldots,f_n(y))}$. In this basis, we have

$$\text{Béz}(g)\cdot\text{Béz}(f) = \sum_{i,j=1}^{m}\sum_{i',j'=1}^{m'} B_{ij}B_{i'j'}'a_ia_{i'}'\otimes a_ja_{j'}',$$

so the Gram matrix of $\deg^{\mathbb{A}^1}(g \times f)$ is the tensor product $(B_{ij}) \otimes (B'_{ij})$. We thus we have an equality $\deg^{\mathbb{A}^1}(g \times f) = \deg^{\mathbb{A}^1}(g) \cdot \deg^{\mathbb{A}^1}(f)$ in $\mathrm{GW}(k)$. $\square$

4.7 Examples

We now give a few remarks and examples about computing the Bézoutian.

Remark 4.7.1. It is not always the case that the determinant $\det(\Delta_{ij}) \in k[X, Y]$ is symmetric. For example, consider the morphism $f : \mathbb{A}^2_k \to \mathbb{A}^2_k$ sending $(x_1, x_2) \mapsto (x_1 x_2, x_1 + x_2)$. Then the Bézoutian is given by

$$\text{Béz}(f) = \det \begin{pmatrix} X_2 & Y_1 \\ 1 & 1 \end{pmatrix} = X_2 - Y_1.$$

However, the Bézoutian is symmetric once we pass to the quotient $\frac{k[X,Y]}{(f(X), f(Y))}$ (BCRS96, 2.12). Continuing the present example, let $\{1, x_2\}$ be a basis for the algebra $Q = k[x_1, x_2]/(x_1 x_2, x_1 + x_2)$. Then we have that

$$\text{Béz}(f) = X_2 - Y_1 = X_2 + Y_2,$$

which is symmetric. Moreover, the Bézoutian bilinear form is represented by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, so $\deg^{\mathbb{A}^1}(f) = \mathbb{H}$.

Example 4.7.2. Let $k = \mathbb{F}_p(t)$, where p is an odd prime, and consider the endomorphism of the affine plane given by

$$f : \mathrm{Spec}\,\mathbb{F}_p(t)[x_1, x_2] \to \mathrm{Spec}\,\mathbb{F}_p(t)[x_1, x_2]$$

$$(x_1, x_2) \mapsto (x_1^p - t, x_1 x_2) .$$

As the residue field of the zero of f is not separable over k, existing strategies for computing the local $\mathbb{A}^1$-degree are insufficient. Our results allow us to compute this $\mathbb{A}^1$-degree. The Bézoutian is given by

$$\begin{aligned} \mathrm{Béz}(f) &= \det \begin{pmatrix} \frac{X_1^p - Y_1^p}{X_1 - Y_1} & 0 \\ \\ X_2 & Y_1 \end{pmatrix} \\ &= X_1^{p-1}Y_1 + X_1^{p-2}Y_1^2 + \ldots + X_1Y_1^{p-1} + Y_1^p \\ &= X_1^{p-1}Y_1 + X_1^{p-2}Y_1^2 + \ldots + X_1Y_1^{p-1} + t. \end{aligned}$$

In the basis $\{1, x_1, \ldots, x_1^{p-1}\}$ of Q, the Bézoutian bilinear form consists of a t in the upper left corner and a 1 in each entry just below the anti-diagonal. Thus

$$\deg^{\mathbb{A}^1}(f) = \deg^{\mathbb{A}^1}_{(t^{1/p},0)}(f) = \langle t \rangle + \frac{p-1}{2}\mathbb{H}.$$

Example 4.7.3. Let $f_1 = (x_1 - 1)x_1x_2$ and $f_2 = (ax_1^2 - bx_2^2)$ for some $a, b \in k^\times$ with $\frac{a}{b}$ not a square in k. Then $f = (f_1, f_2)$ has isolated zeros at $\mathfrak{m} := (x_1 - 0, x_2 - 0)$ and $\mathfrak{n} := (x_1 - 1, x_2^2 - \frac{a}{b})$. We will use Bézoutians to compute the local degrees $\deg^{\mathbb{A}^1}_{\mathfrak{m}}(f)$

and $\deg_{\mathfrak{n}}^{\mathbb{A}^1}(f)$, as well as the global degree $\deg^{\mathbb{A}^1}(f)$. Let

$$Q = \frac{k[x_1, x_2]}{((x_1 - 1)x_1x_2, ax_1^2 - bx_2^2)}.$$

We first compute the global Bézoutian as

$$\begin{aligned}\text{Béz}(f) =& \det \begin{pmatrix} (X_1 + Y_1 - 1)X_2 & a(X_1 + Y_1) \\ \\ Y_1^2 - Y_1 & -b(X_2 + Y_2) \end{pmatrix} \\ =& - a(X_1Y_1^2 - X_1Y_1 + Y_1^3 - Y_1^2) \\ &- b(X_1X_2^2 + X_2^2Y_1 - X_2^2 + X_1X_2Y_2 + X_2Y_1Y_2 - X_2Y_2).\end{aligned}$$

In the basis $\{1, x_1, x_2, x_1^2, x_1x_2, x_1^3\}$ of Q, the Bézoutian is given by

$$\begin{aligned}\text{Béz}(f) =& - a\left(X_1Y_1^2 - X_1Y_1 + Y_1^3 - Y_1^2 + X_1^3 + X_1^2Y_1 - X_1^2\right) \\ &- b(X_1X_2Y_2 + X_2Y_1Y_2 - X_2Y_2).\end{aligned}$$

We now write the Bézoutian matrix given by the coefficients of Béz(f).

	1	X_1	X_2	X_1^2	X_1X_2	X_1^3
1	0	0	0	a	0	$-a$
Y_1	0	a	0	$-a$	0	0
Y_2	0	0	b	0	$-b$	0
Y_1^2	a	$-a$	0	0	0	0
Y_1Y_2	0	0	$-b$	0	0	0
Y_1^3	$-a$	0	0	0	0	0

One may check (e.g. with a computer) that this is equal to $3\mathbb{H}$ in $\mathrm{GW}(k)$.

In $Q_{\mathfrak{m}}$, we have that $x_1^2 x_2 = x_1 x_2 = 0$ and $x_1^3 = \frac{b}{a} x_1 x_2^2 = 0$. In the basis $\{1, x_1, x_2, x_1^2\}$ of $Q_{\mathfrak{m}}$, the global Bézoutian reduces to

$$\mathrm{Béz}(f) = -a \left(X_1 Y_1^2 - X_1 Y_1 - Y_1^2 + X_1^2 Y_1 - X_1^2 \right) + b X_2 Y_2$$

We thus get the Bézoutian matrix at $\mathfrak{m}$.

$$\begin{array}{c|cccc} & 1 & X_1 & X_2 & X_1^2 \\ \hline 1 & 0 & 0 & 0 & a \\ Y_1 & 0 & a & 0 & -a \\ Y_2 & 0 & 0 & b & 0 \\ Y_1^2 & a & -a & 0 & 0 \end{array}$$

This is $\mathbb{H} + \langle a, b \rangle$ in $\mathrm{GW}(k)$.

In $Q_{\mathfrak{n}}$, we have $x_1 = 1$. In the basis $\{1, x_2\}$ for $Q_{\mathfrak{n}}$, the Bézoutian reduces to

$$\mathrm{Béz}(f) = -a - b X_2 Y_2.$$

We can then write the Bézoutian matrix at $\mathfrak{n}$.

$$\begin{array}{c|cc} & 1 & X_2 \\ \hline 1 & -a & 0 \\ Y_2 & 0 & -b \end{array}$$

This is $\langle -a, -b\rangle$ in $\mathrm{GW}(k)$. Note that $\langle -a, -b\rangle$ need not be equal to $\mathbb{H}$. However, this does not contradict (QSW22, Theorem 2), since $\mathfrak{n}$ is a non-rational point.

Putting these computations together, we see that

$$\deg_{\mathfrak{m}}^{\mathbb{A}^1}(f) + \deg_{\mathfrak{n}}^{\mathbb{A}^1}(f) = \mathbb{H} + \langle a, b\rangle + \langle -a, -b\rangle = 3\mathbb{H} = \deg^{\mathbb{A}^1}(f).$$

4.8 Application: the $\mathbb{A}^1$-Euler characteristic of Grassmannians

As an application of Theorem 4.1.2, we compute the $\mathbb{A}^1$-Euler characteristic of various low-dimensional Grassmannians in Example 4.8.2 and Figure 4.1. These computations suggest a recursive formula for the $\mathbb{A}^1$-Euler characteristic of an arbitrary Grassmannian, which we prove in Theorem 4.8.4. This formula is analogous to the recursive formulas for the Euler characteristics of complex and real Grassmannians. Theorem 4.8.4 is probably well-known, and the proof is essentially a combination of results of Hoyois, Levine, and Bachmann–Wickelgren.

4.8.1 The $\mathbb{A}^1$-Euler characteristic

Let X be a smooth, proper k-variety of dimension n with structure map $\pi : X \to \operatorname{Spec} k$. Let $p : T_X \to X$ denote the tangent bundle of X. The $\mathbb{A}^1$-Euler characteristic

$\chi^{\mathbb{A}^1}(X) \in \mathrm{GW}(k)$ is a refinement of the classical Euler characteristic. In particular, if $k = \mathbb{R}$, then $\operatorname{rank} \chi^{\mathbb{A}^1}(X) = \chi(X(\mathbb{C}))$ and $\operatorname{sgn} \chi^{\mathbb{A}^1}(X) = \chi(X(\mathbb{R}))$. There exist several equivalent definitions of the $\mathbb{A}^1$-Euler characteristic (Lev20; LR20; AMBO$^+$22). For example, we may define $\chi^{\mathbb{A}^1}(X)$ to be the π-pushforward of the $\mathbb{A}^1$-Euler class

$$e(T_X) := z^* z_* 1_X \in \widetilde{\mathrm{CH}}^n(X, \omega_{X/k}),$$

of the tangent bundle (Lev20), where $z : X \to T_X$ is the zero section and $\widetilde{\mathrm{CH}}^d(X, \omega_{X/k})$ is the Chow–Witt group defined by Barge–Morel (BM00; Fas08). That is,

$$\chi^{\mathbb{A}^1}(X) := \pi_*(e(T_X)) \in \widetilde{\mathrm{CH}}^0(\operatorname{Spec} k) = \mathrm{GW}(k).$$

Analogous to the classical case (Mil65), the $\mathbb{A}^1$-Euler characteristic can be computed as the sum of local $\mathbb{A}^1$-degrees at the zeros of a general section of the tangent bundle using the work of Kass–Wickelgren (BW21; KW21; Lev20). We now describe this process. Let σ be a section of T_X which only has isolated zeros. For a zero x of σ, choose Nisnevich coordinates[2] $\psi : U \to \mathbb{A}^n_k$ around x. Since ψ is étale, it induces an isomorphism of tangent spaces and thus yields local coordinates around x. Shrinking U if necessary, we can trivialize $T_X|_U \cong U \times \mathbb{A}^n_k$. The chosen Nisnevich coordinates (ψ, U) and trivialization $\tau : T_X|_U \cong U \times \mathbb{A}^n_k$ each define distinguished elements $d_\psi, d_\tau \in \det T_X|_U$. In turn, this yields a distinguished section d of

[2] *Nisnevich coordinates* consist of an open neighborhood U of x and an étale map $\psi : U \to \mathbb{A}^n_k$ that induces an isomorphism of residue fields $k(x) \cong k(\psi(x))$ (KW21, Definition 18).

$\mathcal{H}om(\det T_X|_U, \det T_X|_U)$, which is defined by $d_\psi \mapsto d_\tau$. We say that a trivialization τ is *compatible* with the chosen coordinates (ψ, U) if the image of the distinguished section d under the canonical isomorphism $\rho : \mathcal{H}om(\det T_X|_U, \det T_X|_U) \cong \mathcal{O}_U$ is a square (KW21, Definition 21).

Given a compatible trivialization $\tau : T_X|_U \cong U \times \mathbb{A}^n_k$, the section σ trivializes to $\sigma : U \to \mathbb{A}^n_k$. We can then define the *local index* $\mathrm{ind}_x\sigma$ at x to be the $\mathbb{A}^1$-degree of the composite

$$\frac{\mathbb{P}^n_k}{\mathbb{P}^{n-1}_k} \to \frac{\mathbb{P}^n_k}{\mathbb{P}^n_k \backslash \{\psi(x)\}} \cong \frac{\mathbb{A}^n_k}{\mathbb{A}^n_k \backslash \{\psi(x)\}} \cong \frac{U}{U \backslash \{x\}} \xrightarrow{\sigma} \frac{\mathbb{A}^n_k}{\mathbb{A}^n_k \backslash \{0\}} \cong \frac{\mathbb{P}^n_k}{\mathbb{P}^{n-1}_k}.$$

Here, the first map is the collapse map, the second map is excision, the third map is induced by the Nisnevich coordinates (ψ, U), and the fifth map is purity (see e.g. (BW21, Definition 7.1)). By (KW21, Theorem 3), the $\mathbb{A}^1$-Euler characteristic is then the sum of local indices

$$\chi^{\mathbb{A}^1}(X) = \sum_{x \in \sigma^{-1}(0)} \mathrm{ind}_x\sigma \in \mathrm{GW}(k).$$

By Theorem 4.1.2, we may thus compute the $\mathbb{A}^1$-Euler characteristic by computing the global Bézoutian bilinear form of an appropriate map $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$.

Remark 4.8.1. If all the zeros of $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ are simple, then each local ring $Q_\mathfrak{m}$ in the decomposition of $Q = \frac{k[x_1,\ldots,x_n]}{(f_1,\ldots,f_n)} = Q_{\mathfrak{m}_1} \times \ldots \times Q_{\mathfrak{m}_s}$ is equal to the residue field of the corresponding zero. If each residue field $Q_{\mathfrak{m}_i}$ is a separable extension of k, then the $\mathbb{A}^1$-degree of f is equal to sum of the scaled trace forms $\mathrm{Tr}_{Q_{\mathfrak{m}_i}/k}(\langle J(f)|_{\mathfrak{m}_i}\rangle)$ (see e.g. (BBM$^+$21, Definition 1.2)), where $J(f)|_{\mathfrak{m}_i}$ is the determinant of the Jacobian of f evaluated at the point $\mathfrak{m}_i$. In (Pau20) the last named author uses the scaled trace form for several $\mathbb{A}^1$-Euler number computations. However, Theorem 4.1.2 yields a formula for $\deg^{\mathbb{A}^1}(f)$ for any f with only isolated zeros and without any restriction on the residue field of each zero. Moreover, we can even compute $\deg^{\mathbb{A}^1}(f)$ without solving for the zero locus of f.

4.8.2 The $\mathbb{A}^1$-Euler characteristic of Grassmannians

Let $G := \mathrm{Gr}_k(r,n)$ be the Grassmannian of r-planes in k^n. In order to compute $\chi^{\mathbb{A}^1}(G)$, we first need to describe Nisnevich coordinates and compatible trivializations for G and T_G. We then need to choose a convenient section of T_G and describe the resulting endomorphism $\mathbb{A}_k^{r(n-r)}$. The tangent bundle $T_G \to G$ is isomorphic to $p : \mathcal{H}om(\mathcal{S},\mathcal{Q}) \to G$, where $\mathcal{S} \to G$ and $\mathcal{Q} \to G$ are the universal sub and quotient bundles.

We now describe Nisnevich coordinates on G and a compatible trivialization of T_G, following (SW21). Let $d = r(n-r)$ be the dimension of G, and let $\{e_1,\ldots,e_n\}$ be the standard basis of k^n. Let $\mathbb{A}_k^d = \mathrm{Spec}\, k[\{x_{i,j}\}_{i,j=1}^{r,n-r}] \cong U \subset G$ be the open affine subset consisting of the r-planes

$$H(\{x_{i,j}\}_{i,j=1}^{r,n-r}) := \operatorname{span}\left\{e_{n-r+i} + \sum_{j=1}^{n-r} x_{i,j}e_j\right\}_{i=1}^{r}.$$

The map $\psi : U \to \mathbb{A}_k^d$ given by $\psi\left(H(\{x_{i,j}\}_{i,j=1}^{r,n-r})\right) = (\{x_{i,j}\}_{i,j=1}^{n-r,r})$ yields Nisnevich coordinates (ψ, U) centered at $\psi(\operatorname{span}\{e_{n-r+1}, \ldots, e_n\}) = (0, \ldots, 0)$. For the trivialization of $T_G|_U$, let

$$\widetilde{e}_i = \begin{cases} e_i & i \leq n-r, \\ e_i + \sum_{j=1}^{n-r} x_{i-(n-r),j}e_j & i \geq n-r+1. \end{cases}$$

Then $\{\widetilde{e}_1, \ldots, \widetilde{e}_n\}$ is a basis for k^n, and we denote the dual basis by $\{\widetilde{\phi}_1, \ldots, \widetilde{\phi}_n\}$. Over U, the bundles $\mathcal{S}^*$ and $\mathcal{Q}$ are trivialized by $\{\widetilde{\phi}_{n-r+1}, \ldots, \widetilde{\phi}_n\}$ and $\{\widetilde{e}_1, \ldots, \widetilde{e}_{n-r}\}$, respectively. Since

$$T_G \cong \mathcal{H}om(\mathcal{S}, \mathcal{Q}) \cong \mathcal{S}^* \otimes \mathcal{Q},$$

we get a trivialization of $T_G|_U$ given by $\{\widetilde{\phi}_{n-r+i} \otimes \widetilde{e}_j\}_{i,j=1}^{r,n-r}$. By construction, our Nisnevich coordinates (ψ, U) induce this local trivialization of T_G. It follows that the distinguished element of $\operatorname{Hom}(\det T_G|_U, \det T_G|_U)$ sending the distinguished element of $\det T_G|_U$ (determined by the Nisnevich coordinates) to the distinguished element of $T_G|_U$ (determined by our local trivialization) is just the identity, which is a square.

Next, we describe sections of $T_G \to G$ and the resulting endomorphisms $\mathbb{A}_k^d \to \mathbb{A}_k^d$. Let $\{\phi_1, \ldots, \phi_n\}$ be the dual basis of the standard basis $\{e_1, \ldots, e_n\}$ of k^n. A homogeneous degree 1 polynomial $\alpha \in k[\phi_1, \dots, \phi_n]$ gives rise to a section s of $\mathcal{S}^*$, defined by evaluating α. In particular, given a vector $t = \sum_{i=1}^n t_i \widetilde{e}_i$ in $H(\{x_{i,j}\}_{i,j=1}^{r,n-r})$, we use the dual change of basis

$$\phi_j = \begin{cases} \widetilde{\phi}_j + \sum_{i=1}^r x_{i,j} \widetilde{\phi}_{n-r+i} & j \leq n-r, \\ \widetilde{\phi}_j & j \geq n-r+1 \end{cases}$$

to set

$$s(t) = \alpha\left(t_1 + \sum_{i=1}^r x_{i,1} t_{n-r+i}, \dots, t_{n-r} + \sum_{i=1}^r x_{i,n-r} t_{n-r+i}, t_{n-r+1}, \dots, t_n\right).$$

Note that $t_1 = \cdots = t_{n-r} = 0$ if and only if $t \in H(\{x_{i,j}\}_{i,j=1}^{r,n-r})$, so $s(t) \in k[t_{n-r+1}, \dots, t_n]$. Taking n sections $s_1, \dots, s_n$ of $\mathcal{S}^*$, we get a section of $T_G \cong \mathcal{H}om(\mathcal{S}, \mathcal{Q})$ given by

$$\mathcal{S} \xrightarrow{(s_1,\dots,s_n)} \mathbb{A}_k^n \to \mathcal{Q},$$

where the second map is quotienting by $\{\widetilde{e}_{n-r+1}, \dots, \widetilde{e}_n\}$. We obtain our map $\mathbb{A}_k^d \to \mathbb{A}_k^d$ by applying the trivializations $\{\widetilde{\phi}_{n-r+i} \otimes \widetilde{e}_j\}_{i,j=1}^{r,n-r}$ of T_G. Explicitly, take n sections $s_1, \dots, s_n$ of $\mathcal{S}^*$. Since $e_i = \widetilde{e}_i - \sum_{j=1}^{n-r} x_{i-(n-r),j} e_j$ for $i > n-r$, we have

$$s_j e_j \equiv s_j e_j - \sum_{i=1}^r x_{i,j} s_{n-r+i} e_j \texttt{Mod}(\widetilde{e}_{n-r+1}, \dots, \widetilde{e}_n),$$

for all $j \leq n-r$. Recall that $e_j = \widetilde{e}_j$ for $j \leq n-r$. The coordinate of $\mathbb{A}_k^d \to \mathbb{A}_k^d$ corresponding to $\widetilde{\phi}_{n-r+i} \otimes \widetilde{e}_j$ is thus the coefficient of t_{n-r+i} in $s_j(t) - \sum_{\ell=1}^r x_{\ell,j} s_{n-r+\ell}(t)$.

For a general section σ of $p : T_G \to G$, the finitely many zeros of σ will all lie in U. In this case, the $\mathbb{A}^1$-Euler characteristic of G is equal to the global $\mathbb{A}^1$-degree of the resulting map $\mathbb{A}^d_k \to \mathbb{A}^d_k$, which can computed using the Bézoutian.

Example 4.8.2 $(\mathrm{Gr}_k(2,4))$**.** Let

$$\begin{aligned}
\alpha_1 &= \phi_2 = \widetilde{\phi}_2 + x_{1,2}\widetilde{\phi}_3 + x_{2,2}\widetilde{\phi}_4, \\
\alpha_2 &= \phi_3 = \widetilde{\phi}_3, \\
\alpha_3 &= \phi_4 = \widetilde{\phi}_4, \\
\alpha_4 &= \phi_1 = \widetilde{\phi}_1 + x_{1,1}\widetilde{\phi}_3 + x_{2,1}\widetilde{\phi}_4.
\end{aligned}$$

Evaluating at $t = (0, 0, t_3, t_4)$ in the basis $\{\widetilde{e}_i\}$, we have

$$\begin{aligned}
s_1 &= x_{1,2}t_3 + x_{2,2}t_4, \\
s_2 &= t_3, \\
s_3 &= t_4, \\
s_4 &= x_{1,1}t_3 + x_{2,1}t_4.
\end{aligned}$$

It remains to read off the coefficients of t_3 and t_4 of

$$\begin{aligned}
s_1 - x_{1,1}s_3 - x_{2,1}s_4 &= (x_{1,2} - x_{1,1}x_{2,1})t_3 + (x_{2,2} - x_{1,1} - x_{2,1}^2)t_4, \\
s_2 - x_{1,2}s_3 - x_{2,2}s_4 &= (1 - x_{1,1}x_{2,2})t_3 + (-x_{1,2} - x_{2,1}x_{2,2})t_4.
\end{aligned}$$

$n \backslash r$	1	2	3	4	5
2	$\mathbb{H}$	$\langle 1\rangle$			
3	$\mathbb{H}+\langle 1\rangle$	$\mathbb{H}+\langle 1\rangle$	$\langle 1\rangle$		
4	$2\mathbb{H}$	$2\mathbb{H}+\langle 1,1\rangle$	$2\mathbb{H}$	$\langle 1\rangle$	
5	$2\mathbb{H}+\langle 1\rangle$	$4\mathbb{H}+\langle 1,1\rangle$	$4\mathbb{H}+\langle 1,1\rangle$	$2\mathbb{H}+\langle 1\rangle$	$\langle 1\rangle$
6	$3\mathbb{H}$	$6\mathbb{H}+\langle 1,1,1\rangle$	$10\mathbb{H}$	$6\mathbb{H}+\langle 1,1,1\rangle$	$3\mathbb{H}$
7	$3\mathbb{H}+\langle 1\rangle$	$9\mathbb{H}+\langle 1,1,1\rangle$	$16\mathbb{H}+\langle 1,1,1\rangle$	$16\mathbb{H}+\langle 1,1,1\rangle$	$9\mathbb{H}+\langle 1,1,1\rangle$

Figure 4.1: More examples of $\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(r,n))$

We thus have our endomorphism $\sigma : \mathbb{A}^4_k \to \mathbb{A}^4_k$ defined by

$$\sigma = (x_{1,2} - x_{1,1}x_{2,1}, x_{2,2} - x_{1,1} - x_{2,1}^2, 1 - x_{1,1}x_{2,2}, -x_{1,2} - x_{2,1}x_{2,2}).$$

Using the Sage implementation of the Bézoutian formula for the $\mathbb{A}^1$-degree (BMP21a), we can calculate $\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(2,4)) = \deg^{\mathbb{A}^1}(\sigma) = 2\mathbb{H} + \langle 1,1\rangle$.

Using a computer, we performed computations analogous to Example 4.8.2 for $r \leq 5$ and $n \leq 7$. These $\mathbb{A}^1$-Euler characteristics of Grassmannians are recorded in Figure 4.1.

Recall that the Euler characteristics of real and complex Grassmannians are given by binomial coefficients. In particular, these Euler characteristics satisfy certain recurrence relations related to Pascal's rule. The computations in Figure 4.1 indicate that an analogous recurrence relation is true for the $\mathbb{A}^1$-Euler characteristic of Grassmannians over an arbitrary field. In fact, this recurrence relation is a direct consequence of a result of Levine (Lev20).

Proposition 4.8.3. Let $1 \leq r < n$ be integers. Then

$$\chi^{\mathbb{A}^1}\left(\mathrm{Gr}_k(r,n)\right) = \chi^{\mathbb{A}^1}\left(\mathrm{Gr}_k(r-1,n-1)\right) + \langle -1\rangle^r \chi^{\mathbb{A}^1}\left(\mathrm{Gr}_k(r,n-1)\right).$$

Proof. Fix a line L in k^n. Let Z be the closed subvariety consisting of all r-planes containing L (which is isomorphic to $\mathrm{Gr}_k(r-1,n-1)$), and let U be its open complement (which is isomorphic to an affine rank r bundle over $\mathrm{Gr}_k(r,n-1)$). We then get a decomposition $\mathrm{Gr}_k(r,n) = Z \cup U$. Since $\mathrm{Gr}_k(l,m) \cong \mathrm{Gr}_k(m-l,m)$, we have $\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(l,m)) = \chi^{\mathbb{A}^1}(\mathrm{Gr}_k(m-l,m))$. We can thus apply (Lev20, Proposition 1.4 (3)) to obtain

$$\begin{aligned}
\chi^{\mathbb{A}^1}\left(\mathrm{Gr}_k(r,n)\right) &= \chi^{\mathbb{A}^1}\left(\mathrm{Gr}_k(n-r,n)\right) \\
&= \chi^{\mathbb{A}^1}\left(\mathrm{Gr}_k(n-r,n-1)\right) + \langle -1\rangle^r \chi^{\mathbb{A}^1}\left(\mathrm{Gr}_k(n-r-1,n-1)\right) \\
&= \chi^{\mathbb{A}^1}\left(\mathrm{Gr}_k(r-1,n-1)\right) + \langle -1\rangle^r \chi^{\mathbb{A}^1}\left(\mathrm{Gr}_k(r,n-1)\right). \qquad \square
\end{aligned}$$

We can now apply a theorem of Bachmann–Wickelgren (BW21) to completely characterize $\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(r,n))$.

Theorem 4.8.4. Let k be field of characteristic not equal to 2. Let $n_{\mathbb{C}} := \binom{n}{r}$, and let $n_{\mathbb{R}} := \binom{\lfloor \frac{n}{2} \rfloor}{\lfloor \frac{r}{2} \rfloor}$. Then

$$\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(r,n)) = \frac{n_{\mathbb{C}} + n_{\mathbb{R}}}{2} \langle 1 \rangle + \frac{n_{\mathbb{C}} - n_{\mathbb{R}}}{2} \langle -1 \rangle .$$

Proof. By (BW21, Theorem 5.8), we can restrict this computation to two different possibilities. We will prove by induction that $\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(r,n))\texttt{Mod}\mathbb{H}$ has no $\langle 2 \rangle$ summand. The desired result will then follow from (BW21, Theorem 5.8) by noting that $n_{\mathbb{C}}$ and $n_{\mathbb{R}}$ are the Euler characteristics of $\mathrm{Gr}_{\mathbb{C}}(r,n)$ and $\mathrm{Gr}_{\mathbb{R}}(r,n)$, respectively.

Since $\mathbb{A}^n_k$ is $\mathbb{A}^1$-homotopic to $\operatorname{Spec} k$, we have $\chi^{\mathbb{A}^1}(\mathbb{A}^n_k) = \chi^{\mathbb{A}^1}(\operatorname{Spec} k) = \langle 1 \rangle$. Using this observation and the decomposition $\mathbb{P}^n_k = \bigcup_{i=0}^n \mathbb{A}^i_k$ (and a result analogous to (Lev20, Proposition 1.4 (3))), Hoyois computed the $\mathbb{A}^1$-Euler characteristic of projective space (Hoy14, Example 1.7):

$$\chi^{\mathbb{A}^1}(\mathbb{P}^n_k) = \begin{cases} \frac{n}{2}\mathbb{H} + \langle 1 \rangle & n \text{ is even,} \\ \\ \frac{n+1}{2}\mathbb{H} & n \text{ is odd.} \end{cases}$$

Note that $\mathrm{Gr}_k(0,n) \cong \mathrm{Gr}_k(n,n) \cong \operatorname{Spec} k$ and $\mathrm{Gr}_k(1,n) \cong \mathrm{Gr}_k(n-1,n) \cong \mathbb{P}^{n-1}_k$. In particular, $\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(i,n))\texttt{Mod}\mathbb{H}$ is either trivial or $\langle 1 \rangle$ for $i = 0, 1, n-1$, or n. This forms the base case of our induction, with the inductive step given by Proposition 4.8.3 – namely, if $\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(r-1,n-1))\texttt{Mod}\mathbb{H}$ and $\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(r,n-1))\texttt{Mod}\mathbb{H}$ only have $\langle 1 \rangle$ and $\langle -1 \rangle$ summands, then

$$(\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(r-1,n-1)) + \langle -1 \rangle^r \chi^{\mathbb{A}^1}(\mathrm{Gr}_k(r,n-1)))\texttt{Mod}\mathbb{H}$$

only has $\langle 1\rangle$ and $\langle -1\rangle$ summands. $\square$

4.8.3 Modified Pascal's triangle for $\chi^{\mathbb{A}^1}(\mathbf{Gr}_k(r,n))$

Pascal's triangle gives a mnemonic device for binomial coefficients and hence for the Euler characteristics of complex and real Grassmannians. The recurrence relation of Proposition 4.8.3 indicates that a modification of Pascal's triangle can also be used to calculate the $\mathbb{A}^1$-Euler characteristics of Grassmannians. Explicitly, each entry in the modified Pascal's triangle is an element of $\mathrm{GW}(k)$. The two diagonal edges of this triangle correspond to $\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(0,n)) = \langle 1\rangle$ and $\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(n,n)) = \langle 1\rangle$. Elements of each row of the modified Pascal's triangle are obtained from the previous row by the addition rule illustrated in Figure 4.2.

We rewrite the data recorded in Figure 4.1 in a modified Pascal's triangle in Figure 4.3. The rows correspond to the dimension n of the ambient affine space k^n, while the southwest-to-northeast diagonals correspond to the dimension r of the planes k^r in the ambient space.

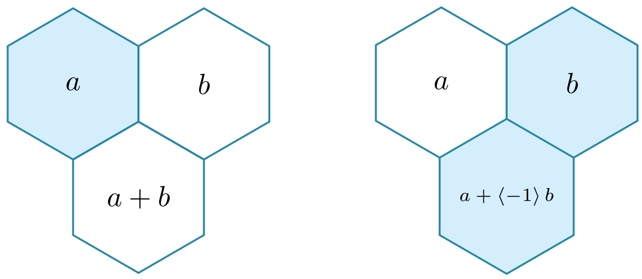

Figure 4.2: Addition rules for modified Pascal's triangle

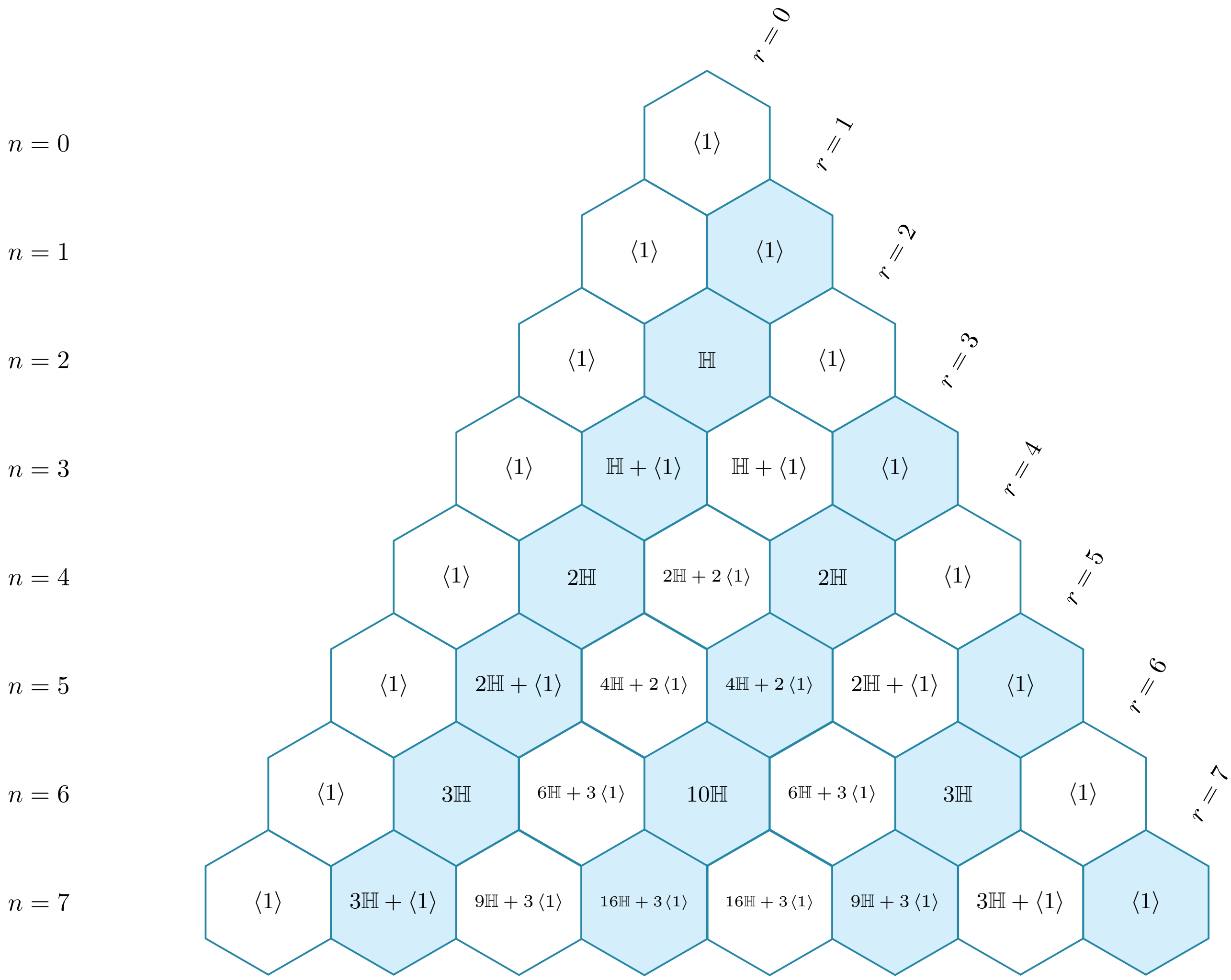

Figure 4.3: Modified Pascal's triangle for $\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(r,n))$ (see Section 4.8.3)

Chapter 5

Lifts, transfers, and degrees of univariate maps

with S. McKean

Abstract: One can compute the local $\mathbb{A}^1$-degree at points with separable residue field by base changing, working rationally, and post-composing with the field trace. We show that for endomorphisms of the affine line, one can compute the local $\mathbb{A}^1$-degree at points with inseparable residue field by taking a suitable lift of the polynomial and transferring its local degree. We also discuss the general set-up and strategy in terms of the six functor formalism. As an application, we show that trace forms of number fields are local $\mathbb{A}^1$-degrees.

5.1 Introduction

Let k be a field. In order to compute the local $\mathbb{A}^1$-Brouwer degree of a map $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ at a closed point p with finite separable residue field $k(p)/k$, one can base change to the field of definition, compute the local degree of $f_{k(p)}$ at the canonical $k(p)$-rational point $\widetilde{p}$ sitting over p, and then apply a field trace (BBM+21). That is, there is an equality

$$\deg_p^{\mathbb{A}^1}(f) = \mathrm{Tr}_{k(p)/k} \deg_{\widetilde{p}}^{\mathbb{A}^1}(f_{k(p)})$$

in the Grothendieck–Witt group $\mathrm{GW}(k)$. For general finite extensions, two issues arise when $k(p)/k$ is inseparable. First, the trace form of an inseparable extension is degenerate, so the field trace does not provide a well-defined transfer $\mathrm{GW}(k(p)) \to \mathrm{GW}(k)$. While alternate transfers are available from motivic homotopy theory, the second issue is simply that base changing f to $k(p)$ and applying a transfer yields a bilinear form whose rank is too large.

We rectify these issues by providing two new ways of lifting f. Assuming that $k(p)/k$ is a finite simple field extension with primitive element t, we consider two transfers arising from $\mathbb{A}^1$-homotopy theory, namely the *geometric transfer*, denoted $\tau_k^{k(p)}(t)$, and the *cohomological transfer*, denoted $\mathrm{Tr}_k^{k(p)}$. Some motivic yoga suggests that the local $\mathbb{A}^1$-degree of f at p is transferred down from the local degree of a suitable lift of f at the $k(p)$-rational point $\widetilde{p}$ (corresponding to the ideal $(x-t)$) above

p. We introduce the *geometric lift* $f_{\mathfrak{g}}$ and the *cohomological lift* $f_{\mathfrak{c}}$ of our polynomial f at the point p. In the separable setting, the cohomological lift agrees with the base change of $f_{k(p)}$, recovering the main result of (BBM$^+$21) in the univariate case.

Theorem 5.1.1. Let $f : \mathbb{A}^1_k \to \mathbb{A}^1_k$ be a morphism with an isolated root at a closed point p. Then

$$\deg_p^{\mathbb{A}^1}(f) = \tau_k^{k(p)}(t) \deg_{\widetilde{p}}^{\mathbb{A}^1}(f_{\mathfrak{g}}) = \mathrm{Tr}_k^{k(p)} \deg_{\widetilde{p}}^{\mathbb{A}^1}(f_{\mathfrak{c}}).$$

The proof of Theorem 5.1.1 will be given in Lemma 5.5.6 and Corollary 5.5.9. In Remark 5.5.7, we discuss how a suitable definition of an *unstable* transfer would imply that Theorem 5.1.1 holds unstably. As a corollary of Theorem 5.1.1, we get an upper bound on the rank of the non-hyperbolic part of the local $\mathbb{A}^1$-degree of a polynomial map $f : \mathbb{A}^1_k \to \mathbb{A}^1_k$.

Corollary 5.1.2. Let $f : \mathbb{A}^1_k \to \mathbb{A}^1_k$ have a root at a closed point p, defined by a monic, irreducible polynomial $m(x)$ of some degree n. Let $t \in k(p)$ be a primitive element for the field extension $k(p)/k$. Then $f(x) = u(x)m(x)^d$ for some polynomial $u \notin m(x) \cdot k[x]$, and

$$\deg_p^{\mathbb{A}^1}(f) = \begin{cases} \frac{nd}{2}\mathbb{H} & d \text{ is even} \\ \frac{n(d-1)}{2}\mathbb{H} + \tau_k^{k(p)}(t)\,\langle u(t)\rangle & d \text{ is odd.} \end{cases}$$

In particular, the rank of the non-hyperbolic part of $\deg_p^{\mathbb{A}^1}(f)$ is bounded above by $[k(p) : k]$.

Another immediate corollary provides a connection between motivic degrees and scaled trace forms or scaled Scharlau forms.

Corollary 5.1.3 (*Scaled trace and Scharlau forms are $\mathbb{A}^1$-degrees*)**.** Let L/k be a finite, separable field extension with primitive element t. Then for any $\langle a \rangle \in \mathrm{GW}(L)$, the scaled trace form $\mathrm{Tr}_k^L \langle a \rangle$ and the scaled Scharlau form $\tau_k^L(t) \langle a \rangle$ are given by the local $\mathbb{A}^1$-degree of an endomorphism of $\mathbb{A}^1_k$.

Combined with the main result of (BMP21b), this provides a method for computing scaled trace forms via Bézoutians.

5.1.1 Outline

We begin with some exposition on purity and the six functor formalism in Section 5.2. In Section 5.3, we recall some basic material on transfers in stable motivic homotopy theory and give evidence suggesting the existence of lifts. We discuss relevant commutative and linear algebraic tools in Section 5.4. Finally, we define geometric and cohomological lifts of univariate polynomials, prove Theorem 5.1.1, and discuss applications to trace forms in Section 5.5.

5.1.2 Acknowledgements

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5.2 Purity and the six functor formalism

In this section, we recall the six functor formalism in stable motivic homotopy theory. We also discuss the (previously established) reformulation of Morel–Voevodsky's purity theorem in terms of the six functors formalism. Throughout this section, we will assume that k is a field finitely generated over a perfect field. This assumption will not be necessary when we arrive at our main results later in the paper.

Assigned to any scheme X, there is a stable symmetric monoidal category $\mathcal{SH}(X)$ of motivic spectra. Given any morphism $f : X \to Y$, there is an adjunction

$$f^* : \mathcal{SH}(Y) \leftrightarrows \mathcal{SH}(X) : f_*,$$

where f^* is symmetric monoidal (in particular, it preserves sphere spectra: $f^*\mathbf{1}_Y = \mathbf{1}_X$). If f is smooth, then f^* admits a left adjoint, denoted $f_\sharp$, which is a "forgetful" functor. Finally, if f is locally of finite type, then there is an exceptional adjunction

$$f_! : \mathcal{SH}(X) \leftrightarrows \mathcal{SH}(Y) : f^!.$$

When f is a sufficiently nice morphism, many of these functors are isomorphic. If f is proper, then there is a natural isomorphism $f_* \simeq f_!$, while if f is étale, we have a natural isomorphism $f_! \simeq f_\sharp$. In particular, if f is proper and étale, then $f_* \simeq f_\sharp$. In the case where f is an open immersion, we have a natural isomorphism $f^* \simeq f^!$. Given a cartesian square, there are various exchange isomorphisms which allow one to interchange various six functors operations. Finally, we have a motivic J-homomorphism $K(X) \to \mathrm{Pic}(\mathcal{SH}(X))$ mapping any ξ to $\Sigma^\xi \mathbf{1}_X$, where Σ^ξ is the *Thom transformation* associated to ξ. If ξ is a vector bundle over X, then Σ^ξ can be seen as smashing with the Thom space $\mathrm{Th}(\xi)$. We refer the reader to (EHK+20, §2) and (BW21, §4.1) for more about the six functor formalism.

We will use the following well-known result.

Proposition 5.2.1. Let $\pi : X \to S$ be a smooth S-scheme. Let $i : Z \hookrightarrow X$ be a closed immersion (not necessarily smooth over S). Then we have a canonical $\mathbb{A}^1$-homotopy equivalence in $\mathcal{SH}(X)$:

$$\Sigma^\infty \frac{X}{X - Z} \simeq i_* \mathbf{1}_Z.$$

Proof. Denote by $j : X - Z \hookrightarrow X$ the open immersion of the complement of Z. The localization theorem (see (MV99, Theorem 2.21, p. 114) and (Hoy21, §1)) then gives an exact sequence

$$j_! j^! \to \mathrm{id} \to i_* i^*.$$

As j is an open immersion, we have that $j_! j^! \simeq j_\sharp j^*$. Applying this exact sequence at the sphere spectrum, we obtain

$$j_\sharp j^* \mathbf{1}_X \to \mathbf{1}_X \to i_* i^* \mathbf{1}_X.$$

We have that $j_\sharp j^* \mathbf{1}_X = j_\sharp \mathbf{1}_{X-Z}$, which is $\Sigma^\infty_+(X - Z)$ in $\mathcal{SH}(X)$. This implies that $i_* \mathbf{1}_Z$ is the cofiber of the natural inclusion $X - Z \hookrightarrow X$. $\square$

Definition 5.2.2. (DJK21, §2.5) Let $f : X \to Y$ be a morphism that is smoothable, local complete intersection (lci), and separated of finite type. Let $\mathcal{L}_f$ be the cotangent complex of f. There is then a natural transformation

$$\mathfrak{p}_f : \Sigma^{\mathcal{L}_f} f^* \to f^!,$$

which is called the *purity transformation.*

If f is smooth, then $\mathfrak{p}_f$ is a natural isomorphism. While the purity transformation generally fails to be a natural isomorphism when f is not smooth, some of its components may still be isomorphisms. That is, there may be spectra E such that the map $\Sigma^{\mathcal{L}_f} f^* E \to f^! E$ is invertible.

Definition 5.2.3. (DJK21, Definition 4.3.7) A spectrum E is called f*-pure* if the component of purity $\Sigma^{\mathcal{L}_f} f^* E \to f^! E$ is invertible.

Proposition 5.2.4. (DJK21, Proposition 4.3.10) Let $f : X \to Y$ be a smoothable, separated morphism of finite type between regular k-schemes. Assume that $E \in \mathcal{SH}(k)$ is a motivic spectrum pulled back from a motivic spectrum defined over a perfect subfield of k. Let $\pi : Y \to \operatorname{Spec} k$ denote the structure map. Then f is lci and $\pi^* E$ is f-pure.

In particular, consider the map $q : \operatorname{Spec} k(p) \to \operatorname{Spec} k$ of regular k-schemes. This map satisfies the conditions of Proposition 5.2.4, and since $\mathbf{1}_k$ is pulled back from any perfect subfield of k, the canonical purity morphism

$$\Sigma^{\mathcal{L}_q} q^* \mathbf{1}_k \xrightarrow{\sim} q^! \mathbf{1}_k \tag{5.2.5}$$

is invertible. It is well-known that the purity isomorphism in Equation 5.2.5 subsumes the foundational theorem of Morel and Voevodsky (MV99, Theorem 2.23, p. 115). We will briefly discuss how to see this. Let S be a scheme, and let X and Z be smooth S-schemes. Consider a (not necessarily smooth) closed immersion $i : Z \hookrightarrow X$:

$$\begin{tikzcd} Z \arrow[r, hook, "i"] \arrow[dr, "g"'] & X \arrow[d, "f"] \\ & S. \end{tikzcd}$$

From the short exact sequence $f^*\mathcal{L}_i \to \mathcal{L}_g \to \mathcal{L}_g$, we have the equality $\mathcal{L}_g = \mathcal{L}_{i \circ f} = i^*\mathcal{L}_f + \mathcal{L}_i$ in $K(Z)$. Let $\mathcal{N}_i$ be the normal bundle of Z in X. Since $\mathcal{N}_i[1] = \mathcal{L}_i$ in

$K(Z)$, we have

$$\Sigma^{-i^*\mathcal{L}_f}\Sigma^{\mathcal{L}_g}g^* = \Sigma^{\mathcal{L}_i}g^* = \Sigma^{-\mathcal{N}_i}g^*.$$

We now apply purity to both g and f to obtain

$$\begin{aligned}\Sigma^{-i^*\mathcal{L}_f}\Sigma^{\mathcal{L}_g}g^* &\cong \Sigma^{-i^*\mathcal{L}_f}g^! = \Sigma^{-i^*\mathcal{L}_f}i^!f^! \\ &\cong i^!\Sigma^{-\mathcal{L}_f}f^! \cong i^!\Sigma^{-\mathcal{L}_f}\Sigma^{\mathcal{L}_f}f^* \cong i^!f^*.\end{aligned}$$

Thus we have a natural isomorphism $\Sigma^{-\mathcal{N}_i}g^* \cong i^!f^*$. Passing to left adjoints, we obtain $g_\sharp\Sigma^{\mathcal{N}_i} \cong f_\sharp i_*$. Finally, we consider the component of this equivalence at the sphere spectrum. By Proposition 5.2.1, we have that $i_*\mathbf{1}_Z = \Sigma^\infty\frac{X}{X-Z}$ as X-motivic spectra. Forgetting along f gives us $f_\sharp i_*\mathbf{1}_Z = f_\sharp\Sigma^\infty\frac{X}{X-Z}$ in $\mathcal{SH}(S)$. Conversely, we have that $\Sigma^{\mathcal{N}_i}\mathbf{1}_Z = \mathrm{Th}(\mathcal{N}_i)$ in $\mathcal{SH}(Z)$. Forgetting along g gives us $g_\sharp\mathrm{Th}(\mathcal{N}_i) = g_\sharp\Sigma^{\mathcal{N}_i}\mathbf{1}_Z$ in $\mathcal{SH}(S)$. Since $g_\sharp\Sigma^{\mathcal{N}_i}\mathbf{1}_Z \simeq f_\sharp i_*\mathbf{1}_Z$, we have the following equivalence in $\mathcal{SH}(S)$:

$$g_\sharp\mathrm{Th}\left(\mathcal{N}_i\right) \simeq f_\sharp\Sigma^\infty\frac{X}{X-Z}.$$

5.3 Transfers

In this section we discuss transfers arising in stable motivic homotopy theory, as well as their algebraic incarnations for Grothendieck–Witt groups.

Given a finite simple extension, residue homomorphisms induce a transfer called the *geometric transfer* (Mor12, §4.2) arising in Milnor–Witt K-theory. The geometric transfer can alternatively be defined using motivic spaces. In an attempt to extend this definition to finite field extensions, one might naively factor a finitely generated field extension $k \subseteq L$ into a composite of simple field extensions, and then compose geometric transfers. However, such a composition of geometric transfers will depend on the choice of factorization, indicating that the geometric transfer is not functorial along arbitrary finite field extensions. This can be rectified by multiplication by a certain rank one bilinear form, built out of the choice of primitive element of the extension, yielding the *cohomological transfer* (Mor12, §4.2). Alternatively, by incorporating all possible such factorizations simultaneously, one obtains a transfer along *twisted* Grothendieck–Witt rings, called the *absolute transfer* (Mor12, §5.1).

Throughout this section, we will maintain our assumption from Section 5.2 that k is finitely generated over a perfect field, which allows us to align the absolute transfer with Gysin maps. This assumption can be dropped in latter sections where our main results are proved.

5.3.1 Geometric transfers

In Milnor K-theory, the *residue homomorphisms* associated to discrete valuations enable the construction of transfers along field extensions. In Milnor–Witt K-theory, first defined by Hopkins and Morel, residue homomorphisms are still available, but ambiguities arise corresponding to a choice of uniformizing parameter. In degree zero, the Milnor–Witt K-theory of a field is the Grothendieck–Witt ring $\mathrm{GW}(k)$, so these residue homomorphisms permit us to define transfers of symmetric bilinear forms along finite simple field extensions.

Suppose that $p \in \mathbb{A}^1_k$ is a closed point, so that $k(p)/k$ is a finite simple field extension. Let $t \in k(p)$ be a primitive element of the extension with minimal polynomial $m(x) \in k[x]$. Considering the affine line as a subspace of the projective line with global sections $k(x)$, the minimal polynomial m of $p \in \mathbb{A}^1_k \subseteq \mathbb{P}^1_k$ defines a discrete valuation on $k(x)$. With $m(x)$ as a uniformizing parameter, we obtain a residue homomorphism

$$K_1^{\mathrm{MW}}(k(x)) \xrightarrow{\partial_p} \mathrm{GW}(k(p)).$$

We additionally have a residue homomorphism $-\partial_\infty : K_1^{\mathrm{MW}}(k(x)) \to \mathrm{GW}(k)$ for the point at infinity on the projective line, corresponding to the uniformizing parameter $-1/x$. Given a class $\alpha \in \mathrm{GW}(k(p))$, we may select an arbitrary preimage of α in $K_1^{\mathrm{MW}}(k(x))$ and then map to $\mathrm{GW}(k)$ along $-\partial_\infty$. It turns out that this defines a well-defined group homomorphism called the *geometric transfer* (Mor12, §4.2).

Definition 5.3.1. The *geometric transfer* for a finite simple extension $k(p)/k$ with primitive element t is defined by

$$\begin{aligned} \tau_k^{k(p)}(t) : \mathrm{GW}(k(p)) &\to \mathrm{GW}(k) \\ \alpha &\mapsto -\partial_\infty\left(\partial_p^{-1}(\alpha)\right). \end{aligned}$$

Turning our attention to motivic spaces, we can alternatively consider the composite of a collapse map and purity isomorphism to obtain a canonical map[1]

$$\mathbb{P}^1_k \to \frac{\mathbb{P}^1_k}{\mathbb{P}^1_k - p} \simeq \mathrm{Th}\left(\mathcal{N}_{p/\mathbb{P}^1_k}\right).$$

The minimal polynomial of p determines a non-canonical trivialization of the normal bundle, yielding an isomorphism $\mathrm{Th}\left(\mathcal{N}_{p/\mathbb{P}^1_k}\right) \simeq \mathrm{Th}\left(\mathcal{O}_{k(p)}\right)$. We now take cohomology (with coefficients in the Grothendieck–Witt sheaf) of the composite $\mathbb{P}^1_k \to \mathrm{Th}\left(\mathcal{O}_{k(p)}\right)$ to get a map $\mathrm{GW}(k(p)) \to \mathrm{GW}(k)$ that agrees with $\tau_k^{k(p)}(t)$. This geometric description motivates the terminology "geometric transfer" (see, for instance, (Mor12, §4.2)).

Remark 5.3.2. Note that any k-linear map $h : L \to k$ along a finite field extension will induce a transfer $h_* : \mathrm{GW}(L) \to \mathrm{GW}(k)$ by post-composition. It turns out that the geometric transfer is induced by a classical map called the *Scharlau form*.

[1]Here we are using our assumption that k is finitely generated over a perfect field in order to apply purity.

Definition 5.3.3. Let L/k be a finite simple extension with primitive element t. Then the *Scharlau form* is the k-linear map $s : L \to k$ defined by

$$s(t^j) = \begin{cases} 1 & j = [L:k] - 1, \\ \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 5.3.4. (CF17, Lemma 2.2), (Hoy14, Lemma 5.10) Let L/k be a finite simple extension with primitive element t, and let $s : L \to k$ be the Scharlau form associated to t. Then $\tau_k^L(t) = s_*$ as homomorphisms $\mathrm{GW}(L) \to \mathrm{GW}(k)$.

This description allows us to understand explicitly the geometric transfer of any rank one form in $\mathrm{GW}(L)$. We first set up some notation.

Notation 5.3.5. Let L/k be a finite simple extension of degree n with primitive element t, so that $B_{L/k} := \{1, t, \dots, t^{n-1}\}$ is a k-vector space basis of L. Given an L-vector space V with basis $B_{V/L} := \{a_1, \dots, a_d\}$, the set $B_{V/k} := \{a_i, a_i t, \dots, a_i t^{n-1}\}_{i=1}^d$ is a k-basis of V.

Lemma 5.3.6. In the context of Notation 5.3.5, let $\beta : V \times V \to L$ be a symmetric bilinear form whose Gram matrix with respect to $B_{V/L}$ is $(\beta_{ij})_{i,j}$, and let $s : L \to k$ be the Scharlau form. Then the Gram matrix of $s_*\beta$ with respect to $B_{V/k}$ is a block matrix whose $(i,j)^{\text{th}}$ block is equal to the Gram matrix of $s_* \langle \beta_{ij} \rangle$ with respect to $B_{L/k}$.

Proof. By definition, the $(i,j)^{\text{th}}$ block of $s_*\beta$ in the basis $B_{V/k}$ is given by

$$\begin{array}{c|cccc}
 & a_j & a_jt & \cdots & a_jt^{n-1} \\
\hline
a_i & s(\beta(a_i,a_j)) & s(\beta(a_i,a_jt)) & \cdots & s(\beta(a_i,a_jt^{n-1})) \\
a_it & s(\beta(a_it,a_j)) & s(\beta(a_it,a_jt)) & \cdots & s(\beta(a_i,ta_jt^{n-1})) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_it^{n-1} & s(\beta(a_it^{n-1},a_j)) & s(\beta(a_it^{n-1},a_jt)) & \cdots & s(\beta(a_it^{n-1},a_jt^{n-1}).
\end{array}$$

Since $t \in L$ and β is L-bilinear, we can rewrite this block as

$$\begin{array}{c|cccc}
 & a_j & a_jt & \cdots & a_jt^{n-1} \\
\hline
a_i & s(\beta_{ij}) & s(t\beta_{ij}) & \cdots & s(t^{n-1}\beta_{ij}) \\
a_it & s(t\beta_{ij}) & s(t^2\beta_{ij}) & \cdots & s(t^n\beta_{ij}) \\
\vdots & \vdots & \vdots & \iddots & \vdots \\
a_it^{n-1} & s(t^{n-1}\beta_{ij}) & s(t^n\beta_{ij}) & \cdots & s(t^{2n-2}\beta_{ij}),
\end{array}$$

which is precisely $s_* \langle \beta_{ij} \rangle$ with respect to the k-basis $\{1, t, \ldots, t^{n-1}\}$ of L. $\square$

5.3.2 Cohomological transfers

Let L/k be a finite simple extension with primitive element $t \in L$, and take $m(x) \in k[x]$ to be the minimal polynomial of t. Let p be the *exponential characteristic* of k, which is defined to be $\operatorname{char} k$ in positive characteristic and 1 in characteristic 0. We may factor the extension L/k as

$$k \subset L_{\mathrm{sep}} = k[t^{p^i}] \subseteq L,$$

for some i, where L_{sep} is the separable closure of k in L. This implies that $m(x) = m_0(x^{p^i})$ for some suitable $m_0(x) \in k[x]$. Note that $m_0(x)$ is the minimal polynomial of t^{p^i} over k, and hence is separable. Moreover, if L/k is separable, then $m_0(x) = m(x)$.

Notation 5.3.7. Using the notation from the previous paragraph, we define a distinguished polynomial $\omega_0(x) \in L[x]$ associated to the extension L/k by

$$\omega_0(x) := \frac{m_0(x)}{x - t^{p^i}}.$$

Note that t^{p^i} is a root of $m_0(x)$ since t is a root of $m(x)$, so $\omega_0(x)$ is indeed a polynomial. Since $m_0(x)$ is separable, we see that $\omega_0(t) \in L^{\times}$. We will use $\omega_0(x)$ to define the cohomological transfer in terms of the geometric transfer in Definition 5.5.8.

Example 5.3.8. Let L/k be a finite purely inseparable extension in characteristic p. Then its minimal polynomial is by definition of the form $x^{p^r} - a$ for some $a \in k$, so $m_0(x) = x - a$ and therefore $\omega_0(x) = 1$.

Example 5.3.9. Let L/k be a finite separable extension with primitive element t, and let $m(x)$ be the minimal polynomial of t. Then $\omega_0(x) = \frac{m(x)}{(x-t)}$, so $\omega_0(t) = m'(t)$ by the product rule.

Definition 5.3.10. (Mor12, Definition 4.26) For a finite simple extension L/k with primitive element t, the *cohomological transfer* Tr_k^L is defined to be the composite

$$\begin{tikzcd} \mathrm{GW}(L) \arrow[r, "\langle \omega_0(t) \rangle"] \arrow[dr, "\mathrm{Tr}_k^L"'] & \mathrm{GW}(L) \arrow[d, "\tau_k^L(t)"] \\ & \mathrm{GW}(k). \end{tikzcd}$$

Under nice conditions, Tr_k^L does not depend on the choice of primitive element t (Mor12, Theorem 4.27). Moreover, *loc. cit.* also implies that the cohomological transfer is functorial along field extensions outside of characteristic two, so we can define the cohomological transfer of an arbitrary finite extension as the composite of cohomological transfers over constituent simple extensions. For finite separable extensions, the cohomological transfer recovers the transfer on Grothendieck–Witt groups induced by the field trace (CF17, Lemma 2.3). For purely inseparable extensions, the cohomological transfer and the geometric transfer coincide by Example 5.3.8.

5.3.3 Absolute transfers

Let L/k be a finite, purely inseparable extension. We can factor this into simple extensions

$$k = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n = L.$$

Let t_i be a primitive element for the simple extension L_i/L_{i-1}. From this we obtain a composite of geometric transfers

$$\tau_k^{L_1}(t_1) \circ \tau_{L_1}^{L_2}(t_2) \circ \cdots \circ \tau_{L_{n-1}}^{L}(t_n) : \mathrm{GW}(L) \to \mathrm{GW}(k).$$

This composite transfer is not independent of the tuple $(t_1, \ldots, t_n)$. However, this transfer depends only on the class of the element $dt_1 \wedge \cdots \wedge dt_n$ in the determinant of the L-vector space of Kähler differentials of L over k (Mor12, §5). Thus any class in $\omega_{L/k} := \det \Omega_{L/k}$ provides a way to transfer from L down to k. This perspective allows us to produce a well-defined *absolute transfer*

$$\mathrm{Tr}_k^{k(p)}\left(\omega_{L/k}\right) : \mathrm{GW}(L, \omega_{L/k}) \to \mathrm{GW}(k),$$

where $\mathrm{GW}(L, \omega_{L/k})$ denotes the twisted Grothendieck–Witt group (Mor12, Definition 5.4). In the simple setting, $\omega_{L/k}$ is a one-dimensional L-vector space, and therefore isomorphic to L, inducing a group isomorphism $\mathrm{GW}(L, \omega_{L/k}) \cong \mathrm{GW}(L)$. This idea can be leveraged to canonically *untwist* the absolute transfer in odd characteristic to obtain a transfer $\mathrm{GW}(L) \to \mathrm{GW}(k)$, which coincides with the cohomological transfer (Mor12, Remark 5.6).

It turns out that the absolute transfer is hiding in the background of the definition of the local $\mathbb{A}^1$-Brouwer degree. We will establish this fact after recalling the definition of the local degree.

Definition 5.3.11. A point $p \in \mathbb{A}^n_k$ is called an *isolated zero* of a morphism $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ is $f(p) = 0$ and p is isolated in its fiber $f^{-1}(0)$.

Let $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ be a morphism of affine space, and let $p \in \mathbb{A}^n_k$ be an isolated zero of f. By viewing $\mathbb{A}^n_k \subseteq \mathbb{P}^n_k$ as a subscheme of projective space via a standard chart, f induces a map

$$\overline{f} : \frac{\mathbb{P}^n_k}{\mathbb{P}^n_k - p} \to \frac{\mathbb{P}^n_k}{\mathbb{P}^n_k - 0} \simeq \frac{\mathbb{P}^n_k}{\mathbb{P}^{n-1}_k}.$$

Precomposing with the collapse map $c_p : \mathbb{P}^n_k/\mathbb{P}^{n-1}_k \to \mathbb{P}^n_k/(\mathbb{P}^n_k - p)$ yields a morphism f_p as in the following diagram:

$$\begin{tikzcd}
\mathbb{P}^n_k/(\mathbb{P}^n_k - p) \arrow[r, "\overline{f}"] & \mathbb{P}^n_k/\mathbb{P}^{n-1}_k \\
\mathbb{P}^n_k/\mathbb{P}^{n-1}_k \arrow[u, "c_p"] \arrow[ur, "f_p"'] &
\end{tikzcd}$$

Definition 5.3.12. Let $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$, and let p be an isolated zero of f. The *local* $\mathbb{A}^1$*-degree* $\deg_p^{\mathbb{A}^1}(f)$ of f at p is the image of the homotopy class of f_p under Morel's degree map

$$\deg^{\mathbb{A}^1} : \left[\mathbb{P}^n_k/\mathbb{P}^{n-1}_k, \mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right]_{\mathcal{SH}(k)} \to \mathrm{GW}(k).$$

If $k(p)/k$ is separable, then the stable class of the collapse map admits a tractable description (KW19, Lemma 13). In particular, since $\operatorname{Spec} k(p)$ is a smooth k-scheme, purity gives an equivalence

$$\frac{\mathbb{P}^n_k}{\mathbb{P}^n_k - p} \simeq \mathrm{Th} T_p \mathbb{P}^n_k \simeq \left(\frac{\mathbb{P}^n_k}{\mathbb{P}^{n-1}_k} \right) \wedge \operatorname{Spec} k(p)_+.$$

From this, one can prove that the collapse map is $\left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k\right) \wedge \eta_{1_k}$, where $\eta : \mathrm{id} \to q_* q^*$ is the unit of the pushforward-pullback adjunction for the structure map $q : \operatorname{Spec} k(p) \to \operatorname{Spec} k$.

If $k(p)/k$ is not separable, we defer to the theory of Gysin maps in order to characterize the collapse map.

Proposition 5.3.13. Let $E \in \mathcal{SH}(k)$ be a motivic spectrum. Then the compactly supported cohomology of $\mathbb{P}^n_k$ on $p \in \mathbb{P}^n_k$ is given by

$$E_p\left(\mathbb{P}^n_k\right) = E\left(\frac{\mathbb{P}^n_k}{\mathbb{P}^n_k - p} \right).$$

Proof. Let $i : \operatorname{Spec} k(p) \to \mathbb{P}^n_k$ be the closed immersion of p into $\mathbb{P}^n_k$. Let $\pi : \mathbb{P}^n_k \to \operatorname{Spec} k$ be the structure map, which is smooth. Cohomology with compact supports (c.f. (BW21, 4.2.1)) is defined to be

$$E_p\left(\mathbb{P}^n_k\right) := \left[\mathbf{1}_k, \pi_* i_! i^! \pi^* E\right]_{\mathcal{SH}(k)}.$$

Since i is a closed immersion, it is a proper map, so we have a canonical natural isomorphism $i_! \simeq i_*$. As π is smooth, $\pi_\sharp$ exists and is left adjoint to π^*. Combining these facts with the basic properties of adjunctions, we have a string of natural isomorphisms:

$$\begin{aligned}
\left[\pi^*\mathbf{1}_k, i_!i^!\pi^*E\right]_{\mathcal{SH}(\mathbb{P}^n_k)} &\cong \left[\mathbf{1}_{\mathbb{P}^n_k}, i_*i^!\pi^*E\right]_{\mathcal{SH}(\mathbb{P}^n_k)} && (i_* \simeq i_!) \\
&\cong \left[i^*\mathbf{1}_{\mathbb{P}^n_k}, i^!\pi^*E\right]_{\mathcal{SH}(k(p))} && (i^* \text{ left adjoint to } i_*) \\
&\cong \left[\mathbf{1}_{k(p)}, i^!\pi^*E\right]_{\mathcal{SH}(k(p))} && (i^* \text{ monoidal}) \\
&\cong \left[i_*\mathbf{1}_{k(p)}, \pi^*E\right]_{\mathcal{SH}(\mathbb{P}^n_k)} && (i_* \simeq i_! \text{ left adjoint to } i^!) \\
&\cong \left[\pi_\sharp i_*\mathbf{1}_{k(p)}, E\right]_{\mathcal{SH}(k)} && (\pi_\sharp \text{ left adjoint to } \pi^*).
\end{aligned}$$

Proposition 5.2.1 states that $i_*\mathbf{1}_{k(p)}$ is the cofiber $\mathbb{P}^n_k/\left(\mathbb{P}^n_k - p\right)$, while $\pi_\sharp$ is the forgetful functor. The result follows from the definition of $E\left(\frac{\mathbb{P}^n_k}{\mathbb{P}^n_k - p}\right)$. □

Proposition 5.3.14 (*The collapse map induces the Gysin transfer*)**.** Let E be any motivic spectrum over k. Let $i : \operatorname{Spec} k(p) \to \mathbb{P}^n_k$ be the inclusion of a closed point p, and let $q : \operatorname{Spec} k(p) \to \operatorname{Spec} k$ denote the structure map. The collapse map $c_p : \frac{\mathbb{P}^n_k}{\mathbb{P}^{n-1}_k} \to \frac{\mathbb{P}^n_k}{\mathbb{P}^n_k - p}$ induces a map $c_p^* : E_p(\mathbb{P}^n_k, \mathcal{O}^n_k) \to E_{\mathbb{A}^n_k}(\mathbb{P}^n_k, \mathcal{O}^n_k)$, and the composite

$$E\left(\operatorname{Spec} k(p), \mathcal{L}_q\right) \xrightarrow{i_!} E_p(\mathbb{P}^n_k, \mathcal{O}^n_k) \xrightarrow{c_p^*} E_{\mathbb{A}^n_k}(\mathbb{P}^n_k, \mathcal{O}^n_k) \simeq E(\operatorname{Spec} k)$$

is equal to the Gysin transfer $q_!$ (EHK[+]20, (2.2.4)).

Proof. This can be seen by the commutativity of the bottom rectangle of (EHK[+]20, (3.2.12)). □

Given a map $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ with an isolated zero p, the class $\overline{f}$ lives in the stable homotopy classes of maps from the cofiber $\mathbb{P}^n_k/\left(\mathbb{P}^n_k - p\right)$ into $\mathbb{P}^n_k/\mathbb{P}^{n-1}_k$. This group admits a nice algebraic description.

Proposition 5.3.15. There is an isomorphism of groups

$$\left[\frac{\mathbb{P}^n_k}{\mathbb{P}^n_k - p}, \frac{\mathbb{P}^n_k}{\mathbb{P}^n_k - 0}\right]_{\mathcal{SH}(k)} \cong \mathrm{GW}\left(k(p), \omega_q\right).$$

Proof. Excision implies that

$$\frac{\mathbb{P}^n_k}{\mathbb{P}^n_k - 0} \simeq \frac{\mathbb{A}^n_k}{\mathbb{A}^n_k - 0} \simeq \mathrm{Th}(\mathcal{O}^n_k),$$

where $\mathcal{O}^n_k$ is the trivial rank n bundle over a point. As an element of $\mathcal{SH}(k)$, we can write $\Sigma^\infty \mathrm{Th}(\mathcal{O}^n_k)$ as $\Sigma^n \mathbf{1}_k$. Let $\widetilde{\pi} : \mathbb{P}^n_{k(p)} \to \operatorname{Spec} k(p)$, $\pi : \mathbb{P}^n_k \to \operatorname{Spec} k$, and $q : \operatorname{Spec} k(p) \to \operatorname{Spec} k$ be structure maps. Let $i : \operatorname{Spec} k(p) \to \mathbb{P}^n_k$ denote the inclusion of p, and let $\iota : \operatorname{Spec} k(p) \to \mathbb{P}^n_{k(p)}$ denote inclusion of the canonical $k(p)$-rational point $\widetilde{p}$ lying over p. These maps fit into the commutative diagram

$$\begin{tikzcd} \operatorname{Spec} k(p) \arrow[r, "\iota"] \arrow[d, equal] & \mathbb{P}^n_{k(p)} \arrow[r, "\widetilde{\pi}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \operatorname{Spec} k(p) \arrow[d, "q"] \\ \operatorname{Spec} k(p) \arrow[r, "i"'] & \mathbb{P}^n_k \arrow[r, "\pi"'] & \operatorname{Spec} k. \end{tikzcd} \tag{5.3.16}$$

By Proposition 5.2.1 and purity, we can rewrite our mapping classes as

$$\left[\frac{\mathbb{P}^n_k}{\mathbb{P}^n_k - p}, \frac{\mathbb{P}^n_k}{\mathbb{P}^n_k - 0}\right]_{\mathcal{SH}(k)} \cong \left[\pi_\sharp i_* \mathbf{1}_{k(p)}, \Sigma^n \mathbf{1}_k\right]_{\mathcal{SH}(k)}.$$

Functoriality implies $(q\widetilde{\pi}\iota)_* = q_*\widetilde{\pi}_*\iota_*$ and $(\pi i)_* = \pi_* i_*$. As π and $\widetilde{\pi}$ are both proper and étale, we have $\pi_* \simeq \pi_\sharp$ and $\widetilde{\pi}_* \simeq \widetilde{\pi}_\sharp$. Since q is proper, we have $q_* \simeq q_!$, with right adjoint $q^!$. Equation 5.3.16 thus allows us to rewrite

$$\begin{aligned}
\left[\pi_\sharp i_* \mathbf{1}_{k(p)}, \Sigma^n \mathbf{1}_k\right]_{\mathcal{SH}(k)} &\cong \left[\pi_* i_* \mathbf{1}_{k(p)}, \Sigma^n \mathbf{1}_k\right]_{\mathcal{SH}(k)} && (\pi_* \simeq \pi_\sharp) \\
&\cong \left[q_* \widetilde{\pi}_* \iota_* \mathbf{1}_{k(p)}, \Sigma^n \mathbf{1}_k\right]_{\mathcal{SH}(k)} && (q\widetilde{\pi}\iota = \pi i) \\
&\cong \left[\widetilde{\pi}_\sharp \iota_* \mathbf{1}_{k(p)}, q^! \Sigma^n \mathbf{1}_k\right]_{\mathcal{SH}(k(p))} && (q^! \text{ right adjoint to } q_*,\ \widetilde{\pi}_* \simeq \widetilde{\pi}_\sharp) \\
&\cong \left[\Sigma^\infty \frac{\mathbb{P}^n_{k(p)}}{\mathbb{P}^n_{k(p)} - \widetilde{p}}, q^! \Sigma^n \mathbf{1}_k\right]_{\mathcal{SH}(k(p))} && (\text{Proposition 5.2.1}) \\
&\cong \left[\Sigma^n \mathbf{1}_{k(p)}, q^! \Sigma^n \mathbf{1}_k\right]_{\mathcal{SH}(k(p))} && (\text{purity}).
\end{aligned}$$

We can now use the isomorphism $q^!\Sigma^n \cong \Sigma^n q^!$, desuspend, and remark that the sphere spectrum is q-pure (Equation 5.2.5) to deduce

$$\left[\Sigma^n \mathbf{1}_{k(p)}, \Sigma^n q^! \mathbf{1}_k\right] \cong \left[\mathbf{1}_{k(p)}, q^! \mathbf{1}_k\right] \cong \left[\mathbf{1}_{k(p)}, \Sigma^{\mathcal{L}_q} \mathbf{1}_{k(p)}\right].$$

Since the unit map $\mathbf{1}_{k(p)} \to H\widetilde{\mathbb{Z}}$ induces an isomorphism on π_0, we have an isomorphism

$$\left[\mathbf{1}_{k(p)}, \Sigma^{\mathcal{L}_q} \mathbf{1}_{k(p)}\right] \cong \left[\mathbf{1}_{k(p)}, \Sigma^{\mathcal{L}_q} H\widetilde{\mathbb{Z}}\right] = \widetilde{\mathrm{CH}}^0\left(\operatorname{Spec} k(p), \omega_q\right).$$

Here $\widetilde{\mathrm{CH}}$ denotes the Chow–Witt groups of a scheme, which are represented by the motivic spectrum $H\widetilde{\mathbb{Z}}$, and $\omega_q = \det \mathcal{L}_q$. We conclude by noting that $\widetilde{\mathrm{CH}}^0\left(\operatorname{Spec} k(p), \omega_q\right) \cong \mathrm{GW}\left(\operatorname{Spec} k(p), \omega_q\right)$ (see e.g. (EHK+20, p. 35)). $\square$

Corollary 5.3.17 (*Precomposition with the collapse map is the absolute transfer*)**.**

The collapse map $c_p : \mathbb{P}^n_k/\mathbb{P}^{n-1}_k \to \mathbb{P}^n_k/\left(\mathbb{P}^n_k - p\right)$ induces a morphism

$$\left[\frac{\mathbb{P}^n_k}{\mathbb{P}^n_k - p}, \frac{\mathbb{P}^n_k}{\mathbb{P}^{n-1}_k}\right]_{\mathcal{SH}(k)} \to \left[\frac{\mathbb{P}^n_k}{\mathbb{P}^{n-1}_k}, \frac{\mathbb{P}^n_k}{\mathbb{P}^{n-1}_k}\right]_{\mathcal{SH}(k)}, \tag{5.3.18}$$

which is a map of the form $\mathrm{GW}(k(p), \omega_q) \to \mathrm{GW}(k)$. This is the absolute transfer.

Proof. By Proposition 5.3.15 and (Mor12, Corollary 1.24), Equation 5.3.18 can be written as a map $\mathrm{GW}(k(p), \omega_q) \to \mathrm{GW}(k)$. Taking $E = \mathbf{1}_k$ to be the sphere spectrum, Proposition 5.3.14 implies that the collapse map induces a Gysin map $\mathrm{GW}(k(p), \omega_q) \to \mathrm{GW}(k)$. By (EHK+20, Proposition 4.3.17), the Gysin map coincides with the absolute transfer. □

5.3.4 Hinting at lifts for transfers

So far, we have discussed transfers in the context of both Grothendieck–Witt rings and motivic spectra. The following result suggests that one can lift the class $\overline{f}$ up to a class $\widetilde{f}$ around the canonical $k(p)$-rational point $\widetilde{p}$. We then ask if the lift $\widetilde{f}$ is compatible with a given transfer τ: is $\tau(\widetilde{f}) = \overline{f}$?

Proposition 5.3.19. Morel's canonical untwisting (in odd characteristic) can be thought of as a map of the form

$$\left[\frac{\mathbb{P}^n_k}{\mathbb{P}^n_k - p}, \frac{\mathbb{P}^n_k}{\mathbb{P}^n_k - 0}\right]_{\mathcal{SH}(k)} \cong \mathrm{GW}(k(p), \omega_q) \xrightarrow{\sim} \mathrm{GW}(k(p)) \cong \left[\frac{\mathbb{P}^n_{k(p)}}{\mathbb{P}^n_{k(p)} - \widetilde{p}}, \frac{\mathbb{P}^n_{k(p)}}{\mathbb{P}^n_{k(p)} - 0}\right]_{\mathcal{SH}(k(p))}.$$

Proof. Since both $\widetilde{p}$ and 0 are $k(p)$-rational, the equivalence $[\frac{\mathbb{P}^n_{k(p)}}{\mathbb{P}^n_{k(p)}-\widetilde{p}}, \frac{\mathbb{P}^n_{k(p)}}{\mathbb{P}^n_{k(p)}-0}]_{\mathcal{SH}(k(p))} \cong$ GW$(k(p))$ follows immediately by purity and (Mor12, Corollary 1.24). The result now follows from Proposition 5.3.15. □

Remark 5.3.20. Suppose that f is an endomorphism of $\mathbb{A}^n_k$ with an isolated root at a closed point p. This induces a class $\overline{f} \in \left[\frac{\mathbb{P}^n_k}{\mathbb{P}^n_k-p}, \frac{\mathbb{P}^n_k}{\mathbb{P}^n_k-0}\right]$ whose absolute transfer is $\deg^{\mathbb{A}^1}_p(f)$. However, Proposition 5.3.19 implies that we can untwist $\overline{f}$ to obtain a class $\widetilde{f} \in \left[\frac{\mathbb{P}^n_{k(p)}}{\mathbb{P}^n_{k(p)}-\widetilde{p}}, \frac{\mathbb{P}^n_{k(p)}}{\mathbb{P}^n_{k(p)}-0}\right]$ whose geometric transfer recovers $\deg^{\mathbb{A}^1}_p(f)$. This leads us to the question of lifts, transfers, and degrees: is there an endomorphism g of $\mathbb{A}^n_{k(p)}$ such that $\overline{g} = \widetilde{f}$? Lemma 5.5.6 answers this question in the affirmative in the univariate setting.

5.4 Bézoutians, Hankel forms, and Horner bases

We now discuss a few algebraic tools used for computing local $\mathbb{A}^1$-degrees. The first tool will be the *Bézoutian* Béz(f) of a map $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$, which is a polynomial in $2n$ variables. The coefficients of Béz(f) determine a bilinear form $k[X_1, \ldots, X_n]/(f) \times k[Y_1, \ldots, Y_n]/(f) \to k$ whose isomorphism class is $\deg^{\mathbb{A}^1}(f)$ (BMP21b). This was first noticed by Cazanave in the univariate case (Caz12). One can also recover the local $\mathbb{A}^1$-degree $\deg^{\mathbb{A}^1}_p(f)$ from Béz(f) in a similar manner (BMP21b).

The second tool will be *Hankel* matrices. In the univariate case, the bilinear forms determined by Bézoutians have a particular structure (namely, they are represented by Hankel matrices). By exploiting this structure, one can easily diagonalize these bilinear forms to better understand their classes in $\mathrm{GW}(k)$.

The final tool will be *Horner bases*, which serve as an alternative to the monomial basis of a quotient $k[x]/(f)$. We will also discuss how Horner bases interact with the Scharlau form when $k[x]/(f)$ is a field. This will be relevant in the proof of Lemma 5.5.6.

5.4.1 Bézoutians and $\mathbb{A}^1$-degrees

Given a map $f/g : \mathbb{P}^1_k \to \mathbb{P}^1_k$, let

$$\mathrm{B\acute{e}z}(f/g) := \frac{f(X)g(Y) - f(Y)g(X)}{X - Y} \in k[X, Y]$$

be its Bézoutian. Writing $\mathrm{B\acute{e}z}(f/g) = \sum_{i,j} c_{ij} X^{i-1} Y^{j-1}$, the matrix of coefficients (c_{ij}) defines the *Bézoutian bilinear form* of f/g, and the class in $\mathrm{GW}(k)$ of this bilinear form recovers $\deg^{\mathbb{A}^1}(f/g)$ (Caz12).

In the univariate case, every local $\mathbb{A}^1$-degree can be expressed as a global $\mathbb{A}^1$-degree of the projective line.

Proposition 5.4.1 (*Univariate local degrees are global degrees*)**.** Let $f : \mathbb{A}^1_k \to \mathbb{A}^1_k$ be a map with an isolated zero at a closed point p, and let $m(x) \in k[x]$ be an irreducible polynomial that generates the maximal ideal corresponding to p. Then $f(x) = u(x)m(x)^d$ for some $u(x) \in k[x]$ that is nonvanishing at p, and

$$\deg_p^{\mathbb{A}^1}(f) = \deg^{\mathbb{A}^1}\left(\mathbb{P}^1_k \xrightarrow{m^d/u} \mathbb{P}^1_k\right).$$

Proof. Combining Cazanave's theorem with (BMP21b), it suffices to show that $\text{Béz}(f) \equiv \text{Béz}(m^d/u) \bmod (f(X), f(Y))$. Moreover, since $u(x)$ is not contained in the ideal $(m(x))$, we have an isomorphism

$$\frac{k[x]_{(m)}}{(f)} \cong \frac{k[x]_{(m)}}{(m^d)}$$

of k-algebras. It thus suffices to show that $\text{Béz}(f) \equiv \text{Béz}(m^d/u) \bmod (m(X)^d, m(Y)^d)$. We compute that

$$\begin{aligned}
\text{Béz}(f) &= \frac{u(X)m(X)^d - u(Y)m(Y)^d}{X-Y} \\
&= \frac{u(X)m(X)^d - u(Y)m(Y)^d}{X-Y} + \frac{u(Y)m(X)^d - u(Y)m(X)^d}{X-Y} \\
&\quad + \frac{u(X)m(Y)^d - u(X)m(Y)^d}{X-Y} \\
&= \frac{u(Y)m(X)^d - u(X)m(Y)^d}{X-Y} + \frac{u(X)-u(Y)}{X-Y}(m(X)^d + m(Y)^d) \\
&\equiv \frac{u(Y)m(X)^d - u(X)m(Y)^d}{X-Y} \bmod (m(X)^d, m(Y)^d). \\
&\equiv \text{Béz}(m^d/u) \bmod (m(X)^d, m(Y)^d).
\end{aligned}$$

$\square$

Corollary 5.4.2. Let $p \in \mathbb{A}^1_k$ be a closed point, and let $m(x) \in k[x]$ be an irreducible polynomial that generates the maximal ideal corresponding to p. This polynomial determines a map $m : \mathbb{A}^1_k \to \mathbb{A}^1_k$, and

$$\deg_p^{\mathbb{A}^1}(m) = \deg^{\mathbb{A}^1}(m) = \deg^{\mathbb{A}^1}(\mathbb{P}^1_k \xrightarrow{m/1} \mathbb{P}^1_k).$$

Proof. The equality $\deg^{\mathbb{A}^1}(m) = \deg^{\mathbb{A}^1}(\mathbb{P}^1_k \xrightarrow{m/1} \mathbb{P}^1_k)$ is a special case of Proposition 5.4.1 (with $d = 1$ and $u = 1$). Morally speaking, $\deg_p^{\mathbb{A}^1}(m) = \deg^{\mathbb{A}^1}(m)$ since p is the only root of m over k. More precisely, the isomorphism

$$\frac{k[x]}{(m)} \cong \frac{k[x]_{(m)}}{(m)}$$

of k-algebras preserves the Bézoutian and basis of $k[x]/(m)$. By (BMP21b, Lemma 4.7), it follows that $\deg_p^{\mathbb{A}^1}(m) = \deg^{\mathbb{A}^1}(m)$. □

5.4.2 Hankel and block Hankel forms

A *Hankel matrix* is a symmetric matrix with constant anti-diagonals. A symmetric bilinear form that can be represented by a Hankel matrix is called a *Hankel form*. Hankel matrices and forms are classical objects of study (Ioh82). In the univariate setting, we may observe that Bézoutian bilinear forms of polynomials can be naturally represented by Hankel matrices. As a motivating example, consider the polynomial $f(x) = x^3 + 3x^2 - 4x + 1$. Its Bézoutian is given by

$$\text{Béz}(f) = \frac{f(X) - f(Y)}{X - Y} = (X^2 + XY + Y^2) + 3(X + Y) - 4.$$

Writing this in monomial basis for the global algebra $k[x]/f(x)$, we obtain

$$\deg^{\mathbb{A}^1}(f) = \left(\begin{array}{c|ccc} & 1 & X & X^2 \\ \hline 1 & -4 & 3 & 1 \\ Y & 3 & 1 & 0 \\ Y^2 & 1 & 0 & 0 \end{array} \right).$$

In particular, $\deg^{\mathbb{A}^1}(f)$ is a Hankel form. Note that all the anti-diagonals below the main anti-diagonal are constantly zero. We call such a form an *upper triangular* Hankel form. The isomorphism class in $\mathrm{GW}(k)$ of an upper triangular Hankel form is well-understood — interestingly, none of the information lying above the main anti-diagonal matters.

Proposition 5.4.3. (KW20, Lemma 6) Let $s_1, \ldots, s_d \in k$ with $s_d \neq 0$. Then the matrix

$$\begin{pmatrix} s_1 & s_2 & \cdots & s_{d-1} & s_d \\ s_2 & s_3 & \cdots & s_d & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ s_{d-1} & s_d & \cdots & 0 & 0 \\ s_d & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

represents the $\mathrm{GW}(k)$ class

$$\begin{cases} \frac{d}{2}\mathbb{H} & d \text{ is even} \\ \frac{d-1}{2}\mathbb{H} + \langle s_d \rangle & d \text{ is odd.} \end{cases}$$

The global $\mathbb{A}^1$-degree of any polynomial map $\mathbb{P}^1_k \to \mathbb{P}^1_k$ is an upper triangular Hankel form, so Proposition 5.4.3 characterizes such $\mathbb{A}^1$-degrees. This characterization alternatively follows from the fact that any univariate polynomial can be naïvely $\mathbb{A}^1$-homotoped to its leading term (Caz12, Example 2.4).

One might ask whether local $\mathbb{A}^1$-degrees of univariate polynomials exhibit a similar symmetry. Since localizing the global algebra $k[x]/(f)$ at a maximal ideal $m(x) \cdot k[x]$ (corresponding to an isolated zero p of f) can decrease its rank, the monomials $\{1, x, \ldots, x^{\deg(f)}\}$ may not form a basis of $k[x]_{(m)}/(f)$. In a suitable basis, we will show that the Gram matrix of $\deg_p^{\mathbb{A}^1}(f)$ is a block upper triangular matrix with constant blocks on each anti-diagonal. We call such a form a *block Hankel form*. We will also see that each block in this Gram matrix for $\deg_p^{\mathbb{A}^1}(f)$ is itself a Hankel matrix.[2]

As in Proposition 5.4.3, we will demonstrate that information above the main off-diagonal of blocks does not affect the $\mathrm{GW}(k)$ class of a block Hankel form. We first introduce some notation before proving this general result.

[2]We have elected to not call this a *Hankel block Hankel form*.

Notation 5.4.4. Let V be an algebra over a field K. Let

$$\mathcal{B} := \{a_1b_1, \ldots, a_1b_n, \ldots, a_db_1, \ldots, a_db_n\}$$

be a vector space basis for V. Let β be a bilinear form on V. The $dn \times dn$ Gram matrix for β in the basis $\mathcal{B}$ can be written as

$$\beta_{\mathcal{B}} = \begin{array}{c|c} & \begin{matrix} a_1 & a_2 & \cdots & a_d \end{matrix} \\ \hline \begin{matrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{matrix} & \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1d} \\ A_{21} & A_{22} & \cdots & A_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ A_{d1} & A_{d2} & \cdots & A_{dd} \end{pmatrix} \end{array},$$

where each A_{ij} is a block matrix of the form

$$A_{ij} = \begin{array}{c|c} & \begin{matrix} a_jb_1 & a_jb_2 & \cdots & a_jb_n \end{matrix} \\ \hline \begin{matrix} a_ib_1 \\ a_ib_2 \\ \vdots \\ a_ib_n \end{matrix} & \begin{pmatrix} \beta_{ij}^{11} & \beta_{ij}^{12} & \cdots & \beta_{ij}^{1n} \\ \beta_{ij}^{21} & \beta_{ij}^{22} & \cdots & \beta_{ij}^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{ij}^{n1} & \beta_{ij}^{n2} & \cdots & \beta_{ij}^{nn} \end{pmatrix} \end{array}.$$

That is, $\beta_{ij}^{\ell k}$ is the coefficient appearing on $a_ib_\ell \otimes a_jb_k$ in β.

Lemma 5.4.5. Let V, $\mathcal{B}$, and β be as in Notation 5.4.4. Assume that $\mathrm{char}K \neq 2$.

Suppose that β is non-degenerate, and that $\beta_{\mathcal{B}}$ is a block Hankel matrix

$$\beta_{\mathcal{B}} = \begin{pmatrix} A_1 & A_2 & \cdots & A_{d-1} & A_d \\ A_2 & A_3 & \cdots & A_d & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ A_{d-1} & A_d & \cdots & 0 & 0 \\ A_d & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Also suppose that each A_i is an $n \times n$ Hankel matrix

$$A_i = \begin{pmatrix} \beta_i^1 & \beta_i^2 & \cdots & \beta_i^n \\ \beta_i^2 & \beta_i^3 & \cdots & \beta_i^{n+1} \\ \vdots & \vdots & \iddots & \vdots \\ \beta_i^n & \beta_i^{n+1} & \cdots & \beta_i^{2n-1} \end{pmatrix}.$$

Then the class in $\mathrm{GW}(K)$ of β is $\frac{nd}{2}\mathbb{H}$ if d is even and $\frac{n(d-1)}{2}\mathbb{H} + \hat{A}_d$ if d is odd.[3]

Proof. The goal here is to exhibit a basis $\mathcal{B}'$ such that the Gram matrix $\beta_{\mathcal{B}'}$ is block diagonal. In the basis $\mathcal{B}$, the Gram matrix for β can be written as

$$\beta_{\mathcal{B}} = \sum_{i,j=1}^{d} \sum_{\ell,k=1}^{n} \beta_{i+j-1}^{\ell+k-1} a_i b_\ell \otimes a_j b_k$$

for some scalars $\beta_{i+j-1}^{\ell+k-1} \in K$. We will recursively use the rows of $\beta_{\mathcal{B}}$ to construct the basis $\mathcal{B}'$. See Section 5.6 for the intuition behind the following details. For

[3] Here we are abusing notation to conflate the Gram matrix A_d with the isomorphism class of forms it represents in $\mathrm{GW}(k)$.

$1 \leq i \leq \lfloor \frac{d}{2} \rfloor$ and $1 \leq \ell \leq n$, let

$$\psi_i^\ell = \frac{\beta_i^{2\ell-1}}{2} a_i b_\ell + \sum_{k=\ell+1}^{n} \beta_i^{2\ell-1+k} a_i b_k + \sum_{j=i+1}^{d} \sum_{k=1}^{n} \beta_j^{k+\ell-1} a_j b_k.$$

Now let

$$\mathcal{B}' = \{a_1 b_1, \psi_1^1, a_1 b_2, \psi_1^2, \ldots, a_1 b_n, \psi_1^n, \ldots, a_{\lfloor d/2 \rfloor} b_n, \psi_{\lfloor d/2 \rfloor}^n\}$$
$$\cup \begin{cases} \varnothing & d \text{ is even} \\ \left\{ a_{\frac{d+1}{2}} b_1, \ldots, a_{\frac{d+1}{2}} b_n \right\} & d \text{ is odd.} \end{cases}$$

The assumption that β is non-degenerate implies that the elements of $\mathcal{B}'$ are linearly independent, so $\mathcal{B}'$ is a K-basis for V. We now rewrite $\beta_{\mathcal{B}}$ in terms of $\mathcal{B}'$:

$$\beta_{\mathcal{B}} = \sum_{i=1}^{\lfloor d/2 \rfloor} \sum_{\ell=1}^{n} \left(a_i b_\ell \otimes \psi_i^\ell + \psi_i^\ell \otimes a_i b_\ell \right) + \begin{cases} 0 & d \text{ is even} \\ \sum_{\ell,k=1}^{n} \beta_d^{\ell+k-1} a_{\frac{d+1}{2}} b_\ell \otimes a_{\frac{d+1}{2}} b_k & d \text{ is odd.} \end{cases}$$

It follows that $\beta_{\mathcal{B}'}$ is block diagonal. For $1 \leq i \leq \lfloor \frac{d}{2} \rfloor$, the i^{th} block of $\beta_{\mathcal{B}'}$ (corresponding to the basis elements $\{a_ib_1, \psi_i^1, \dots, a_ib_n, \psi_i^n\}$) is

$$\begin{array}{c|ccccccc}
 & a_ib_1 & \psi_i^1 & a_ib_2 & \psi_i^2 & \cdots & a_ib_n & \psi_i^n \\
\hline
a_ib_1 & 0 & 1 & & & & & \\
\psi_i^1 & 1 & 0 & & & & & \\
a_ib_2 & & & 0 & 1 & & & \\
\psi_i^2 & & & 1 & 0 & & & \\
\vdots & & & & & \ddots & & \\
a_ib_n & & & & & & 0 & 1 \\
\psi_i^n & & & & & & 1 & 0.
\end{array}$$

This is a block sum of n copies of the hyperbolic form $\mathbb{H}$. If d is odd, the final block of $\beta_{\mathcal{B}'}$ (corresponding to the basis elements $\{a_{\frac{d+1}{2}}b_1, \dots, a_{\frac{d+1}{2}}b_n\}$) is simply A_d. It follows that β is the direct sum of hyperbolic forms, along with a direct summand of $\hat{A}_d$ when d is odd. $\square$

In Section 5.5, we will use Lemma 5.4.5 to compare the local $\mathbb{A}^1$-degree of a function f with the transfer of the local $\mathbb{A}^1$-degree of the lift of f.

5.4.3 Horner bases

Many of our calculations in Section 5.5 involve choosing convenient bases of quotients of polynomial rings. The *Horner basis*, defined below, is a basis which is dual to the monomial basis with respect to the Scharlau form (see Proposition 5.4.7); this fact will be useful when we prove Lemma 5.5.6. We will collect a few definitions and results from (BPR06) for later use.

Definition 5.4.6. (BPR06, Notation 8.6) Let $m(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0 \in k[x]$. Define the *Horner polynomials*

$$\mathrm{Hor}_i(m,x) := \begin{cases} 1 & i = 0, \\ x\,\mathrm{Hor}_{i-1}(m,x) + a_{n-i} & 1 \le i < n. \end{cases}$$

The set $\{\mathrm{Hor}_{n-1}(m,x), \mathrm{Hor}_{n-2}(m,x), \ldots, \mathrm{Hor}_0(m,x)\}$ forms a k-basis of $k[x]/(m)$, which is called the *Horner basis.*

Let $s : k[x]/m(x) \to k$ be the Scharlau form associated to the primitive element x. The following proposition states that s is a dualizing form for the monomial and Horner bases, in the sense of (BMP21b, Definition 2.1).

Proposition 5.4.7. (BPR06, Proposition 9.18) Let $0 \le i, j \le n-1$. Then

$$s(x^i\,\mathrm{Hor}_{n-1-j}(m,x)) = \begin{cases} 1 & i = j, \\ 0 & i \ne j. \end{cases}$$

Proof. Since we have assumed that $m(x)$ is monic, the Kronecker form mentioned in *loc. cit.* is equal to the Scharlau form. $\square$

By (BMP21b, Proposition 3.5(2)), the Scharlau form gives a straightforward way to write down elements of $k[x]/m(x)$ in terms of the Horner basis. This is also proved directly in (BPR06, Corollary 9.19).

Corollary 5.4.8. For any $g \in k[x]/m(x)$, we have

$$g(x) \equiv \sum_{i=0}^{n-1} s(x^i g(x)) \operatorname{Hor}_{n-1-i}(m, x) \bmod (m(x)).$$

We now show that there is a close connection between the Bézoutian of m and the Horner basis associated to m. In the language of (BMP21b, Definition 3.8), we will demonstrate that the bilinear form induced by the Scharlau form is in fact a Bézoutian bilinear form.

Proposition 5.4.9. We have an equality in $k[X, Y]$ of the form

$$\frac{m(X) - m(Y)}{X - Y} = \sum_{i=0}^{n-1} X^i \operatorname{Hor}_{n-1-i}(m, Y).$$

Proof. Since $m(x) = \sum_{i=0}^{n} a_i x^i$ is a polynomial, its Bézoutian can be written as

$$\frac{m(X) - m(Y)}{X - Y} = \sum_{\ell=1}^{n} a_\ell \left(\sum_{i+j=\ell-1} X^i Y^j \right) = \sum_{i+j=0}^{n-1} a_{i+j+1} X^i Y^j.$$

Next, the coefficient of Y^j in $\operatorname{Hor}_i(m, Y)$ is a_{n+j-i} when $i \geq j$, and is zero otherwise. In particular, the coefficient of $X^i Y^j$ in $X^i \operatorname{Hor}_{n-1-i}(m, Y)$ is a_{i+j+1}. Thus the coefficients of $X^i Y^j$ in $\operatorname{Béz}(m)$ and $\sum_{i=0}^{n-1} X^i \operatorname{Hor}_{n-1-i}(m, Y)$ agree. $\square$

To conclude this section, we will relate the coefficients of the Bézoutian in the Horner basis to the coefficients of the Scharlau transfer in the monomial basis. Since the Scharlau transfer is equal to the geometric transfer for finite simple extensions (Lemma 5.3.4), the following result will be useful when computing a geometric transfer in Lemma 5.5.6. See also (BPR06, Proposition 9.20).

Proposition 5.4.10. Let L/k be a finite simple extension with primitive element t, and let $m(x) \in k[x]$ be the minimal polynomial of t. Given any $u(x) \in L[x]$, the coefficient matrix of $u(X)\frac{m(X)-m(Y)}{X-Y}$ in the Horner basis is equal to the coefficient matrix of $s_*\langle u(t)\rangle$ in the monomial basis.

Proof. By Proposition 5.4.9, we have

$$\frac{m(X)-m(Y)}{X-Y} = \sum_{i=0}^{n-1} X^i \operatorname{Hor}_{n-1-i}(m,Y).$$

Multiplying both sides by $u(X)$, we obtain

$$u(X)\frac{m(X)-m(Y)}{X-Y} = \sum_{i=0}^{n-1} u(X)X^i \operatorname{Hor}_{n-1-i}(m,Y). \tag{5.4.11}$$

Since $u(X)X^i = \sum_j s\left(u(X)X^{i+j}\right) \operatorname{Hor}_{n-1-j}(m,X)$ by Corollary 5.4.8, we can rewrite Equation 5.4.11 as

$$\sum_{i,j=0}^{n-1} s\left(u(X)X^{i+j}\right) \operatorname{Hor}_{n-1-i}(m,X) \operatorname{Hor}_{n-1-j}(m,Y).$$

On the other hand, the coefficient matrix of $s_*\langle u(t)\rangle$ in the monomial basis is given by $\big(s(u(t)t^{i+j})\big)_{i,j=0}^{n-1}$. Since $k(t) = k[X]/m(X)$, we have $s(u(X)X^{i+j}) = s(u(t)t^{i+j}) \in k$, as desired. $\square$

5.5 Lifts of univariate maps and transfers of local degrees

Given a map $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ with a non-rational isolated zero p, we would like to compute the local degree $\deg_p(f) \in \mathrm{GW}(k)$ by lifting f to a map $\widetilde{f} : \mathbb{A}^n_{k(p)} \to \mathbb{A}^n_{k(p)}$ with rational isolated zero $\widetilde{p}$, computing $\deg_{\widetilde{p}}(\widetilde{f}) \in \mathrm{GW}(k(p))$, and applying the appropriate transfer $\mathrm{GW}(k(p)) \to \mathrm{GW}(k)$. If $k(p)/k$ is a finite, separable extension, one may take $\widetilde{f}$ to be the base change $f_{k(p)}$ (BBM+21). However, if $k(p)/k$ is finite and purely inseparable, lifting f to $f_{k(p)}$ yields a local degree whose rank is too large, as illustrated in Example 5.5.1.

Example 5.5.1. Let $k = \mathbb{F}_p(t)$ for some prime $p > 2$, and let $f : \mathbb{A}^1_k \to \mathbb{A}^1_k$ be given by $f(x) = (x^p - t)^d$, where $d \geq 1$ is an integer. Take $q \in \mathbb{A}^1_k$ to be the non-rational point defined by the ideal $(x^p - t) \subset \mathbb{F}_p(t)[x]$, and note that $k(q) = \mathbb{F}_p(t^{1/p})$. Let $\widetilde{q} = (x - t^{1/p})$ be the $k(q)$-rational lift of q. By (SS75, p. 182) (and e.g. (BMP21b, Theorem 5.1)), we have

$$\operatorname{rank}(\deg_q(f)) = \dim_k \frac{k[x]_q}{(f)},$$
$$\operatorname{rank}(\deg_{\widetilde{q}}(f_{k(q)})) = \dim_{k(q)} \frac{k(q)[x]_{\widetilde{q}}}{(f_{k(q)})}.$$

Since f is a polynomial of degree pd lying in the maximal ideal $(x^p - t)$, we observe that $\dim_k k[x]_q/(f) = pd$. The freshman's dream implies $f_{k(q)} = (x - t^{1/p})^{pd}$, so it follows that $\dim_{k(q)} k(q)[x]_{\widetilde{q}}/(f_{k(q)}) = pd$ as well. Applying the geometric (equivalently, Scharlau) transfer $\tau_k^{k(q)}(t^{1/p}) = s_* : \mathrm{GW}(k(q)) \to \mathrm{GW}(k)$ scales rank by $[k(q) : k]$, so

$$\operatorname{rank}(s_* \deg_{\widetilde{q}}(f_{k(q)})) > \operatorname{rank}(\deg_q(f)).$$

This too-high rank issue arises from the splitting of the minimal polynomial $m(x)$ of q. Any morphism $f : \mathbb{A}^1_k \to \mathbb{A}^1_k$ vanishing at q must be a multiple of m. If $k(q)/k$ is purely inseparable, then all linear factors of $m_{k(q)}$ are contained in the ideal $\widetilde{q}$ and are hence not invertible in $k(q)[x]_{\widetilde{q}}$. This stands in contrast with the separable case, where all but one linear factor of $m_{k(q)}$ are not contained in $\widetilde{q}$ and are hence invertible in the relevant local ring. The invertibility of these factors of $m_{k(p)}$ causes the desired drop in dimension when constructing the quotient ring $k(q)[x]_{\widetilde{q}}/(f_{k(q)})$.

As motivated by Proposition 5.3.19, we would like to look for a suitable lift of f.

Notation 5.5.2. Throughout Section 5.5, let $p \in \mathbb{A}^1_k$ be a closed point with corresponding minimal polynomial $m(x) \in k[x]$. Since $\mathbb{A}^1_k = \operatorname{Spec} k[x]$, the residue field

$L := k(p)$ is a finite simple extension of k. Let t be a primitive element of L/k. The canonical point $\widetilde{p} \in \mathbb{A}^1_L$ is the point corresponding to the ideal $(x - t) \subset L[x]$. We fix $f(x) \in k[x]$ to be a polynomial vanishing at p, written uniquely as $f(x) = u(x)m(x)^d$, where $u(x)$ is not contained in the ideal corresponding to p (that is, u is non-vanishing at p).

5.5.1 Geometric lifts of univariate polynomials

We now describe how to lift univariate polynomials relative to geometric and cohomological transfers. We begin with geometric lifts.

Definition 5.5.3. Let $f(x) = u(x)m(x)^d$ and p be as in Notation 5.5.2. The *geometric lift* of f at the point p is the polynomial

$$f_{\mathfrak{g}}(x) := u(x)(x - t)^d \in L[x].$$

Now that we have defined the geometric lift of f at p, we can compute its local $\mathbb{A}^1$-degree.

Lemma 5.5.4. Let $f(x) = u(x)m(x)^d$ and p be as in Notation 5.5.2. Then, as elements of $\mathrm{GW}(L)$, we have

$$\deg^{\mathbb{A}^1}_{\widetilde{p}}(f_{\mathfrak{g}}) = \begin{cases} \frac{d}{2}\mathbb{H} & d \text{ is even} \\ \langle u(t)\rangle + \frac{d-1}{2}\mathbb{H} & d \text{ is odd.} \end{cases}$$

Proof. The Bézoutian of $f_{\mathfrak{g}}$ at $\widetilde{p}$ will be an element of the algebra

$$\frac{L[X]_{(X-t)}}{(u(X)(X-t)^d)} \otimes \frac{L[Y]_{(Y-t)}}{(u(Y)(Y-t)^d)} \cong \frac{L[X]_{(X-t)}}{((X-t)^d)} \otimes \frac{L[Y]_{(Y-t)}}{((Y-t)^d)}.$$

We expand the Bézoutian as

$$\begin{aligned}
\text{Béz}(f_{\mathfrak{g}}) &= \frac{u(X)(X-t)^d - u(Y)(Y-t)^d}{X-Y} \\
&= \frac{u(X)(X-t)^d - u(Y)(Y-t)^d}{X-Y} + \frac{u(X)(Y-t)^d - u(X)(Y-t)^d}{X-Y} \\
&= u(X)\frac{(X-t)^d - (Y-t)^d}{(X-t)-(Y-t)} + \frac{u(X)-u(Y)}{X-Y}(Y-t)^d \\
&\equiv u(X)\frac{(X-t)^d - (Y-t)^d}{(X-t)-(Y-t)} \bmod ((X-t)^d, (Y-t)^d) \\
&= u(X)\left(\sum_{i=0}^{d-1}(X-t)^i(Y-t)^{d-1-i}\right).
\end{aligned}$$

Our next goal is to write $\text{Béz}(f_{\mathfrak{g}})$ with respect to the basis $\{(x-t)^{d-1}, (x-t)^{d-2}, \ldots, (x-t), 1\}$ of $L[x]_{(x-t)}/((x-t)^d)$. In order to do so, we must expand $u(x) \mod (x-t)^d$ in this basis. This is done using a truncated Taylor series expansion. Let $u^{(i)}$ denote the i^{th} Hasse derivative of $u(x)$. Then $\sum_{i=0}^{d-1} u^{(i)}(t)(x-t)^i \equiv u(x) \bmod (x-t)^d$, so

$$\text{Béz}(f_{\mathfrak{g}}) = \left(\sum_{i=0}^{d-1} u^{(i)}(t)(X-t)^i\right)\left(\sum_{j=0}^{d-1}(X-t)^j(Y-t)^{d-1-j}\right).$$

It follows that the Bézoutian bilinear form of $f_{\mathfrak{g}}$ with respect to the basis $\{(x -$

$t)^{d-i}\}_{i=1}^{d}$ is

$$\begin{array}{c|cccccc} & (X-t)^{d-1} & (X-t)^{d-2} & \cdots & (X-t) & 1 \\ \hline (Y-t)^{d-1} & u^{(d-1)}(t) & u^{(d-2)}(t) & \cdots & u^{(1)}(t) & u(t) \\ (Y-t)^{d-2} & u^{(d-2)}(t) & u^{(d-3)}(t) & \cdots & u(t) & 0 \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ (Y-t) & u^{(1)}(t) & u(t) & \cdots & 0 & 0 \\ 1 & u(t) & 0 & \cdots & 0 & 0. \end{array} \tag{5.5.5}$$

Since $u(x)$ is not an element of the maximal ideal $m(x) \cdot k[x]$, it cannot be an element of the maximal ideal $(x-t) \cdot L[x]$. In particular, $u(t) \neq 0$, so the result follows from Proposition 5.4.3. □

Corollary 5.1.2 now follows from Lemma 5.5.4.

Proof of Corollary 5.1.2. Apply Lemma 5.3.6 to Equation 5.5.5. Conclude with Lemma 5.4.5 to block diagonalize the bilinear form. □

Since we have computed $\deg_{\widetilde{p}}^{\mathbb{A}^1}(f_{\mathfrak{g}})$, we can compare its geometric transfer to $\deg_p^{\mathbb{A}^1}(f)$.

Lemma 5.5.6. The geometric lift is compatible with the local degree and geometric transfer. That is, $\tau_k^{k(p)}(t)\left(\deg_{\widetilde{p}}^{\mathbb{A}^1}(f_{\mathfrak{g}})\right) = \deg_p^{\mathbb{A}^1}(f)$ in $\mathrm{GW}(k)$.

Proof. Using the same idea as in the proof of Lemma 5.5.4, we have

$$\begin{aligned}
\mathrm{Béz}(f) &= \frac{u(X)m(X)^d - u(Y)m(Y)^d}{X-Y} \\
&\equiv u(X)\frac{m(X)^d - m(Y)^d}{X-Y} \bmod (m(X)^d, m(Y)^d) \\
&= u(X)\frac{m(X)^d - m(Y)^d}{m(X)-m(Y)} \cdot \frac{m(X)-m(Y)}{X-Y}.
\end{aligned}$$

For $0 \leq j < n$, let $H_j(x) := \mathrm{Hor}_j(m, x)$ be the j^{th} Horner polynomial associated to $m(x)$ (as defined in Definition 5.4.6), and let

$$\mathcal{B}_i(x) = \{H_{n-1}(x)m(x)^{d-1-i}, H_{n-2}(x)m(x)^{d-1-i}, \ldots, H_0(x)m(x)^{d-1-i}\}.$$

Note that $\mathcal{B}(x) := \bigcup_{i=0}^{d-1} \mathcal{B}_i(x)$ is a k-basis of $k[x]_{(m)}/(f) \cong k[x]_{(m)}/(m^d)$, since all elements of this set have distinct polynomial degree. Collecting powers of $m(X)$ and $m(Y)$, we have

$$\mathrm{Béz}(f) \equiv u(X)\frac{m(X)-m(Y)}{X-Y}\left(\sum_{i=0}^{d-1} m(X)^i m(Y)^{d-1-i}\right) \bmod (m(X)^d, m(Y)^d).$$

In this expansion, each summand of $\mathrm{Béz}(f)$ is divisible by $m(X)^i m(Y)^{d-1-i}$. In particular, in the basis $\mathcal{B}(X) \times \mathcal{B}(Y)$, the matrix of coefficients of $\mathrm{Béz}(f)$ is block upper left triangular, where the $(i,j)^{\text{th}}$ block corresponds to the coefficients of the basis elements $\mathcal{B}_i(X) \times \mathcal{B}_j(Y)$. By Lemma 5.4.5, it suffices to compare the blocks of the coefficient matrix of $\mathrm{Béz}(f)$ along the main anti-diagonal to those appearing in $\tau_k^L(t)(\deg_{\widetilde{p}}^{\mathbb{A}^1}(f_{\mathfrak{g}}))$. The blocks appearing along this diagonal consists of the coefficients of $u(X)\frac{m(X)-m(Y)}{X-Y}$ mod $(m(X),m(Y))$ expanded in the Horner basis $\{H_{n-1}(X),\ldots,H_0(X)\}\times\{H_{n-1}(Y),\ldots,H_0(Y)\}$, because the coefficients of any terms of $u(X)\frac{m(X)-m(Y)}{X-Y}$ that are divisible by $m(X)$ or $m(Y)$ will be shifted to blocks above the main anti-diagonal. This is exactly the Gram matrix of $\tau_k^L(t)\langle u(t)\rangle$ (see Equation 5.5.5) by Proposition 5.4.10. The desired result now follows from Lemma 5.5.4. $\square$

Remark 5.5.7 (*Unstable degree*)**.** We expect that Lemma 5.5.6 holds *unstably.* While Morel's $\mathbb{A}^1$-degree homomorphism

$$\deg^{\mathbb{A}^1} : [(\mathbb{P}^1_k)^{\wedge n},(\mathbb{P}^1_k)^{\wedge n}]_{\mathcal{H}_\bullet(k)} \to \mathrm{GW}(k)$$

is an isomorphism for $n \geq 2$, this map is only an epimorphism for $n = 1$ (Mor12). Building on the work of Morel (Mor06, p. 1037), Cazanave showed that

$$(\deg^{\mathbb{A}^1}, \det \mathrm{B\acute{e}z}) : [\mathbb{P}^1_k,\mathbb{P}^1_k]_{\mathcal{H}_\bullet(k)} \to \mathrm{GW}(k) \times_{k^\times/k^{\times 2}} k^\times$$

is an isomorphism (Caz12), where $\mathrm{B\acute{e}z}(f)$ is the Bézoutian bilinear form of the rational map f. Moreover, the $\mathbb{A}^1$-degree of f is the isomorphism class of $\mathrm{B\acute{e}z}(f)$, so $(\mathrm{B\acute{e}z}, \det \mathrm{B\acute{e}z})$ can be regarded as the *unstable* $\mathbb{A}^1$*-degree*.

In the proof of Lemma 5.5.6, we showed that $\mathrm{B\acute{e}z}_p(f)$ and $\tau_k^{k(p)}(t)\,(\mathrm{B\acute{e}z}_t(f_{\mathfrak{g}}))$ represent the same class in $\mathrm{GW}(k)$. However, we also showed that $\det \mathrm{B\acute{e}z}_p(f) = (\det \mathrm{B\acute{e}z}_t(f_{\mathfrak{g}}))^{[k(p):k]}$. Thus if the geometric transfer $\tau_k^{k(p)}(t) : \mathrm{GW}(k(p)) \to \mathrm{GW}(k)$

can be extended to an "unstable transfer"

$$\left(\tau_k^{k(p)}(t), \phi\right) : \mathrm{GW}(k(p)) \times_{k(p)^\times/k(p)^{\times 2}} k(p)^\times \to \mathrm{GW}(k) \times_{k^\times/k^{\times 2}} k^\times$$

such that $\phi(a) = a^{[k(p):k]}$ for any $a \in k^\times$, then the geometric lift will be compatible with the unstable local degree and unstable transfer:

$$\left(\tau_k^{k(p)}(t)(\deg_{\widetilde{p}}^{\mathbb{A}^1}(f_{\mathfrak{g}})), \phi(\det \mathrm{Béz}_{\widetilde{p}}(f_{\mathfrak{g}}))\right) = (\deg_p^{\mathbb{A}^1}(f), \det \mathrm{Béz}_p(f)).$$

5.5.2 Cohomological lifts of univariate polynomials

As discussed earlier, geometric transfers do not behave well with respect to composite field extensions. One can rectify this issue by twisting geometric transfers, which leads to the notion of cohomological transfers. In Lemma 5.5.6, we saw that the geometric transfer of the local $\mathbb{A}^1$-degree at $\widetilde{p}$ of the geometric lift of f is the local degree of f at p. Analogously, we will define the *cohomological lift* of f by twisting the geometric lift. We will also prove that the cohomological lift is compatible with the cohomological transfer.

Definition 5.5.8. Let $f(x) = u(x)m(x)^d$ and p be as in Notation 5.5.2. The *cohomological lift* of f at p is the polynomial

$$f_{\mathfrak{c}}(x) := \omega_0(x)^d u(x)(x-t)^d \in L[x],$$

where $\omega_0(x)$ is the polynomial associated to the extension L/k defined in Notation 5.3.7.

Corollary 5.5.9. The cohomological lift is compatible with the local $\mathbb{A}^1$-degree and cohomological transfer. That is, $\mathrm{Tr}_k^{k(p)} \deg_{\widetilde{p}}^{\mathbb{A}^1}(f_{\mathfrak{c}}) = \deg_p^{\mathbb{A}^1}(f)$.

Proof. Since $m_0(x)$ is a separable polynomial, $\omega_0(x)$ is non-vanishing at t. Lemma 5.5.4 thus implies that

$$\begin{aligned}
\deg_{\widetilde{p}}^{\mathbb{A}^1}(f_{\mathfrak{c}}) &= \begin{cases} \frac{d}{2}\mathbb{H} & d \text{ is even} \\ \left\langle \omega_0(t)^d u(t) \right\rangle + \frac{d}{2}\mathbb{H} & d \text{ is odd} \end{cases} \\
&= \begin{cases} \frac{d}{2}\mathbb{H} & d \text{ is even} \\ \langle \omega_0(t) u(t) \rangle + \frac{d}{2}\mathbb{H} & d \text{ is odd} \end{cases} \\
&= \langle \omega_0(t) \rangle \deg_{\widetilde{p}}^{\mathbb{A}^1}(f_{\mathfrak{g}}).
\end{aligned}$$

The result now follows from Definition 5.3.10 and Lemma 5.5.6. □

Proposition 5.5.10. Assume that $k(p)/k$ is separable. Then the cohomological lift of f at p is the base change $f_{k(p)}$.

Proof. This follows from the observation that $\omega_0(x)(x-t) = m_0(x) = m(x)$ in this setting. □

For finite separable extensions, the cohomological transfer is equal to the field trace on Grothendieck–Witt groups (CF17, Lemma 2.3). By Proposition 5.5.10, we have that Corollary 5.5.9 recovers the main result of (BBM^{+}21) for univariate maps.

Example 5.5.11. The cohomological lift and geometric lift of a polynomial agree at a point with purely inseparable residue field by Example 5.3.8.

Example 5.5.12. Consider the polynomial $f(x) = (x+2)(x-2)(x^2+1)^3 \in \mathbb{R}[x]$, vanishing at (x^2+1). We have that the geometric lift of f is

$$f_{\mathfrak{g}} = (x+2)(x-2)(x-i)^3,$$

while $\omega_0(x) = (x+i)$, so that $f_{\mathfrak{c}}(x) = f_{\mathbb{C}}(x)$.

5.5.3 Trace forms and Scharlau forms

Given a finite separable extension L/k, the *trace form* $(x,y) \mapsto \mathrm{Tr}_{L/k}(xy)$ is an important invariant of the extension; see (CP84) for a survey. Post-composition with the field trace induces a homomorphism $\mathrm{GW}(L) \to \mathrm{GW}(k)$, which coincides with the cohomological transfer.

Proposition 5.5.13. (CF17, Lemma 2.3) Let L/k be a finite separable field extension. Then post-composition with the field trace $\mathrm{Tr}_{L/k} : L \to k$ induces the cohomological transfer

$$\begin{aligned} \mathrm{Tr}_k^L : \mathrm{GW}(L) &\to \mathrm{GW}(k) \\ \left[V \times V \xrightarrow{\beta} L\right] &\mapsto \left[V \times V \xrightarrow{\beta} L \xrightarrow{\mathrm{Tr}_{L/k}} k\right]. \end{aligned}$$

Similarly, associated to each $a \in L^\times$ is the *scaled trace form* $(x, y) \mapsto \mathrm{Tr}_{L/k}(axy)$. Since the field trace induces the cohomological transfer for finite separable extensions, (scaled) trace forms are of the form $\mathrm{Tr}_k^L\langle a\rangle$.

Definition 5.5.14. Let L/k be a finite separable extension with primitive element t. Recall that the geometric transfer is equal to the Scharlau transfer (Lemma 5.3.4). In analogy with (scaled) trace forms, we define the *(scaled) Scharlau form* associated to $a \in L^\times$ as $\tau_k^L(t)\,\langle a\rangle$.

We will show that the isomorphism class of any (scaled) trace form or Scharlau form along a finite separable field extension L/k is given by a local $\mathbb{A}^1$-degree. Paired with the main result of (BMP21b), we obtain a straightforward computational formula for the isomorphism class of any scaled trace form or Scharlau form in the separable setting. We first recall a result that allows us to relate cohomological and geometric transfers in the separable setting.

Proposition 5.5.15. (Hoy14, Lemma 5.8) Let L/k be a finite separable extension with primitive element t. Let $m(x) \in k[x]$ be the minimal polynomial of t. Then for any $\beta \in \mathrm{GW}(L)$, we have $\mathrm{Tr}_k^L(\beta) = \tau_k^L(t)\left(\langle m'(t)\rangle \cdot \beta\right)$.

Proof. Since L/k is separable, we have $\omega_0(t) = m_0(t) = m'(t)$. The result thus follows from Definition 5.3.10. $\square$

After giving a definition, we will be ready to show that scaled trace forms are in fact local $\mathbb{A}^1$-degrees.

Definition 5.5.16. Let L/k be a finite simple field extension with primitive element t. Given $a \in L$, we then have $a = \sum_{i=0}^{[L:k]-1} a_i t^i$, with $a_i \in k$ uniquely determined (since t is fixed). Define $a(x) := \sum_{i=0}^{[L:k]-1} a_i x^i \in k[x]$.

Proposition 5.5.17 (*Scaled Scharlau forms are $\mathbb{A}^1$-degrees*)**.** Let L/k be a finite separable extension with primitive element t, and let $m(x) \in k[x]$ be the minimal polynomial of t. Let $p \in \mathbb{A}_k^1$ be the closed point defined by $m(x)$. Let $a \in L^\times$. Then

$$\tau_k^L(t)\langle a\rangle = \deg_p^{\mathbb{A}^1}(a(x)m(x)).$$

Proof. Let $h(x) = a(x)m(x)$. By Proposition 5.5.10, we have that $m(x) = \omega_0(x)(x - t)$. Since $a(x)$ is non-vanishing at t, the cohomological lift of $h(x)$ is simply the base change $h_{\mathfrak{c}}(x) = h_L(x)$. By (KW19, Proposition 15), its local degree at t is

$$\begin{aligned}
\deg_t^{\mathbb{A}^1}(h_L) &= \left\langle \frac{d}{dx} h_L(x)\Big|_{x=t} \right\rangle \\
&= \langle a'(x)m(x) + a(x)m'(x)|_{x=t}\rangle \\
&= \langle a(t)m'(t)\rangle .
\end{aligned}$$

Applying the cohomological transfer and invoking Corollary 5.5.9, we have $\deg_p^{\mathbb{A}^1}(h) = \operatorname{Tr}_k^L \deg_t^{\mathbb{A}^1}(h_{\mathfrak{c}})$. Combining this with Proposition 5.5.15 concludes the proof. $\square$

Example 5.5.18. The Scharlau form $\tau_k^{k(p)} \langle 1\rangle$ is the local degree of the minimal polynomial of p at the point p. This is also equal to the global degree of the minimal polynomial by Corollary 5.4.2. This indicates that unscaled Scharlau forms are uninteresting, in the sense that they are either entirely hyperbolic or hyperbolic plus a summand of $\langle 1\rangle$.

Proposition 5.5.19 (*Scaled trace forms are $\mathbb{A}^1$-degrees*)**.** Let L/k be a finite separable extension with primitive element t, and let $m(x) \in k[x]$ be the minimal polynomial of t. Let $a \in k(p)^\times$. Then

$$\operatorname{Tr}_k^L \langle a\rangle = \deg_p^{\mathbb{A}^1}(a(x)m'(x)m(x)).$$

Proof. Let $h(x) = a(x)m'(x)m(x)$. The geometric lift is given by $h_{\mathfrak{g}}(x) = a(x)m'(x)(x-$

t), so the local degree of $h_{\mathfrak{g}}$ at t is

$$\begin{aligned}\deg_t^{\mathbb{A}^1}(h_{\mathfrak{g}}) &= \left\langle \frac{d}{dx} a(x)m'(x)(x-t)\bigg|_{x=t} \right\rangle \\ &= \langle a(t)m'(t)\rangle .\end{aligned}$$

Combining this with Proposition 5.5.15, we have that

$$\deg_p^{\mathbb{A}^1}(h) = \tau_k^L(t)\left(\deg_t^{\mathbb{A}^1}(h_{\mathfrak{g}})\right) = \mathrm{Tr}_k^L \langle a\rangle . \qquad \square$$

Example 5.5.20. Let $K = \mathbb{Q}(\sqrt[3]{2})$ with minimal polynomial $m(x) = x^3 - 2$. The extension $K/\mathbb{Q}$ has trace form

$$\begin{aligned}\mathrm{Tr}_{K/\mathbb{Q}}\langle 1\rangle &= \left(\mathrm{Tr}_{K/\mathbb{Q}}(\sqrt[3]{2}^i \cdot \sqrt[3]{2}^j)\right)_{0\leq i,j\leq 2} \\ &= \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 6 \\ 0 & 6 & 0 \end{pmatrix} \\ &= \langle 3\rangle + \mathbb{H}.\end{aligned}$$

Using the code provided in (BMP21a), we verify that $\deg_{\sqrt[3]{2}}^{\mathbb{A}^1}(m'(x)\cdot m(x)) = \langle 3\rangle + \mathbb{H}$.

Remark 5.5.21. Given any irreducible polynomial $m(x) \in k[x]$ (defining a finite simple field extension L/k) and any unit $a \in L^\times$, we can readily compute the scaled trace form $\mathrm{Tr}_k^L \langle a\rangle$ using Proposition 5.5.19 together with the Sage code provided in (BMP21a).

5.6 Appendix: Pictorial intuition for diagonalization arguments

Suppose we are given a symmetric bilinear form that can be represented by an upper left triangular Hankel matrix. The intuition behind the proof of Proposition 5.4.3 is that the data of the matrix can be repackaged into "upper-left corners." To illustrate what we mean by this, consider the 5×5 example illustrated in Figure 5.1.

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_2 & a_3 & a_4 & a_5 & \\ a_3 & a_4 & a_5 & & \\ a_4 & a_5 & & & \\ a_5 & & & & \end{pmatrix}$$

Figure 5.1: Upper triangular Hankel matrix

Given a vector space basis $\{x_1, \ldots, x_5\}$, this matrix defines a bilinear form by

$$\sum_{i,j} a_{i+j-1} x_i \otimes x_j.$$

Consider all the terms with a factor of x_1, illustrated in red (or the darkest shade in grayscale). The top row is given by $x_1 \otimes (a_1 x_1 + \ldots + a_5 x_5)$; by symmetry, the first column is given by $(a_1 x_1 + \ldots + a_5 x_4) \otimes x_1$. Note that $a_1 x_1^{\otimes 2}$ is double counted, so we define a new basis element

$$\psi_1 = \frac{a_1}{2} x_1 + a_2 x_2 + \ldots + a_5 x_5.$$

In this terminology, the first corner of the matrix (highlighted in red) can be rewritten as $x_1 \otimes \psi_1 + \psi_1 \otimes x_1$. Similarly, for the second corner (highlighted in cyan, or the medium shade in grayscale), we can define

$$\psi_2 = \frac{a_3}{2} x_2 + a_4 x_3 + a_5 x_4.$$

Then the cyan portion of the form is $x_2 \otimes \psi_2 + \psi_2 \otimes x_2$. Finally, we are left with the lone term in yellow (or the lightest shade in grayscale), which is $a_5 x_3^{\otimes 2}$. We can thus define a new basis $\{x_1, \psi_1, x_2, \psi_2, x_3\}$. In this basis, our form can be written as

$$x_1 \otimes \psi_1 + \psi_1 \otimes x_1 + x_2 \otimes \psi_2 + \psi_2 \otimes x_2 + a_5 x_3^{\otimes 2},$$

so the isomorphism class of this form is $2\mathbb{H} + \langle a_5 \rangle$.

Note that the Hankel structure was not used in this discussion — we only needed symmetry and upper left triangularity.

Remark 5.6.1. The proof of Proposition 5.4.3 holds when the matrix is symmetric and upper left triangular, so the Hankel assumption is unnecessary.

Passing to a more general case, replace the each a_i with a block matrix A_i (see Figure 5.2). We will use the same idea to diagonalize this matrix. If there is an odd number of blocks along the diagonal, we will stop our modifications short of the central block.

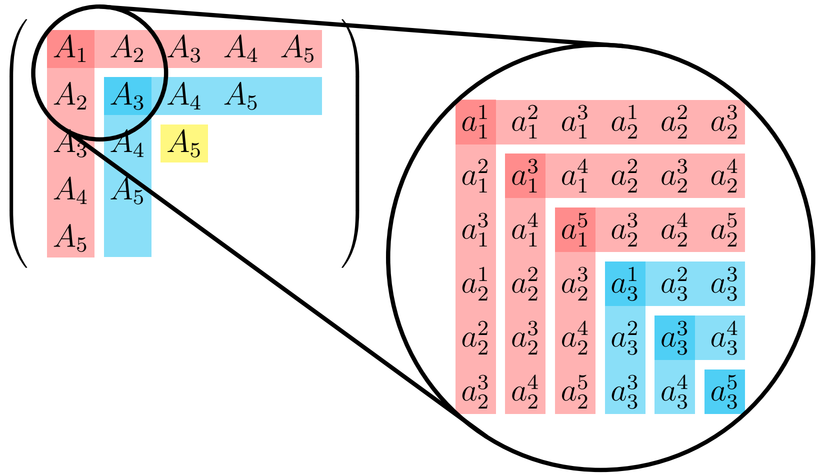

Figure 5.2: Block upper triangular Hankel matrix

We can now clarify the intuition behind the choice of

$$\psi_i^\ell = \underbrace{\frac{\beta_i^{2\ell-1}}{2} a_i b_\ell}_{\text{(i)}} + \underbrace{\sum_{k=\ell+1}^{d} \beta_i^{2\ell-1+k} a_i b_k}_{\text{(ii)}} + \underbrace{\sum_{j=i+1}^{n} \sum_{k=1}^{d} \beta_j^{k+\ell-1} a_j b_k}_{\text{(iii)}},$$

which we used to diagonalize the block form in Lemma 5.4.5. The term (i) is the term lying on the diagonal in the i^{th} block on the ℓ^{th} row. The sum (ii) travels horizontally from the term on the diagonal until it reaches the edge of the block. Finally, the double sum (iii) continues the row to the right across all the other remaining blocks.

We can now decompose our form as a sum of hyperbolic forms $\sum_{i,\ell} a_i b_\ell \otimes \psi_i^\ell + \psi_i^\ell \otimes a_i b_\ell$. If there is an odd number of blocks, this decomposition will leave the central block (in this example, a copy of A_5) alone.

Remark 5.6.2. Again, we did not use any Hankel structure in this argument. In particular, the statement of Lemma 5.4.5 holds when the matrix is any symmetric matrix that is block upper left triangular.

Chapter 6

An enriched degree of the Wronski map

Abstract: Given mp different p-planes in general position in $(m+p)$-dimensional space, a classical problem is to ask how many p-planes intersect all of them. For example when $m = p = 2$, this is precisely the question of "lines meeting four lines in 3-space" after projectivizing. The Brouwer degree of the Wronski map provides an answer to this general question, first computed by Schubert over the complex numbers and Eremenko and Gabrielov over the reals. We provide an enriched degree of the Wronski map for all m and p even, valued in the Grothendieck–Witt ring of a field, using machinery from $\mathbb{A}^1$-homotopy theory. We further demonstrate in all parities that the local contribution of an m-plane is a determinantal relationship between certain Plücker coordinates of the p-planes it intersects.

6.1 Introduction

Given m functions $f_1(t), \ldots, f_m(t)$ of degree equal to $m+p-1$, we define their *Wronskian*

$$\mathrm{Wr}(f_1, \ldots, f_m)(t) := \begin{vmatrix} f_1(t) & f_2(t) & \cdots & f_m(t) \\ f_1'(t) & f_2'(t) & \cdots & f_m'(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(m-1)}(t) & f_2^{(m-1)}(t) & \cdots & f_m^{(m-1)}(t) \end{vmatrix}.$$

This is a polynomial of degree at most mp. Let $k_{m+p-1}[t]$ denote the vector space of polynomials of degree at most $m+p-1$ over a field k. We observe that if s is a root of the Wronskian, then the m-plane span $\{f_1, \ldots, f_m\} \subseteq k_{m+p-t}[t]$ intersects the p-plane $E_p(s) = \mathrm{span}\,\{(t-s)^{m+p-1}, \ldots, (t-s)^m\}$ nontrivially. Thus the fiber of the Wronski counts certain m-planes intersecting mp different p-planes. For example when $m = p = 2$, we recover the classical statement that there are two lines meeting four lines in three-space.

We could also envision these polynomials f_i as defining a rational curve by $\mathbb{P}^1 \to \mathbb{P}^{m-1}$, given by $t \mapsto [f_1(t) : \ldots : f_m(t)]$. In this case s is a root of the Wronski if and only if the vectors $\phi(s), \phi'(s), \ldots, \phi^{(m-1)}(s)$ do not span all of $\mathbb{P}^{m-1}$.[1] We say that

[1] This is analogously phrased as inflection of a linear series $V \subseteq \Gamma(\mathbb{P}^1, \mathcal{O}(m+p-1))$. For an investigation of arithmetically-enriched inflection on elliptic and hyperelliptic curves, see (CGL22; CDH20).

ϕ *inflects* at such a point. Thus the fiber of the Wronski counts rational curves of degree $(m+p-1)$ with mp prescribed inflection points. Viewing the polynomials f_i as spanning an m-plane in the $(m+p)$-dimensional vector space of polynomials over k of degree at most $(m+p-1)$, we can consider the Wronski map as a morphism between mp-dimensional varieties

$$\mathrm{Wr} : \mathrm{Gr}_k(m, m+p) \to \mathbb{P}_k^{mp} = \mathrm{Proj}\,(k_{mp}[t]). \tag{6.1.1}$$

In 1886, Schubert (Sch86) formulated the number of m-planes meeting mp general p-planes in $(m+p)$-dimensional space as

$$n_{\mathbb{C}} = \frac{1!2!\cdots(p-1)!(mp)!}{m!(m+1)!\cdots(m+p-1)!}. \tag{6.1.2}$$

This admits a combinatorial description in that it counts the number of *standard Young tableaux* of size $m \times p$. It is also the Brouwer degree of the complex Wronski map (Equation 6.1.1 when $k = \mathbb{C}$). Over the reals, orientation data prevents producing a well-defined integer value for the Brouwer degree of the real Wronski, nonetheless by working on an affine open cell, Eremenko and Gabrielov computed the Brouwer degree of the real Wronski map (Equation 6.1.1 when $k = \mathbb{R}$) (EG01; EG02), which also admits a combinatorial description, being the number of *semi-shifted* standard Young tableaux of size $m \times p$ (HH92; Whi01).

$$n_{\mathbb{R}} = \pm \begin{cases} \frac{1!2!\cdots(p-1)!(m-1)!(m-2)!\cdots(m-p+1)!(mp/2)!}{(m-p+2)!(m-p+4)!\cdots(m+p-2)!\left(\frac{m-p+1}{2}\right)!\left(\frac{m-p+3}{2}\right)!\cdots\left(\frac{m+p-1}{2}\right)!} & m+p \text{ odd} \\ 0 & m+p \text{ even.} \end{cases}$$

We attach the first few values of these for the reader's reference:

p \ m	2	3	4	5
2	2	5	14	42
3	5	42	462	6006
4	14	462	24024	1662804
5	42	6006	1662804	701149020

Figure 6.1: Values of $n_{\mathbb{C}}$

p \ m	2	3	4	5
2	0	1	0	2
3	1	0	2	0
4	0	2	0	12
5	2	0	12	0

Figure 6.2: Values of $|n_{\mathbb{R}}|$

In this paper we unify these two computations into a single enriched Brouwer degree in the case when m and p are both even. The algebrao-geometric analogue of the Brouwer degree that we use is called the $\mathbb{A}^1$-*Brouwer degree*, first defined by Morel (Mor06), which is valued in the Grothendieck–Witt group of symmetric bilinear forms over k. This tool has been instrumental in the development of $\mathbb{A}^1$-*enumerative geometry* (or *enriched enumerative geometry*). This program has grown in recent years due to seminal work of Levine (Lev20), Kass and Wickelgren (KW19), Bachmann and Wickelgren (BW21), among others.

Theorem A. (As Theorem 6.3.29) Let k be any field in which $(m+p-1)!$ is invertible, and let m and p both be even. Then the $\mathbb{A}^1$-degree of the Wronski $\mathrm{Wr}: \mathrm{Gr}_k(m, m+p) \to \mathbb{P}_k^{mp}$ computed on an open affine cell is

$$\deg^{\mathbb{A}^1} \mathrm{Wr} = \frac{n_{\mathbb{C}}}{2}\mathbb{H},$$

where $n_{\mathbb{C}}$ is the Brouwer degree of the Wronski over the complex numbers, and $\mathbb{H}$ denotes the hyperbolic form $\langle 1, -1\rangle$.

Given a closed point $W = \operatorname{span}\{f_1, \ldots, f_m\}$ with Wronskian having roots at distinct scalars $s_1, \ldots, s_{mp} \in k$, we may compute the local degree of the Wronski map in any parities.

Theorem B. (As Theorem 6.4.3) Let k be any field in which $(m+p-1)!$ is invertible, and let $W \in \mathrm{Gr}_k(m, m+p)$ be a closed point whose Wronskian is of the form $\mathrm{Wr}(W)(t) = \prod_{i=1}^{mp}(t - s_i)$ for distinct $s_i \in k$. Then we have that

$$\deg_W^{\mathbb{A}^1}(\mathrm{Wr}) = \langle C \cdot \det \mathcal{B}\rangle\,,$$

where C is a fixed constant depending only on m, p, and the s_i's, and $\mathcal{B}$ is a matrix of distinguished Plücker coordinates of the p-planes $E_p(s_1), \ldots, E_p(s_{mp})$.

The ℓth column of $\mathcal{B}$ consists of mp distinguished Plücker coordinates of the plane $E_p(s_\ell)$, and each row corresponds to the same coordinate. Thus considering the columns as vectors over k, we have that $\det \mathcal{B}$ is a signed volume of vectors determined by the p-planes that span $\{f_1, \ldots, f_m\}$ intersects.

As the Wronski map also counts rational curves with prescribed inflection data, we provide evidence that the local $\mathbb{A}^1$-degree encodes information about the geometry of the associated rational curve. In Corollary 6.4.10 we demonstrate that the local degree at a planar quartic aligns with an enriched *Welschinger invariant* in the sense of (KLSW22).

6.1.1 Outline

In Section 6.2, we provide some historical background for studying the Brouwer degree of the Wronski map, before exploring in greater detail the technical machinery. We discuss the rational normal curve, Grassmann duality, and Plücker coordinates, before providing relevant background from $\mathbb{A}^1$-enumerative geometry. We discuss relative orientations of vector bundles and how the formalism of $\mathbb{A}^1$-enumerative geometry allows one to associate to them a well-defined Euler number valued in Grothendieck–Witt of a ground field.

In Section 6.3, we compare the Wronski map to a section of an appropriate vector bundle over an affine chart on the Grassmannian, and demonstrate that their Brouwer degrees agree up to some global constant. In the case where m and p are both even, we can compute the global $\mathbb{A}^1$-degree of the Wronski map on an affine patch using the fact that the Euler classes of relatively oriented vector bundles with odd rank summands are hyperbolic.

Finally, in Section 6.4, we provide an arithmetic formula for the local $\mathbb{A}^1$-degree of the Wronski map that holds in all parities. We demonstrate that this local index at an m-plane can be interpreted as a "signed volume" of the p-planes that this m-plane intersects. This agrees with and generalizes the local index computed by (SW21). We provide some very preliminary evidence towards a connection between the local $\mathbb{A}^1$-degree of the Wronski map and arithmetic Welschinger invariants a la (KLSW22).

6.1.2 Acknowledgements

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6.2 Preliminaries

We will begin by delving into the Wronski map, understanding its geometric interpretation as counting planes meeting planes of the correct codimension osculating the rational normal curve. By mapping a plane of covectors to the plane it annihilates, we have a natural duality on Grassmannians, and it will benefit us to be able to translate information through this duality, and discuss how it relates to things like Plücker coordinates. After this, we establish some of the foundations of $\mathbb{A}^1$-enumerative geometry, from which we collect the tools to explore the local degree of the Wronski in greater detail.

6.2.1 The rational normal curve

Over the complex numbers, the degree of the Wronski map provides a count of planes which meet a collection of planes, which are said to *osculate the rational normal curve*. We will define these terms, and provide a rough outline of this argument over any field here, but for a more rigorous version of this statement over the complex numbers, we refer the reader to (Sot11, §10.1).

We may view affine space $\mathbb{A}_k^{m+p}$ as the space $k_{m+p-1}[t]$ of polynomials of degree at most $m+p-1$ with coefficients in k by considering a rational point $(a_0, \ldots, a_{m+p-1}) \in \mathbb{A}_k^n$ as a polynomial $a_0 + a_1 t + \ldots + a_{m+p-1} t^{m+p-1} \in k[t]$. We then let $\gamma : \mathbb{A}_k^1 \to \mathbb{A}_k^{m+p}$

denote the *rational normal curve*, also referred to the *moment curve* γ, defined to be the image of the map

$$s \mapsto (1, s, s^2, s^3, \ldots, s^{m+p-1}),$$

where as above we are identifying affine space with a space of polynomials. That is

$$\gamma(s) = 1 + st + s^2t^2 + \ldots + s^{m+p-1}t^{m+p-1} \in k_{m+p-1}[t].$$

We may define the derivative of the rational normal curve by deriving termwise, to obtain

$$\gamma'(s) = (0, 1, 2s, 3s^2, \ldots, (m+p-1)s^{m+p-2}),$$

which corresponds to the polynomial

$$\gamma'(s) = t + 2st^2 + 3s^2t^3 + \ldots + (m+p-1)s^{m+p-2}t^{m+p-1} \in k_{m+p-1}[t].$$

Higher derivatives are defined analogously. One may check that, for any s, the elements $\gamma(s)$, $\gamma'(s)$, ..., $\gamma^{(m+p-1)}(s)$ yield a basis of $k_{m+p-1}[t]$. Thus we obtain an *osculating flag* $F_\bullet(s)$ along the rational curve whose i-plane at any time s is the span:

$$F_i(s) := \text{span}\left\{\gamma(s), \gamma'(s), \ldots, \gamma^{(i-1)}(s)\right\}. \tag{6.2.1}$$

In this setting, we say that the i-plane $F_i(s)$ *osculates the rational normal curve* at the point $\gamma(s)$. We will see in Remark 6.2.5 that $F_m(s)$ is dual in a sense to the planes $E_p(s)$ defined in the introduction.

The monomial basis for polynomials provides an isomorphism between $k_{m+p-1}[t]$ and its dual $k_{m+p-1}[t]^*$, given by sending a polynomial g to g^*, where $g^*(f)$ is defined to be the dot product of the coefficients of g and f. Under this isomorphism we may view each $\gamma^{(i)}(s)$ as a covector, from which perspective it admits an interesting interpretation.

Proposition 6.2.2. Considering $\gamma^{(i)}(s)$ as a covector, we see that it has the interpretation of mapping a polynomial to its ith derivative evaluated at s:

$$\left(\gamma^{(i)}(s)\right)^* (f) = f^{(i)}(s).$$

Proof. We may compute explicitly for $0 \leq j \leq m+p-1$ that

$$\gamma^{(j)}(s) = \sum_{r=j}^{m+p-1} \frac{r!}{(r-j)!} s^{r-j} t^r \in k_{m+p-1}[t].$$

Therefore for any $f(t) = \sum_{i=0}^{m+p-1} a_i t^i$, we have that

$$\left(\gamma^{(j)}(s)\right)^* (f) = \sum_{r=j}^{m+p-1} \frac{r!}{(r-j)!} a_r s^{r-j} = f^{(j)}(s).$$

□

We note that we may write the Wronskian as a determinant of matrices built out of the rational normal curve and the input polynomials. Let $f_1(t), \ldots, f_m(t) \in k_{m+p-1}(t)$ be m linearly independent polynomials of degree at most $m+p-1$, so that their span defines a point on $\mathrm{Gr}_k(m, m+p)$. Let $f_i(t) = \sum_{j=0}^{m+p-1} a_{i,j} t^j$, and define a matrix M comprised of the coefficients of the polynomials f_i:

$$M = \begin{pmatrix} \text{coefficients of } f_1 \\ \text{coefficients of } f_2 \\ \vdots \\ \text{coefficients of } f_m \end{pmatrix} = \begin{pmatrix} a_{1,0} & a_{1,1} & \cdots & a_{1,m+p-1} \\ a_{2,0} & a_{2,1} & \cdots & a_{2,m+p-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,0} & a_{m,1} & \cdots & a_{m,m+p-1} \end{pmatrix}.$$

Let $\Gamma(s)$ denote the matrix $k^m \to k_{m+p-1}[t]$ whose jth column is given by the coefficients of the polynomial $\gamma^{(j-1)}(s) \in k_{m+p-1}[t]$

$$\Gamma(s) = \left(\begin{array}{c|c|c|c} \gamma(s) & \gamma'(s) & \cdots & \gamma^{(m-1)}(s) \end{array} \right).$$

Phrased differently, the columns of $\Gamma(s)$ are the basis vectors spanning the m-plane $F_m(s)$ osculating the rational normal curve at $\gamma(s)$.

Proposition 6.2.3. In the previous notation, one may express the Wronskian of $f_1, \ldots, f_m$ evaluated at a point s as a determinant:

$$\det(M \cdot \Gamma(s)) = \mathrm{Wr}(f_1, \ldots, f_m)(s).$$

Proof. Multiplying a row of M with a column of $\Gamma(s)$ is the same as taking the dot product of $\gamma^{(i)}(s)$ with $f_j(t)$, yielding $f^{(i)}(s)$ by Proposition 6.2.2. It follows then that the determinant of the product of M and $\Gamma(s)$ yields the Wronskian evaluated at s. $\square$

Corollary 6.2.4. Consider $f_1, \dots, f_m$ as covectors, let H be the p-plane defined by their simultaneous vanishing, and let $s \in k$ be a fixed scalar. Then the Wronskian $\mathrm{Wr}(f_1, \dots, f_m)(t)$ vanishes at s if and only if H meets $F_m(s)$ non-trivially.

Proof. Linear dependence in the columns of $M \cdot \Gamma(s)$ implies that there is a non-trivial linear combination of the covectors $\gamma(s), \dots, \gamma^{(m-1)}(s)$ which vanishes on each $f_i(t)$. This linear combination provides a point on the intersection of $F_m(s)$, which is the span of these covectors, and on H, the plane of covectors annihilating $\mathrm{span}\{f_1, \dots, f_m\}$. □

We can rephrase this result slightly to state that the plane defined by the span of $f_1, \dots, f_m$ intersects a p-plane dual to $F_m(s)$ nontrivially. In order to make this precise, we must discuss a natural duality arising on Grassmannian varieties.

6.2.2 Grassmann Duality

Let $\mathrm{Gr}_k(m, (k_{m+p-1}[t])^*)$ denote the collection of m-planes in the space of linear forms on $k_{m+p-1}[t]$, and let $\mathrm{span}\{h_1, \dots, h_m\}$ denote a point on this Grassmannian. We may consider the action of the h_i's on $k_{m+p-1}[t]$. The subspace of $k_{m+p-1}[t]$ given by those polynomials f so that $h_1(f) = 0$ is a subspace of codimension 1. The vanishing locus of m linearly independent linear forms imposes m linearly independent conditions, and thus produces a subspace of $k_{m+p-1}[t]$ of codimension m. That is to say, each such point $\{h_1, \ldots, h_m\}$ canonically defines a p-plane in $k_{m+p-1}[t]$. This yields a canonical (i.e. *basis-independent*) isomorphism

$$\mathrm{Gr}_k\left(m, (k_{m+p-1}[t])^*\right) \cong \mathrm{Gr}_k(p, k_{m+p-1}[t]).$$

This isomorphism is called *Grassmann duality.* It is an important property of Grassmannians, and for instance can be used to explain the fact that $d(m,p) = d(p,m)$. Grassmann duality is a crucial tool for our geometric interpretation of the Wronski map, and shall be used heavily in this paper.

Remark 6.2.5. We may define a flag $E_\bullet(s)$ in k_{m+p-1} where $E_i(s) = \mathrm{span}\left\{(t-s)^{m+p-1}, \ldots, (t-s)^{m+p-}\right.$ is the space of those polynomials which vanish at s to order $\geq m+p-i$. Then for a polynomial f, the following are equivalent:

1. $f \in E_i(s)$
2. $(t-s_i)^{m+p-i} \big| f(t)$
3. f, viewed as a linear form, annihilates $F_{m+p-i}(s)$, the osculating plane to the rational normal curve at s.

Thus the flags $E_\bullet(s)$ and $F_\bullet(s)$ are dual (Sot11, Theorem 10.8). This allows us to revisit Corollary 6.2.4 to say that $\mathrm{Wr}(f_1, \ldots, f_m)(t)$ vanishes at s if and only if the m-plane $W = \mathrm{span}\{f_1, \ldots, f_m\}$ intersects $E_p(s)$ non-trivially. This perspective allows us to develop a geometric intuition for the Wronski.

Proposition 6.2.6. *(Geometric interpretation of the Wronski map)* Let $s_1, \ldots, s_{mp}$ be distinct scalars in k. Then $\mathrm{Wr}^{-1}\left(\prod_{i=1}^{mp}(t - s_i)\right)$ consists of those m-planes which intersect each of $E_p(s_1), \ldots, E_p(s_{mp})$ non-trivially.

In particular this tells us that the Brouwer degree of the Wronski map provides a solution to an enumerative problem.

6.2.3 Schubert cells and the Plücker embedding

Frequently the Grassmannian can be understood better once it has been embedded in projective space. Viewing the vectors spanning a plane as a wedge power, we can sit the Grassmannian inside a suitably large projective space.

Definition 6.2.7. The *Plücker embedding* for the Grassmannian $\mathrm{Gr}_k(m, k_{m+p-1}[t])$ is defined to be the closed embedding

$$\mathrm{Pl} : \mathrm{Gr}(m, k_{m+p-1}[t]) \hookrightarrow \mathrm{Proj}\left(\wedge^m k_{m+p-1}[t]\right)$$

$$\mathrm{span}\left\{f_1, \ldots, f_m\right\} \mapsto \left[f_1 \wedge \cdots \wedge f_m\right].$$

Notation 6.2.8. We denote by $\binom{[m+p]}{m}$ the following set of integer sequences:

$$\binom{[m+p]}{m} = \left\{(\alpha_1, \ldots, \alpha_m) \ : \ 1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_m \leq m+p\right\}.$$

Provided we have *chosen a basis* $e_1, \ldots, e_{m+p}$ for $k_{m+p-1}[t]$, we see that the projective space $\mathrm{Proj}\left(\wedge^m k_{m+p-1}[t]\right) = \mathbb{P}^{\binom{m+p}{m}-1}$ inherits a basis consisting of the coordinates

$$P_\alpha = e_{\alpha_1} \wedge e_{\alpha_2} \wedge \cdots \wedge e_{\alpha_m},$$

where α is varying over all multiindices in $\binom{[m+p]}{m}$. Given any $W = \operatorname{span}\{f_1, \ldots, f_m\}$ on $\mathrm{Gr}_k(m, m+p)$, we may embed it in projective space, where it must be expressible as a k-linear sum over the P_α's. We refer to the coefficients appearing in this sum as the *Plücker coordinates* of W, and denote them by $z_\alpha(W)$:

$$f_1 \wedge \cdots \wedge f_m = \sum_{\alpha \in \binom{[m+p]}{m}} z_\alpha(W) P_\alpha.$$

How do we compute these $z_\alpha(W)$'s? We remark that we may write the coefficients of $f_1, \ldots, f_m$ in the basis $e_1, \ldots, e_{m+p}$, yielding an $m \times (m+p)$ matrix over k. The coefficient $z_\alpha(W)$ associated to a multiindex $\alpha = (\alpha_1, \ldots, \alpha_m)$ is precisely the determinant of the $m \times m$-minor of this matrix given by the α_ith columns. As the image of the Plucker embedding is a projective space, any ambiguities arising in expressing W as a matrix are resolved; that is, the Plücker coordinates corresponding to W are well-defined.

Remark 6.2.9. It is a classical fact that the Plücker embedding is injective; that is, a point on the Grassmannian can be recovered from its Plücker coordinates.

Grassmann duality translates to duality on Plücker coordinates as well. In order to demonstrate this, we must first define the dual of a multiindex.

Definition 6.2.10. Let $\alpha \in \binom{[m+p]}{m}$ be a multiindex $(\alpha_1, \ldots, \alpha_m)$. Denote by $\alpha^c \in \binom{[m+p]}{p}$ the complement of α in $(1, \ldots, m+p)$. Then we define the *dual* multiindex α^* whose entries are $m+p+1-(\alpha^c)_i$.

Example 6.2.11. If $m = 2$ and $p = 3$, let $\alpha = (1, 4)$. Then $\alpha^c = (2, 3, 5)$, and $\alpha^* = (1, 3, 4)$.

Proposition 6.2.12. Consider the Grassmann duality isomorphism

$$\mathrm{Gr}_k(m, k_{m+p-1}[t]^*) \xrightarrow{\sim} \mathrm{Gr}_k(p, k_{m+p-1}[t]),$$

given by sending an m-plane of covectors to the p-plane it annihilates. Let $\alpha \in \binom{[m+p]}{m}$, and fix a basis $\{e_i\}$ of $k_{m+p-1}[t]$ with dual basis $\{e_i^*\}$. Then for any $W^* \in \mathrm{Gr}_k(m, k_{m+p-1}[t]^*)$, where W is the plane it annihilates, we have that

$$z_\alpha(W^*) = z_{\alpha^*}(W).$$

That is, the αth Plücker coordinate of W^* in the dual basis e_i^* is the α^*th Plücker coordinate of W in the basis $\{e_i\}$.

Example 6.2.13. We have that $z_\alpha(F_m(s)) = z_{\alpha^*}(E_p(s))$ for any scalar s and any multiindex α.

6.2.4 Background from $\mathbb{A}^1$-enumerative geometry

Solving an enumerative problem can often be reduced to the computation of a certain characteristic number of a vector bundle, under certain orientation data and expected dimension assumptions. We first begin with a moduli space of possible solutions to the enumerative problem (for the example of lines on a cubic surface, our moduli space would simply be the Grassmannian of lines in projective 3-space). Following this, we construct an appropriate vector bundle over the moduli space together with a section of the bundle whose zeros are precisely the solutions to the enumerative problem at hand, and which are assumed to be isolated points. In the presence of certain orientation data for the bundle, the solution to our enumerative problem is the Euler class of the bundle, which by the Poincaré–Hopf theorem can be thought of as a sum of local indices of the section at points in its zero locus. On a coordinate patch which is compatible with our orientation data, these local indices can be computed as local Brouwer degrees of our section at points in the vanishing locus. Over the complex numbers, the local Brouwer degree at any simple zero will be equal to 1, which we read as a Boolean value informing us that this point on the moduli space is a solution to the enumerative problem (at non-simple points, it will encode the multiplicity of the solution as a natural number). Over other fields, a richer definition of Brouwer degree can produce a wider variety of data at a single solution to an enumerative problem, often revealing deep information about the ambient geometry that was invisible in the complex setting.

The algebrao-geometric analogue of the Brouwer degree that we use is called the $\mathbb{A}^1$-*Brouwer degree*, first defined by Morel (Mor06), which is valued in the Grothendieck–Witt group of symmetric bilinear forms over k. This tool has been instrumental in the development of $\mathbb{A}^1$-*enumerative geometry* (or *enriched enumerative geometry*). This program has grown in recent years due to seminal work of Levine (Lev20), Kass and Wickelgren (KW19), Bachmann and Wickelgren (BW21), among others. Recent results include an enriched Bézout's theorem (McK21), an enriched count of lines on a quintic threefold (Pau22), and a count of conics meeting eight lines (DGGM21). For further reading on this field we refer the reader to the survey papers (Bra21; PW21).

In order to compute local $\mathbb{A}^1$-degrees of sections of vector bundles, we will first need some analogue of charts from differential topology. This is provided by *Nisnevich coordinates*, defined by (KW21, Definition 17).

Definition 6.2.14. Let X be a smooth n-scheme, $p \in X$ a closed point, and $U \ni p$ an open neighborhood. Then we say that an étale map

$$\phi : U \to \mathbb{A}^n_k,$$

which induces an isomorphism on the residue field at p, defines *Nisnevich coordinates near p*. We say this defines Nisnevich coordinates *centered at p* if $\phi(p) = 0$.

Definition 6.2.15. Let X be a smooth n-scheme admitting Nisnevich coordinates $\phi : U \to \mathbb{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n]$ near a point $p \in X$. Affine space admits a standard trivialization, given by the basis elements $\frac{d}{dx_1}, \ldots, \frac{d}{dx_n}$ on $T\mathbb{A}^n_k$. Since ϕ is étale, it induces an isomorphism

$$TX|_U \xrightarrow{\sim} T\mathbb{A}^n_k,$$

and by pulling back the basis elements $\frac{d}{dx_i}$, we obtain a basis for $TX|_U$. We refer to these basis elements as the *distinguished trivialization of* $TX|_U$ arising from the Nisnevich coordinates ϕ.

Example 6.2.16. If $f : \mathbb{A}^n_k \to X$ is a Zariski open immersion, then by denoting $U := \operatorname{im}(f)$, the function $U \to \mathbb{A}^n_k$ given by $y \mapsto f^{-1}(y)$ is étale, and moreover defines Nisnevich coordinates.

Definition 6.2.17. (c.f. (OT14), (KW21), (Mor12, §4.3)) Suppose $E \to X$ is a vector bundle of rank n over a smooth, projective n-scheme over a field k. Then we say E is *relatively oriented* over X if there is an isomorphism

$$j : \mathcal{H}om(\det TX, \det E) \cong \mathscr{L}^{\otimes 2},$$

for $\mathscr{L} \to X$ a line bundle. Any such choice of isomorphism j is called a *relative orientation*.

Definition 6.2.18. For an open set $U \subseteq X$, and a relatively oriented bundle (E, j) we say a section $s \in \Gamma(U, \mathcal{H}om(\det TX, \det E))$ is *a square* if its image in $\Gamma(U, \mathscr{L}^{\otimes 2})$ is a square, meaning it is of the form $s' \otimes s'$ for some $s' \in \Gamma(U, \mathscr{L})$.

Now suppose we had Nisnevich coordinates $\phi : U \to \mathbb{A}^n_k$ near a point $p \in X$, and a relative orientation $j : \mathcal{H}om(\det TX, \det E) \xrightarrow{\sim} \mathscr{L}^{\otimes 2}$. As in Definition 6.2.15, the coordinates ϕ induce a trivialization of $TX|_U$, and by restricting U we may assume that there is a trivialization of the vector bundle E over U, meaning an isomorphism $\psi : E|_U \cong \mathbb{A}^n_k$.

Definition 6.2.19. In the situation above, we say the trivialization ψ is *compatible* with the Nisnevich coordinates ϕ *and* the relative orientation (E, j) if the associated element in

$$\mathcal{H}om(\det TX|_U, \det E|_U)$$

taking our distinguished basis of $\det TX|_U$ to the distinguished basis of $\det E|_U$ is a square.

If $\sigma : X \to E$ is a section, p is an isolated zero of σ, and $U \ni p$ an open neighborhood not containing any other points of $Z(\sigma)$, we can pull back the map $\psi \circ \sigma_U$ by ϕ to an endomorphism of affine space, which we denote by $(f_1, \ldots, f_n)$, yielding the following diagram:

$$\begin{tikzcd} U \arrow[r, "\sigma|_U"] \arrow[d, "\phi"'] & E|_U \arrow[d, "\psi"] \\ \mathbb{A}^n_k \arrow[r, dashed, "(f_1,\ldots,f_n)"'] & \mathbb{A}^n_k. \end{tikzcd}$$

Definition 6.2.20. The *local index* of σ at p is defined by

$$\mathrm{ind}_p\sigma = \deg^{\mathbb{A}^1}_{\phi(p)}(f_1, \ldots, f_n),$$

where $\deg^{\mathbb{A}^1}_{\phi(p)}(f)$ is the local $\mathbb{A}^1$-Brouwer degree of f at $\phi(p)$, that is, it is a class in the Grothendieck–Witt group $\mathrm{GW}(k)$.

For techniques and code for computing such local degrees, we refer the reader to (BMP21b).

Definition 6.2.21. (KW21, Definition 33) Let $E \to X$ be a relatively oriented vector bundle of rank r over a smooth r-dimensional scheme $X \in \mathtt{Sch}_k$, and let $\sigma : X \to E$ be a section of the bundle with isolated zeros so that Nisnevich coordinates exist near every zero. We define the *Euler number*

$$e(E, \sigma) = \sum_{p \in Z(\sigma)} \mathrm{ind}_p\sigma,$$

where we are summing over closed points p where σ vanishes.

Proposition 6.2.22. ((BW21, Theorem 1.1)) The Euler class of a relatively oriented vector bundle $e(E, \sigma)$ over a smooth and proper scheme X is independent of the choice of section.

We can also ask about how we might wield Nisnevich coordinates to compute a global $\mathbb{A}^1$-degree of a morphism between suitably nice smooth n-schemes. This notion is based off forthcoming work of Kass, Levine, Solomon and Wickelgren (KLSW22), and was discussed briefly in the expository paper (PW21, §8). See also (MS20, 2.53) for the degree of an endomorphism of projective space, and (Mor12) for the degree of an endomorphism of $\mathbb{P}^n/\mathbb{P}^{n-1}$.

Definition 6.2.23. ((KLSW22)) Let $f : X \to Y$ be a finite map of smooth n-schemes over a field k. We say that f is *oriented* if $\mathcal{H}om(\det TX, \det f^*TY)$ is a relatively oriented vector bundle over X. Phrased differently, a *relative orientation* for f is a choice of isomorphism

$$j : \mathcal{H}om(\det TX, \det f^*TY) \xrightarrow{\sim} \mathscr{L}^{\otimes 2},$$

for $\mathscr{L} \to X$ a line bundle over X.

Definition 6.2.24. ((KLSW22)) Let $U \subseteq X$ and $V \subseteq Y$ be open sets such that $f(U) \subseteq V$. We say that Nisnevich coordinates $\phi : U \to \mathbb{A}^n_k$ and $\psi : V \to \mathbb{A}^n_k$ are *compatible with the relative orientation* j if the distinguished section of

$$\Gamma(U, \mathcal{H}om(\det TX|_U, \det f^*TY|_U)$$

is a square.

Theorem 6.2.25. ((KLSW22), c.f. (PW21, 8.7)) For a finite oriented map $f : X \to Y$ of smooth k-schemes, with Y an $\mathbb{A}^1$-connected scheme, we have a well-defined degree valued in $\mathrm{GW}(k)$, defined by $\deg^{\mathbb{A}^1}(f) = \sum_{p \in f^{-1}(q)} \deg_p^{\mathbb{A}^1}(f)$.

6.3 The $\mathbb{A}^1$-degree of the Wronski map

Our strategy for computing the $\mathbb{A}^1$-degree of the Wronski is to exhibit a section σ of a particular vector bundle $\mathcal{V} \to \mathrm{Gr}_k(m, m+p)$, and on a suitable open chart, equate σ with the Wronski map up to another morphism of constant $\mathbb{A}^1$-degree. In this way, we will be able to equate the local index of the section σ with a constant multiple of the local $\mathbb{A}^1$-degree of the Wronski map in all possible parities (Lemma 6.3.27). In the case when m and p are even, the bundle $\mathcal{V}$ will be relatively orientable, and its Euler class will therefore be an integer multiple of the hyperbolic form in $\mathrm{GW}(k)$ (Lemma 6.3.28), as will the global $\mathbb{A}^1$-degree of the Wronski map on an affine chart. Deferring to the classical computation of Schubert, as we know the rank of this form over $\mathbb{C}$, we are able to provide the global $\mathbb{A}^1$-degree of the Wronski map in Theorem 6.3.29.

6.3.1 Nisnevich coordinates on the Grassmannian, distinguished bases

We will begin by establishing the existence of Nisnevich coordinates on an arbitrary Grassmannian. Let $W \in \mathrm{Gr}_k(m, m+p)$ be an arbitrary point, and pick a basis $e_1, \ldots, e_{m+p}$ of $k_{m+p-1}[t]$ so that

$$W = \mathrm{span}\left\{e_{p+1}, \ldots, e_{m+p}\right\}.$$

Definition 6.3.1. We define a *moving basis around* W, denoted by $\{\widetilde{e}_1, \ldots, \widetilde{e}_{m+p}\}$, to be a basis of $k_{m+p-1}[t]$, parametrized by $\mathbb{A}_k^{mp} = \mathrm{Spec}\,[x_{i,j}]_{1\le i\le m,\ 1\le j\le p}$:

$$\widetilde{e}_i = \begin{cases} e_i & 1 \le i \le p \\ e_i + \sum_{j=1}^{p} x_{i-p,j} e_j & p+1 \le i \le m+p. \end{cases} \tag{6.3.2}$$

Consider the morphism

$$\begin{aligned} \mathbb{A}_k^{mp} = \mathrm{Spec}\,[x_{i,j}]_{1\le i\le m,\ 1\le j\le p} &\to \mathrm{Gr}_k(m, m+p) \\ (x_{i,j})_{i,j} &\mapsto \mathrm{span}\left\{\widetilde{e}_{p+1}, \ldots, \widetilde{e}_{m+p}\right\}. \end{aligned}$$

Another way to phrase the image of this map is that $(x_{i,j})$ is sent to

$$\mathrm{RowSpace}\begin{pmatrix} x_{1,1} & \cdots & x_{1,p} & 1 & 0 & \cdots & 0 \\ x_{2,1} & \cdots & x_{2,p} & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{m,1} & \cdots & x_{m,p} & 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{6.3.3}$$

where the columns correspond to the basis elements $e_1, \dots, e_{m+p}$. This define a Zariski open immersion by the content of the argument in (KW21, Lemma 40). Letting U denote its image, we obtain Nisnevich coordinates centered around W by Example 6.2.16.

Remark 6.3.4. We can provide a more classical description of the Nisnevich coordinates above. We note that the m-plane W lives inside the ambient vector space, so we have a short exact sequence

$$W \hookrightarrow k_{m+p-1}[t] \to k_{m+p-1}[t]/W.$$

Picking a splitting for this is equivalent to picking a complementary p-plane to W. We remark that, due to the construction of the tangent space to the Grassmannian at W, we have an isomorphism

$$T_W \mathrm{Gr}_k(m, k_{m+p-1}[t]) \cong \mathrm{Hom}(W, k_{m+p-1}[t]/W).$$

By fixing such a complementary plane (i.e. choosing a basis $e_1, \dots, e_p$), we can identify $\mathrm{Hom}(W, k_{m+p-1}[t]/W)$ with $\mathbb{A}_k^{mp}$, by sending a homomorphism to its graph. This graph is precisely given by the matrix in Equation 6.3.3. In this sense, our open cell U is the subspace of m-planes in the Grassmannian which only intersect the p-plane span $\{e_1, \dots, e_p\}$ trivially (c.f. (EH16, §3.2.2)).

Remark 6.3.5. We remark that the $x_{i,j}$'s can be recovered as particular *Plücker coordinates*. Let W denote the m-plane corresponding to $(x_{i,j})$. Consider the $k \times k$ minor of columns $(j, p{+}1, \ldots, \widehat{p+i}, \ldots, p{+}m)$. Expanding along the first column, we see that everything vanishes until we hit the ith row, at which point the determinant yields $(-1)^i x_{i,j}$. That is,

$$x_{i,j} = (-1)^i p_{(j,p+1,\widehat{p+i},\ldots,p+m)}(W).$$

Here the Plücker coordinates are taken with respect to the basis $\{e_1, \ldots, e_{m+p}\}$. For concision we will introduce new notation to correspond to this multiindex:

$$\alpha(i,j) := (j, p+1, \ldots, \widehat{p+i}, \ldots, p+m). \tag{6.3.6}$$

Nisnevich coordinates defined by a moving basis induce distinguished basis elements on the tangent space $T\mathrm{Gr}_k(m, m+p)|_U$. In order to describe these distinguished basis elements, we first must discuss the structure of the tangent space of the Grassmannian.

Definition 6.3.7. The *tautological bundle* on the Grassmannian, denoted $\mathcal{S} \to \mathrm{Gr}_k(m, m+p)$, is the m-plane bundle whose fiber over the point W is the vector space W itself.

The line bundle $\mathcal{O}(1)$ on the Grassmannian, defined to be the pullback of $\mathcal{O}(1)$ under the Plücker embedding, is precisely $\det \mathcal{S} = \wedge^m \mathcal{S}$. Including the tautological bundle into the trivial rank mp bundle, we obtain the *quotient bundle*, defined as the cokernel

$$0 \to \mathcal{S} \to \mathbb{A}_k^{mp} \twoheadrightarrow \mathcal{Q} \to 0.$$

We can therefore express the tangent bundle of the Grassmannian by

$$T\mathrm{Gr}_k(m, m+p) \cong \mathcal{H}om(\mathcal{S}, \mathcal{Q}) = \mathcal{S}^* \otimes \mathcal{Q}. \tag{6.3.8}$$

Proposition 6.3.9. Given Nisnevich coordinates $U \to \mathbb{A}_k^{mp}$ corresponding to a moving basis $\widetilde{e}_1, \ldots, \widetilde{e}_{m+p}$, then one has the following distinguished bases over U:

1. $\{\widetilde{e}_{p+1}, \ldots, \widetilde{e}_{p+m}\}$ is a distinguished basis for the tautological bundle $\mathcal{S}|_U$
2. Letting $\widetilde{\phi}_i$ denote the cobasis element to $\widetilde{e}_i$, we see that $\left\{\widetilde{\phi}_{p+1}, \ldots, \widetilde{\phi}_{m+p}\right\}$ provides a distinguished basis for the dual tautological bundle $\mathcal{S}^*|_U$.
3. $\{\widetilde{e}_1, \ldots, \widetilde{e}_p\}$ provides a distinguished basis for the quotient bundle $\mathcal{Q}|_U$.
4. The tensor products of vectors

$$\left\{\widetilde{\phi}_j \otimes \widetilde{e}_i \ : \ 1 \leq i \leq p \text{ and } p+1 \leq j \leq m+p\right\}$$

provide a distinguished basis for the tangent bundle $T\mathrm{Gr}_k(m, m+p)|_U$.

Lemma 6.3.10. If m and p are both even, there is a global orientation of the tangent bundle $T\mathrm{Gr}_k(m, m+p)$ which is compatible with any Nisnevich coordinates defined by moving bases.

Proof. This is a direct generalization of (SW21, Lemma 8). Let $(e_1, \ldots, e_{m+p})$ and $(e'_1, \ldots, e'_{m+p})$ denote two bases of $k_{m+p-1}[t]$, and let $\{\widetilde{e}_i\}$, $\{\widetilde{e}'_i\}$ denote the associated moving bases parametrizing open cells U and U' of the Grassmannian. If $U \cap U' \neq \varnothing$, we have that

$$\operatorname{span}\{\widetilde{e}_{p+1}, \ldots, \widetilde{e}_{m+p}\} = \operatorname{span}\{\widetilde{e}'_{p+1}, \ldots, \widetilde{e}'_{m+p}\} \qquad \text{on } U \cap U'. \tag{6.3.11}$$

Letting $\widetilde{\phi}_i$ and $\widetilde{\phi}'_i$ denote the dual basis elements, respectively, we obtain canonical trivializations for $T\mathrm{Gr}_k(m, m+p)|_{U \cap U'}$, given by:

$$\{\widetilde{\phi}_j \otimes \widetilde{e}_i \ : \ 1 \leq i \leq p, \ p+1 \leq j \leq m+p\},$$
$$\{\widetilde{\phi}'_j \otimes \widetilde{e}'_i \ : \ 1 \leq i \leq p, \ p+1 \leq j \leq m+p\}.$$

Denote by B the change of basis matrix from $\{\widetilde{e}_1, \ldots, \widetilde{e}_p\}$ and $\left\{\widetilde{e}'_1, \ldots, \widetilde{e}'_p\right\}$ on the quotient bundle $\mathcal{Q}|_{U \cap U'}$, and denote by A the change of basis matrix on $\mathcal{S}^*|_{U \cap U'}$ from the basis $\left\{\widetilde{\phi}_{p+1}, \ldots, \widetilde{\phi}_{m+p}\right\}$ to $\left\{\widetilde{\phi}'_{p+1}, \ldots, \widetilde{\phi}'_{m+p}\right\}$. Then the change of basis matrix on $T\mathrm{Gr}_k(m, m+p)|_{U \cap U'}$ is given by $A \otimes B$. The determinant of this matrix is $\det(A)^m \det(B)^p$. As m and p are both even, this is a square in $\mathcal{O}(U \cap U')^\times$. $\square$

6.3.2 Relative orientations of bundles over the Grassmannian; even-even parity

We will discuss a bundle $\mathcal{V}$ over the Grassmannian, which is relatively orientable in the case where m and p are both even. We will additionally construct a section $\sigma : \mathrm{Gr}_k(m, m+p) \to \mathcal{V}$, whose local degree at any simple zero is related to the local $\mathbb{A}^1$-degree of the Wronski map in any parities.

Notation 6.3.12. We denote by $\mathcal{V}$ the mp-dimensional line bundle

$$\mathcal{V} = \bigoplus_{i=1}^{mp} \bigwedge^m \mathcal{S}^* \to \mathrm{Gr}_k(m, m+p). \tag{6.3.13}$$

Proposition 6.3.14. The vector bundle $\mathcal{V} \to \mathrm{Gr}_k(m, m+p)$ is relatively orientable if and only if m and p are both even.

Proof. For our bundle $\mathcal{V}$, we compute that

$$\begin{aligned}\mathcal{H}om(\det T\mathrm{Gr}(m, m+p), \det \mathcal{V}) &\cong \mathcal{H}om(\mathcal{O}(m+p), \prod_{i=1}^{mp} \det(\mathcal{E})) \cong \mathcal{H}om(\mathcal{O}(m+p), \mathcal{O}(mp)) \\ &\cong \mathcal{O}(-m-p) \otimes \mathcal{O}(mp) \cong \mathcal{O}(mp-m-p).\end{aligned}$$

We note that $\mathcal{O}(mp-m-p)$ is a square of a line bundle if and only if $mp-m-p \equiv 0 \pmod 2$, that is, m and p are both even. $\square$

Proposition 6.3.15. If m and p are both even, then there is a relative orientation of the vector bundle $\mathcal{V} \to \mathrm{Gr}_k(m, m+p)$ which is compatible with Nisnevich coordinates defined by moving bases.

Proof. Take two cells U and U' on $\mathrm{Gr}_k(m, m+p)$ with non-empty intersection, parametrized respectively by the moving bases $\widetilde{e}_i$ and $\widetilde{e}'_i$, and assume as before that

$$\mathrm{span}\left\{\widetilde{e}_{p+1}, \dots, \widetilde{e}_{m+p}\right\} = \mathrm{span}\left\{\widetilde{e}'_{p+1}, \dots, \widetilde{e}'_{m+p}\right\} \text{ on } U \cap U'.$$

The trivializations $\left\{\widetilde{\phi}_{p+1}, \dots, \widetilde{\phi}_{m+p}\right\}$ and $\left\{\widetilde{\phi}'_{p+1}, \dots, \widetilde{\phi}'_{m+p}\right\}$ on the dual tautological bundle $\mathcal{S}^*|_{U\cap U'}$ induce associated trivializations $\widetilde{\phi}_{p+1}\wedge\cdots\wedge\widetilde{\phi}_{m+p}$ and $\widetilde{\phi}'_{p+1}\wedge\cdots\wedge\widetilde{\phi}'_{m+p}$, respectively, for $\wedge^m\mathcal{S}^*|_{U\cap U'}$. If A denotes the change of basis matrix on $\mathcal{S}^*|_{U\cap U'}$ as above, then $\det(A)$ denotes the change of basis on $\wedge^m\mathcal{S}^*|_{U\cap U'}$. Since $\mathcal{V} = \oplus_{i=1}^{mp}\wedge^m\mathcal{S}^*$, we have that the change of basis on $\mathcal{V}|_{U\cap U'}$ is given by a block sum of mp copies of $\det(A)$. Thus the change of basis matrix in $\mathcal{H}om(\det T\mathrm{Gr}_k(m, m+p), \det\mathcal{V}) \cong (\det T\mathrm{Gr}_k(m, m+p))^* \otimes \det\mathcal{V}$ over $U \cap U'$ is given by

$$\det\left(A \otimes B\right)^{-1} \otimes \det\left(\bigoplus_{i=1}^{mp}\det(A)\right) = \det(A)^{-m}\det(B)^{-p}\det(A)^{mp}.$$

As m and p are both even, this is a square. $\square$

6.3.3 Interpreting the Wronski map as a section of a line bundle

We now construct a section σ of the bundle $\mathcal{V}$ which is intimately related to the Wronski. For this section we fix $s_1, \ldots, s_{mp}$ to be distinct scalars in k — the reader is invited to think of these scalars as timestamps on $\mathbb{A}^1_k$, yielding positions on the rational normal curve at each time, as well as osculating planes.

Recall from Proposition 6.2.2 the covector $\left(\gamma^{(j)}(s)\right)^*$, which mapped a polynomial f to $f^{(j)}(s)$. We would like to consider these covectors as j ranges from 0 to $m-1$, and as s varies over our set of scalars $\{s_1, \ldots, s_{mp}\}$. To that end, it will be beneficial to introduce some more compact notation.

Notation 6.3.16. We denote by $\sigma_{i,j}$ the covector $\left(\gamma^{(j-1)}(s_i)\right)^*$, given by

$$\begin{aligned} \sigma_{i,j} : k_{m+p-1}[t] &\to k \\ f &\mapsto f^{(j-1)}(s_i). \end{aligned}$$

Note the indexing on $\sigma_{i,j}$: the index i is running from 1 to mp, keeping track of the time on the rational normal curve, while j is running from 1 to m, indicating the extent to which the input is being differentiated.

Remark 6.3.17. For a fixed i, the covectors $\{\sigma_{i,1}, \sigma_{i,2}, \ldots, \sigma_{i,m}\}$ cut out a p-plane under Grassmannian duality. This plane is precisely $E_p(s_i)$, as defined in Remark 6.2.5.

Notation 6.3.18. We denote by σ_i the wedge of covectors $\sigma_{i,1} \wedge \cdots \wedge \sigma_{i,m}$. This is a section of $\mathcal{O}(1)$ over the Grassmannian, that is, $\sigma_i : \mathrm{Gr}_k(m, m+p) \to \wedge^m \mathcal{S}^*$. Letting i vary from 1 to mp, we obtain mp sections of $\wedge^m \mathcal{S}^*$, that is, a section of our bundle $\mathcal{V}$. We denote by σ this section:

$$\sigma := \bigoplus_{i=1}^{mp} \sigma_i = \bigoplus_{i=1}^{mp} \left(\wedge_{j=1}^{m} \sigma_{i,j}\right) : \mathrm{Gr}_k(m, m+p) \to \mathcal{V} = \bigoplus_{i=1}^{mp} \wedge^m \mathcal{S}^*.$$

We may also write $\sigma = \sigma(s_1, \ldots, s_{mp})$ if we wish to indicate dependence of σ on the initial choice of scalars s_i.

Proposition 6.3.19. We see that $W = \operatorname{span}\{f_1, \ldots, f_m\}$ is a zero of σ_i if and only if the Wronskian $\mathrm{Wr}(f_1, \ldots, f_m)(t)$ vanishes at s_i.

Proof. We observe that $\sigma_i(W)$ vanishes if and only if $(\sigma_{i,1} \wedge \cdots \wedge \sigma_{i,m})(f_1 \wedge \cdots \wedge f_m) = 0$. This evaluation of wedges of covectors can be computed as

$$\begin{aligned}
\sigma_i(W) &= (\sigma_{i,1} \wedge \cdots \wedge \sigma_{i,m})(f_1 \wedge \cdots \wedge f_m) \\
&= \begin{vmatrix} f_1(s_i) & f_2(s_i) & \cdots & f_m(s_i) \\ f_1'(s_i) & f_2'(s_i) & \cdots & f_m'(s_i) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(m-1)}(s_i) & f_2^{(m-1)}(s_i) & \cdots & f_m^{(m-1)}(s_i) \end{vmatrix} \\
&= \mathrm{Wr}(f_1, \ldots, f_m)(s_i).
\end{aligned}$$

□

Corollary 6.3.20. Consider the class of the polynomial $\Phi(t) = \prod_{i=1}^{mp}(t - s_i)$ in projective space $\mathbb{P}_k^{mp}$. We have that the following are equivalent for a point $W = \text{span}\{f_1, \ldots, f_m\} \in \text{Gr}_k(m, m+p)$:

1. W has nonempty intersection with each of $E_p(s_1), \ldots, E_p(s_{mp})$.

2. W is a zero of the section $\sigma : \text{Gr}_k(m, m+p) \to \mathcal{V}$.

3. $\text{Wr}(f_1, \ldots, f_m)(t)$, as a polynomial in t, has a root at each s_i for $1 \leq i \leq mp$.

4. W lives in the fiber $\text{Wr}^{-1}(\Phi(t))$.

6.3.4 Big open cells

Let $Y \subseteq \mathbb{P}_k^{mp}$ denote the collection of monic polynomials in Proj $k_{mp}[t]$ of degree equal to mp. This defines an open affine cell of projective space, of dimension mp. Denote by $X = \text{Wr}^{-1}(Y)$ the preimage of this cell in the Grassmannian.

Remark 6.3.21. We refer to X as the *big open cell*, and remark a few properties about it.

1. This is is a coordinate patch parametrized around the point span $\{t^p, \ldots, t^{m+p-1}\}$, and therefore $X \cong \mathbb{A}_k^{mp}$.

2. This is the *big open cell* as defined in (EG02, p.5), from where we took the terminology.

3. A point $W \in \mathrm{Gr}_k(m, m+p)$ lies in the open cell X if and only its Wronskian is of degree mp.

By this very last point, if $\Phi(t) := \prod_{i=1}^{mp}(t - s_i)$, then in order to study the fiber $\mathrm{Wr}^{-1}(\Phi(t))$, it suffices to restrict our attention to the big open cell X. Let $W \in X$ be an arbitrary point, and fix $e_{p+1}, \ldots, e_{p+m}$ to be monic polynomials which span W. Extending this to a basis $e_1, \ldots, e_{m+p}$ of $k_{m+p-1}[t]$, we can parametrize an open cell $U \cong \mathbb{A}_k^{mp}$ centered around W. For degree reasons, we observe that $U \subseteq X$, so that we have an induced map $\mathrm{Wr}|_U : U \to Y$.

Remark 6.3.22. Let $W \in X$, and let U be an affine cell parametrized around W by a moving basis. Then the restricted Wronski map $\mathrm{Wr}|_U : U \to Y$ admits an orientation induced by the trivializations of TU and TY.

Thus we see that $\mathrm{Wr}|_U$ is a map of the form $\mathbb{A}_k^{mp} \to \mathbb{A}_k^{mp}$. What does this map look like? If (x_{ij}) is a point on $\mathbb{A}_k^{mp} \cong U$, we have that its Wronskian is a degree mp polynomial of the form

$$\mathrm{Wr}\left(\widetilde{e}_{p+1}(x), \ldots, \widetilde{e}_{p+m}(x)\right)(t) = \sum_{i=0}^{mp} h_i t^i.$$

This is by definition a point $[h_0 : \ldots : h_{mp}]$ in projective space. In order to take the affine chart Y we must divide out by h_{mp}, which we know to be non-zero by Remark 6.3.21 since W lies on X. Moreover since we picked the e_{p+i}'s to be monic, we know exactly what h_{mp} is! By only picking out the highest degree terms in the Wronskian, we can observe that h_{mp} is the coefficient on the monomial $\mathrm{Wr}\left(t^p, t^{p+1}, \ldots, t^{m+p-1}\right)$, which is well-defined over k via our hypothesis that $(m+p-1)!$ is invertible over k. It is well-known that this is the Vandermonde $V(p, p+1, \ldots, m+p-1)$ (BD10, Lemma 1), and a simple induction argument shows that this is equal to $\prod_{i=1}^{m-1} i!$. We will now compare the local section σ to the Wronski map. In order to do this, we must first introduce some notation.

Notation 6.3.23. We define the following maps from $\mathbb{A}_k^{mp}$ to itself:

- By abuse of notation, denote by $V_{m,p} : \mathbb{A}_k^{mp} \to \mathbb{A}_k^{mp}$ the map which multiplies each coordinate by the scalar $\prod_{i=1}^{m-1} i!$.
- Denote by $\mathrm{ev}_s : \mathbb{A}_k^{mp} \to \mathbb{A}_k^{mp}$ the map which sends a tuple $(a_0, \ldots, a_{m+p-1})$, viewed as a polynomial $g(t) = \sum_{i=0}^{mp-1} a_i t^i$ to the tuple $(g(s_1), \ldots, g(s_{mp}))$.
- Finally, denote by tr_s the translation map

$$\begin{aligned} \mathrm{tr}_s : \mathbb{A}_k^{mp} &\to \mathbb{A}_k^{mp} \\ (x_1, \ldots, x_{mp}) &\mapsto \left(x_1 + V_{m,p} s_1^{mp}, \ldots, x_{mp} + V_{m,p} s_{mp}^{mp}\right). \end{aligned}$$

Lemma 6.3.24. Let $W \in X$, and let U be an open cell parametrized around X, determined by a monomial basis as above. Then the following diagram commutes:

$$\begin{tikzcd}
 & & \mathcal{V}|_U \arrow[drr, "\oplus_{i=1}^{mp}(\widetilde{e}_{p+1}\wedge\cdots\wedge\widetilde{e}_{m+p})"] & & \\
U \arrow[urr, "\sigma|_U"] \arrow[dr, "\mathrm{Wr}|_U"'] & & & & \mathbb{A}_k^{mp} \\
 & Y \arrow[r, "V_{m,p}"'] & \mathbb{A}^{mp} \arrow[r, "\mathrm{ev}_s"'] & \mathbb{A}^{mp} \arrow[ur, "\mathrm{tr}_s"'] &
\end{tikzcd}$$

Proof. Fix $e_1, \ldots, e_{mp}$ as desired, and begin with a point (x_{ij}) on the affine space U. Let its Wronski be written as

$$\mathrm{Wr}\left(\widetilde{e}_{p+1}(x), \ldots, \widetilde{e}_{p+m}(x)\right) = \sum_{i=0}^{mp} h_i t^i,$$

where we have that $h_{mp} = \prod_{i=1}^{m-1} i!$ as above. Landing in Y, we have that $(x_{i,j})$ is mapped to the mp-tuple

$$\left(\frac{h_0}{h_{mp}}, \ldots, \frac{h_{mp-1}}{h_{mp}}\right).$$

Applying the map $V_{m,p}$, we multiply each factor through by h_{mp} (which we remark is a constant which is independent of (x_{ij})), which clears denominators and maps us to

$$(h_0, h_1, \ldots, h_{mp-1}).$$

Applying the evaluation map ev_s, we arrive at

$$\left(\sum_{i=0}^{mp-1} h_i s_1^i, \sum_{i=0}^{mp-1} h_i s_2^i, \ldots, \sum_{i=0}^{mp-1} h_i s_{mp}^i\right).$$

Finally applying our translation map, we obtain

$$\left(\sum_{i=0}^{mp} h_i s_1^i, \ldots, \sum_{i=0}^{mp} h_i s_{mp}^i\right) = \left(\mathrm{Wr}\left(\widetilde{e}_{p+1}(x), \ldots, \widetilde{e}_{p+m}(x)\right)(s_1), \ldots, \mathrm{Wr}\left(\widetilde{e}_{p+1}(x), \ldots, \widetilde{e}_{p+m}(x)\right)(s_{mp})\right).$$

However we remark that by Proposition 6.3.19 this is exactly what we obtain by applying σ to the point (x_{ij}) and trivializing $\mathcal{V}$ over U. $\square$

Remark 6.3.25. The global $\mathbb{A}^1$-degree of the map $V_{m,p}$ is $\left\langle \left(\prod_{i=1}^{m-1} i!\right)^{mp} \right\rangle$, since we are simply multiplying the scalar $\prod_{i=1}^{m-1} i!$ into each of the mp coordinates. The global degree of the translation map tr_s is just $\langle 1 \rangle$, since translation is $\mathbb{A}^1$-homotopic to the identity.

Lemma 6.3.26. The global $\mathbb{A}^1$-degree of the evaluation map $\mathrm{ev}_{s_1,\ldots,s_{mp}}$ is precisely

$$\deg^{\mathbb{A}^1} \mathrm{ev}_{(s_1,\ldots,s_{mp})} = \langle V(s) \rangle,$$

where $V(s) := V(s_1, \ldots, s_{mp})$ denotes the Vandermonde determinant. As a result, since this is a rank one element of $\mathrm{GW}(k)$, this is the local $\mathbb{A}^1$-degree at any root of $\mathrm{ev}_{(s_1,\ldots,s_{mp})}$.

Proof. Let $\mathrm{ev}_{(s_1,\ldots,s_{mp})} = (\mathrm{ev}_1, \ldots, \mathrm{ev}_{mp})$, and let $(a_0, \ldots, a_{mp-1})$ correspond to $a_0 + a_1 t + \ldots + a_{mp-1} t^{mp-1} + t^{mp}$. Then we can see $\frac{\partial \mathrm{ev}_j}{\partial a_i} = s_j^{i-1}$. $\square$

Lemma 6.3.27. Let $s_1, \ldots, s_{mp}$ be distinct, and let $\Phi(t) = \prod_{i=1}^{mp}(t - s_i)$. For any

$[W] \in \mathrm{Wr}^{-1}(\Phi(t))$, we have that

$$\mathrm{ind}_W \sigma = \left\langle V(s) \cdot \left(\prod_{i=1}^{m-1} i! \right)^{mp} \right\rangle \cdot \deg_W^{\mathbb{A}^1} \mathrm{Wr}.$$

Proof. The proof follows from applying the local degree to the commutative diagram in Lemma 6.3.24, and deferring to the computation in Lemma 6.3.26. □

An explicit formula for $\mathrm{ind}_W \sigma$ at any simple root will be provided in Section 6.4.

Lemma 6.3.28. If $\mathcal{V}$ is relatively orientable, then its Euler class $e(\mathcal{V}) \in \mathrm{GW}(k)$ is an integer multiple of the hyperbolic element $\mathbb{H}$.

For a proof of this lemma, see (Lev20, 4.3) as well as the discussion in (SW21, Section 4).

Theorem 6.3.29. Let m and p be even. Then the $\mathbb{A}^1$-degree of the Wronski map computed on the big open cell X is

$$\deg^{\mathbb{A}^1} \mathrm{Wr}|_X = \frac{d(m,p)}{2} \mathbb{H}.$$

Proof. Let $s_1, \ldots, s_{mp}$ be distinct, and let $V(s) = V(s_1, \ldots, s_{mp})$ denote their Vandermonde determinant. Via Lemma 6.3.27 the degree of the Wronski is precisely

$$\begin{aligned} \deg^{\mathbb{A}^1} \mathrm{Wr} = \sum_{W \in Z(\mathrm{Wr})} \deg_W^{\mathbb{A}^1} \mathrm{Wr} &= \left\langle V(s) \cdot \left(\prod_{i=1}^{m-1} i! \right)^{mp} \right\rangle \sum_{W \in Z(\sigma)} \mathrm{ind}_W \sigma \\ &= \left\langle V(s) \cdot \left(\prod_{i=1}^{m-1} i! \right)^{mp} \right\rangle e(\mathcal{V}, \sigma). \end{aligned}$$

By Lemma 6.3.28, the Euler class is a multiple of $\mathbb{H}$, and since we know the rank of the bilinear form $\deg^{\mathbb{A}^1} \mathrm{Wr}$ in the case where m and p are both even via the classical computation of Schubert, we can determine which integer multiple of the hyperbolic element it must be.

Finally we remark that the choice of orientation of an affine patch is well-defined up to a square class in the ground field (this is precisely the issue we see in (EG02) where the Brouwer degree of the Wronski is well-defined up to a sign). Since the degree produced here is hyperbolic, this ambiguity vanishes, since $\langle a \rangle \mathbb{H} = \mathbb{H}$ for any $a \in k^{\times}$. □

This global count unifies the real and complex degrees of the Wronski map into one computation in these parities — that is, we recover the complex degree by taking the rank of this form, and the real degree by taking the signature. Contained within the local degree of the Wronski map is further geometric information, which we can now explore.

6.4 A formula for the local index

In this section we will provide a formula for the local degree $\deg^{\mathbb{A}^1}_W \mathrm{Wr}$, when the Wronski map has a simple root at the point W. To parametrize an affine open cell around W, we first fix a basis $e_1, \ldots, e_{m+p}$ of $k_{m+p-1}[t]$ so that $W = \operatorname{span}\{e_{p+1}, \ldots, e_{m+p}\}$.

Let ϕ_k denote the dual basis element to e_k. We may then rewrite the covectors $\sigma_{\ell,1}, \ldots, \sigma_{\ell,m}$ in this dual basis. That is, for any $1 \le j \le m$, we write

$$\sigma_{\ell,j} := \sum_{k=1}^{m+p} e_k^{(j-1)}(s_\ell)\phi_k, \tag{6.4.1}$$

It is easy to see by acting on e_k by $\sigma_{\ell,j}$, that $e_k^{(j-1)}(s_\ell)$ will be the coefficient on ϕ_k. By Equation 6.2.1, we have that span $\{\sigma_{\ell,1}, \ldots, \sigma_{\ell,m}\} = F_m(s_\ell)$, thus by a forgivable abuse of notation we refer to the matrix of coefficients of these vectors as $F_m(s_\ell)$:

$$F_m(s_\ell) = \begin{pmatrix} e_1(s_\ell) & e_1'(s_\ell) & \cdots & e_1^{(m-1)}(s_\ell) \\ e_2(s_\ell) & e_2'(s_\ell) & \cdots & e_2^{(m-1)}(s_\ell) \\ \vdots & \vdots & \ddots & \vdots \\ e_{m+p}(s_\ell) & e_{m+p}'(s_\ell) & \cdots & e_{m+p}^{(m-1)}(s_\ell) \end{pmatrix} = \left(\text{ coeffs of } \sigma_{\ell,1} \;\middle|\; \cdots \;\middle|\; \text{coeffs of } \sigma_{\ell,m} \right). \tag{6.4.2}$$

We will define the following notation to identify a *distinguished minor* of this matrix. Namely we want to take minors consisting of all the bottom m rows except one, and one row from higher in the matrix. Explicitly, let $1 \le \gamma \le m$ and $1 \le k \le p$. Then we denote by $\alpha(\gamma, \kappa)$ the multiindex

$$\alpha(\gamma, k) := \{k, p+1, \ldots, p+\gamma-1, p+\gamma+1, \ldots, p+m\}.$$

In particular this gives us $z_{\alpha(\gamma,k)}(F_m(s_\ell))$,which is the $\alpha(\gamma,k)$th Plücker coordinate of $F_m(s_\ell)$:

$$z_{\alpha(\gamma,k)}(F_m(s_\ell)) = \det\begin{pmatrix} e_k(s_\ell) & e_k'(s_\ell) & \cdots & e_k^{(m-1)}(s_\ell) \\ e_{p+1}(s_\ell) & e_{p+1}'(s_\ell) & \cdots & e_{p+1}^{(m-1)}(s_\ell) \\ e_{p+2}(s_\ell) & e_{p+2}'(s_\ell) & \cdots & e_{p+2}^{(m-1)}(s_\ell) \\ \vdots & \vdots & \ddots & \vdots \\ e_{p+\gamma-1}(s_\ell) & e_{p+\gamma-1}'(s_\ell) & \cdots & e_{p+\gamma-1}^{(m-1)}(s_\ell) \\ e_{p+\gamma+1}(s_\ell) & e_{p+\gamma+1}'(s_\ell) & \cdots & e_{p+\gamma+1}^{(m-1)}(s_\ell) \\ \vdots & \vdots & \ddots & \vdots \\ e_{m+p}(s_\ell) & e_{m+p}'(s_\ell) & \cdots & e_{m+p}^{(m-1)}(s_\ell) \end{pmatrix}$$

$$= \mathrm{Wr}\left(e_k, e_{p+1}, \ldots, \widehat{e_{p+\gamma}}, \ldots, e_{m+p}\right)(s_\ell).$$

We recall that the fiber of the Wronski map over $\prod_{i=1}^{mp}(t-s_i)$ counts the number of m-planes meeting $E_p(s_1), \ldots, E_p(s_{mp})$ non-trivially. Here we can state a new geometric interpretation for the local index of the Wronski — namely it picks up a determinantal relation between distinguished Plücker coordinates of the planes $F_m(s_i)$ (under duality these can be considered as distinguished Plücker coordinates of the $E_p(s_i)$'s). We remark that while our computation of the global degree of the Wronski only held when m and p were both even, the following result holds in all parities and over arbitrary fields, subject to the ongoing assumption that

$(m+p-1)!^{-1} \in k$.

Theorem 6.4.3. Let W be a simple preimage of the Wronski map in the fiber $\mathrm{Wr}^{-1}\left(\prod_{i=1}^{mp}(t-s_i)\right)$, and let $e_1,\ldots,e_{mp}$ be a basis chosen so that $W = \operatorname{span}\{e_{p+1},\ldots,e_{m+p}\}$. Then we have that

$$\deg_W^{\mathbb{A}^1} \mathrm{Wr} = \langle C \cdot \det \mathcal{B} \rangle ,$$

where C is the global constant

$$C = V(s_1,\ldots,s_{mp}) \left(\prod_{i=1}^{m-1} i!\right)^{mp} (-1)^{m(m-1)p/2},$$

where $V(s_1,\ldots,s_{mp})$ is the Vandermonde determinant of the s_i's, and $\mathcal{B}$ is the $mp \times mp$-matrix defined by

$$\mathcal{B} = \begin{pmatrix} z_{\alpha(1,1)}(F_m(s_1)) & z_{\alpha(1,1)}(F_m(s_2)) & \cdots & z_{\alpha(1,1)}(F(s_{mp})) \\ z_{\alpha(1,2)}(F_m(s_1)) & z_{\alpha(1,2)}(F_m(s_2)) & \cdots & z_{\alpha(1,2)}(F(s_{mp})) \\ \vdots & \vdots & \ddots & \vdots \\ z_{\alpha(1,p)}(F_m(s_1)) & z_{\alpha(1,p)}(F_m(s_2)) & \cdots & z_{\alpha(1,p)}(F(s_{mp})) \\ z_{\alpha(2,1)}(F_m(s_1)) & z_{\alpha(2,1)}(F_m(s_2)) & \cdots & z_{\alpha(2,1)}(F(s_{mp})) \\ \vdots & \vdots & \ddots & \vdots \\ z_{\alpha(2,p)}(F_m(s_1)) & z_{\alpha(2,p)}(F_m(s_2)) & \cdots & z_{\alpha(2,p)}(F(s_{mp})) \\ \vdots & \vdots & \ddots & \vdots \\ z_{\alpha(m,p)}(F_m(s_1)) & z_{\alpha(m,p)}(F_m(s_2)) & \cdots & z_{\alpha(m,p)}(F(s_{mp})) \end{pmatrix},$$

where these Plücker coordinates are written in the basis $\{\phi_i\}$.

Proof. Since $Z(\sigma) = Z(\mathrm{Wr})$, we may suppose that σ has a simple zero at the top point $W = \mathrm{span}\{e_{p+1}, \ldots, e_{m+p}\} \in \mathrm{Gr}_k(m, m+p)$, and rewrite the covectors of σ in the associated cobasis, as in Equation 6.4.1. Then we have an affine coordinate chart U around W, and we can trivialize $\mathcal{V}$ over U by direct sums of $\widetilde{\phi}_{p+1} \wedge \cdots \wedge \widetilde{\phi}_{m+p}$ We then obtain functions $F_1, \ldots, F_{mp}$ on U defined by

$$\wedge_{j=1}^{m} \sigma_{\ell,j} = F_\ell \cdot \widetilde{\phi}_{p+1} \wedge \cdots \wedge \widetilde{\phi}_{m+p}. \tag{6.4.4}$$

The F_i's are local representations of σ in the chart U, centered around W. As W is a simple zero, then in order to compute $\mathrm{ind}_W \sigma$ it suffices to compute the partial derivatives of the functions F_i at the origin of U (which is the point $W = e_{p+1} \wedge \cdots \wedge e_{m+p}$). By the definition of the moving basis in Equation 6.3.2, we have a change of basis formula[2]

$$\phi_k = \begin{cases} \widetilde{\phi}_k + \sum_{n=1}^{m} x_{n,k} \widetilde{\phi}_{p+n} & 1 \le k \le p \\ \\ \widetilde{\phi}_k & p+1 \le k \le m+p. \end{cases} \tag{6.4.5}$$

For any fixed ℓ, we may then write

$$\begin{aligned} \bigwedge_{j=1}^{m} \sigma_{\ell,j} &= \bigwedge_{j=1}^{m} \left(\sum_{k=1}^{m+p} e_k^{(j-1)}(s_\ell) \phi_k \right) \\ &= \bigwedge_{j=1}^{m} \left(\sum_{k=1}^{p} e_k^{(j-1)}(s_\ell) \left(\widetilde{\phi}_k + \sum_{n=1}^{m} x_{n,k} \widetilde{\phi}_{p+n} \right) + \sum_{q=p+1}^{m+p} e_q^{(i-1)}(s_\ell) \widetilde{\phi}_q \right). \end{aligned} \tag{6.4.6}$$

[2]By allowing ϕ_i to act on $\widetilde{e}_j$, we get the coefficient of $\widetilde{\phi}_j$ in ϕ_i.

Since we will be evaluating this at $e_p \wedge \cdots \wedge e_{m+p-1}$ we only need to worry about terms which are of the form $\widetilde{\phi}_{p+1} \wedge \cdots \wedge \widetilde{\phi}_{m+p-1}$. In particular we can forget about the $\widetilde{\phi}_k$ terms for $1 \leq k \leq p$, and we obtain

$$\begin{aligned}
&\bigwedge_{j=1}^{m} \left(\sum_{k=1}^{p} e_k^{(j-1)}(s_\ell) \left(\sum_{n=1}^{m} x_{n,k} \widetilde{\phi}_{p+n} \right) + \sum_{k=p+1}^{m+p} e_k^{(i-1)}(s_\ell) \widetilde{\phi}_k \right) \\
&= \bigwedge_{j=1}^{m} \left(\sum_{k=1}^{p} \sum_{n=p+1}^{m+p} e_k^{(j-1)}(s_\ell) x_{n-p,k} \widetilde{\phi}_n + \sum_{q=p+1}^{m+p} e_q^{(j-1)}(s_\ell) \widetilde{\phi}_q \right) \\
&\quad = \bigwedge_{j=1}^{m} \left(\sum_{n=p+1}^{m+p} \left(e_n^{(j-1)}(s_\ell) + \sum_{k=1}^{p} e_k^{(j-1)}(s_\ell) x_{n-p,k} \right) \widetilde{\phi}_n \right) \\
&= \det(\mathcal{C}) \cdot \widetilde{\phi}_{p+1} \wedge \cdots \wedge \widetilde{\phi}_{m+p},
\end{aligned} \tag{6.4.7}$$

where $\mathcal{C}_{j,\gamma}$ is the coefficient on $\widetilde{\phi}_{p+\gamma}$ in the jth exterior power above. Explicitly,

$$\mathcal{C}_{j,\gamma} = e_{p+\gamma}^{(j-1)}(s_\ell) + \sum_{k=1}^{p} e_k^{(j-1)}(s_\ell) x_{\gamma,k}.$$

Since we will evaluate partials at the origin, we only need to pick out linear terms in the $x_{\gamma,k}$'s, so we can forget higher order terms as well as constant terms. Thus, we see that

$$\det(\mathcal{C}) = \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) \prod_{\gamma=1}^{m} \left(e_{p+\gamma}^{(\sigma(\gamma)-1)}(s_\ell) + \sum_{k=1}^{p} e_k^{(\sigma(\gamma)-1)}(s_\ell) x_{\gamma,k} \right).$$

For a fixed $x_{\gamma,k}$ the constant coefficient on $x_{\gamma,k}$ is

$$\left. \frac{\partial F_\ell}{\partial x_{\gamma,k}} \right|_0 = \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) e_k^{(\sigma(\gamma)-1)}(s_\ell) \prod_{\substack{1 \leq a \leq m \\ a \neq \gamma}} e_{p+a}^{(\sigma(a)-1)}(s_\ell), \tag{6.4.8}$$

and we can recognize this as a Plücker coordinate!

$$\begin{aligned}\left.\frac{\partial F_\ell}{\partial x_{\gamma,k}}\right|_0 &= \mathrm{Wr}(e_{p+1},\ldots,e_{p+\gamma-1},e_k,e_{p+\gamma+1},\ldots,e_{m+p})(s_\ell)\\ &= (-1)^{\gamma-1}\mathrm{Wr}\left(e_k,e_{p+1},\ldots,\widehat{e_{p+\gamma}},\ldots,e_{m+p}\right)(s_\ell)\\ &= (-1)^{\gamma-1}z_{\alpha(\gamma,k)}\left(F_m(s_\ell)\right).\end{aligned}$$

In particular by Remark 6.3.5 we have that $(-1)^\gamma z_{\alpha(\gamma,k)}(F_m(s_\ell))$ is the (γ,k)th affine coordinate of the plane $F_m(s_\ell)$. Varying over all (γ,k) and ℓ, we obtain the local index as

$$\begin{aligned}\mathrm{ind}_W\sigma &= \left\langle \det\left(\frac{\partial F_\ell}{\partial x_{\gamma,k}}\right)_{(\gamma,k),\ell}\right\rangle\\ &= \left\langle \det(-1)^{\gamma-1}\left(z_{\alpha(\gamma,k)}(F_m(s_\ell))\right)_{(\gamma,k),\ell}\right\rangle.\end{aligned}$$

As γ and k vary, we can pull a $(-1)^{\gamma-1}$ out of p different rows, where γ is varying from 1 to m. So we have to pull out $(-1)^{p\left(\sum_{\gamma=1}^m \gamma-1\right)} = (-1)^{m(m-1)p/2}$. This is the coefficient on (-1) we are seeing in the constant for C. Finally by Lemma 6.3.27 we have that the local degree of the Wronski and the index of σ agree up to these Vandermonde constants. $\square$

Reality check 6.4.9. In (SW21, Proposition 9), the authors demonstrated a formula for the local index of an analogous section in the specific case where $m = 2$ and $p = n-1$ for n odd. For the section $\sigma = \oplus_{i=1}^{2n-2}\alpha_i \wedge \beta_i$, they expressed $\alpha_i = \sum_j \alpha_{i,j}\phi_j$

and $\beta_i = \sum_j b_{i,j}\phi_j$, and demonstrated that the local index at $W = e_n \wedge e_{n+1}$ is given by (both in their notation and in the notation from this paper):

$$\operatorname{ind}_W \sigma = \left\langle \det \begin{vmatrix} \cdots & (a_{i,1}b_{i,n+1} - a_{i,n+1}b_{i,1}) & \cdots \\ & \vdots & \\ \cdots & (a_{i,j}b_{i,n+1} - a_{i,n+1}b_{i,j}) & \cdots \\ & \vdots & \\ \cdots & (a_{i,n-1}b_{i,n+1} - a_{i,n+1}b_{i,n-1}) & \cdots \\ \cdots & (a_{i,n}b_{i,1} - a_{i,1}b_{i,n}) & \cdots \\ & \vdots & \\ \cdots & (a_{i,n}b_{i,j} - a_{i,j}b_{i,n}) & \cdots \\ & \vdots & \\ \cdots & (a_{i,n}b_{i,n-1} - a_{i,n-1}b_{i,n}) & \cdots \end{vmatrix} \right\rangle$$

We note that each of the entries in the ith column of this matrix $\mathcal{B}$ is obtained by taking the matrix $\begin{pmatrix} a_{i,n} & b_{i,n} \\ a_{i,n+1} & b_{i,n+1} \end{pmatrix}$, swapping out a row for something suitable (as in our construction above), and then taking a determinant. Rewriting this local index in the notation from this paper, we can see that it takes the following form:

$$\mathrm{ind}_W\sigma = \left\langle \det \begin{vmatrix} \cdots & z_{\alpha(1,1)}(F_2(s_i)) & \cdots \\ & \vdots & \\ \cdots & z_{\alpha(1,j)}(F_2(s_i) & \cdots \\ & \vdots & \\ \cdots & z_{\alpha(1,n-1)}(F_2(s_i)) & \cdots \\ \cdots & (-1)z_{\alpha(2,1)}(F_2(s_i)) & \cdots \\ & \vdots & \\ \cdots & (-1)z_{\alpha(2,j)}(F_2(s_i)) & \cdots \\ & \vdots & \\ \cdots & (-1)z_{\alpha(2,n-1)}(F_2(s_i)) & \cdots \end{vmatrix} \right\rangle.$$

6.4.1 Maximally inflected curves

Given an m-plane $W = \operatorname{span}\{f_1, \ldots, f_m\}$ with Wronskian $\mathrm{Wr}(f_1, \ldots, f_m)(t) = \prod_{i=1}^{mp}(t - s_i)$, we can consider it as a rational curve $\phi : \mathbb{P}^1 \to \mathbb{P}^{m-1}$, given by mapping $t \mapsto [f_1(t) : \ldots : f_m(t)]$. The statement that the Wronskian vanishes at s_i is equivalent to the statement that the vectors $\phi(s), \phi'(s), \ldots, \phi^{m-1}(s)$ do not span $\mathbb{P}^{m-1}$ at time $t = s_i$ (c.f. (Sot11; KS03)). Equivalently one says that the curve *ramifies* or *inflects* at time s_i. The degree of the Wronski map then admits another interpretation: it counts how many rational curves of degree $\leq m + p - 1$ have *prescribed* *inflection* at times $t = s_1, \ldots, s_{mp}$. With our refined local index in hand, we can ask the following question: *How does the local degree* $\deg_W^{\mathbb{A}^1} \mathrm{Wr}$ *of the Wronski map relate to the topology (or geometry) of the associated rational curve* ϕ*?*

We don't claim any general answer to this question. Indeed studying topological constraints on inflected curves is a difficult problem in general. In the case when $m = 3$ and $p = 1, 2, 3$, we are looking at planar cubics, quartics, and quintics, respectively. Kharlamov and Sottile (KS03) have studied real inflection data in this setting (by the Shapiro–Shapiro conjecture, when the inflection points are real, the rational curve will be real as well). We can present some very preliminary observations that tie our local degree to their work.

In the case of quartics, there are five different quartics with six flexes (this five is the complex degree of the Wronski map whose domain is $\mathrm{Gr}_{\mathbb{C}}(2, 5)$). The graphs of these, pulled from (KS03), are included below.[3] While the curves look topologically distinct due to the nodal singularities, it is perhaps more telling to look at the number of isolated points (real ordinary points with complex conjugate tangent directions).

[3]One remarks that the three leftmost curves have two flexes at the point of self-intersection. This is purely an accident, due to the symmetry on the projective line of the prescribed flex points. In general we shouldn't expect this to happen, and our computations are not impacted by this coincidence.

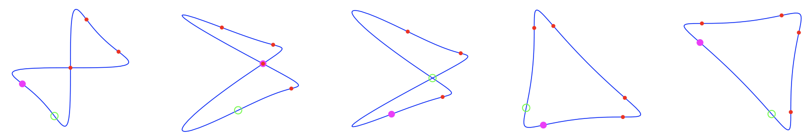
Curve					
# isolated points	2	2	2	3	3
Local $\mathbb{A}^1$-index over $\mathbb{R}$	$+1$	$+1$	$+1$	-1	$-1.$

Figure 6.3: Maximally inflected real quartics, (KS03, p. 23)

Welschinger (Wel05) remarked that $(-1)^{\#I}$ is a revealing invariant to consider for planar curves, where I is the set of isolated points. Kass–Levine–Solomon–Wickelgren have extended this to define an arithmetic Welschinger invariant valued in $\mathrm{GW}(k)$ (KLSW22) (see also (Lev18) and (PW21)). We may compute that Welschinger's original invariant agrees with the local index of σ following the formula in Theorem 6.4.3.

Corollary 6.4.10. When $(f_1 : f_2 : f_3)$ defines a real planar quartic, we have that

$$\operatorname{sgn} \deg^{\mathbb{A}^1}_{\operatorname{span}\{f_1,f_2,f_3\}} \mathrm{Wr} = (-1)^{\#I},$$

where $(-1)^{\#I}$ is Welschinger's invariant.

It is possible that the arithmetic Welschinger invariant provides a local $\mathbb{A}^1$-degree for the Wronski map, which could potentially shine light on the classification of maximally inflected curves in higher degrees and higher dimensions. We plan to explore this idea in greater detail in a future paper.

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