

TOPICS IN MIRROR SYMMETRY FOR FANO VARIETIES AND  
MEROMORPHIC DDP CORRESPONDENCE

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# ABSTRACT

## TOPICS IN MIRROR SYMMETRY FOR FANO VARIETIES AND MEROMOPRHIC DDP CORRESPONDENCE.

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This thesis consists of two parts, each of which can be read independently.

The first part is about mirror symmetry of Fano varieties and related topics. We introduce the notion of a hybrid Landau-Ginzburg (LG) model, which is a mirror partner of a Fano variety with a chosen anti-canonical divisor. We formulate Kontsevich's homological mirror symmetry conjecture [Kon95] of such mirror pairs and show that it implies the mirror  $P=W$  conjecture, a refined Hodge number relation between associated mirror log Calabi-Yau varieties [HKP18]. Next, we discuss the deformation theory of hybrid LG models and related Hodge numbers.

The second part is based on a joint work with Jia-choon Lee [LL20]. We study the relation between Hitchin system and Calabi-Yau integrable system in the meromorphic setting of type A, motivated by the work of Diaconescu-Donagi-Pantev [DDP07]. We consider a symplectization of the meromorphic Hitchin integrable system, which is a semi-polarized integrable system in the sense of Kontsevich and

Soibelman [KS14]. On the Hitchin side, we show that the moduli space of unordered diagonally framed Higgs bundles forms an integrable system in this sense and recovers the meromorphic Hitchin system as the fiberwise compact quotient. Then we construct a new family of quasi-projective Calabi-Yau threefolds and show that its relative intermediate Jacobian fibration, as a semi-polarized integrable system, is isomorphic to the moduli space of unordered diagonally framed Higgs bundles.

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# Chapter 1

## Topics in Mirror Symmetry for Fano Varieties

### 1.1 Introduction

Mirror Symmetry relates two compact  $n$ -dimensional Calabi-Yau manifolds  $X$  and  $X^\vee$ : the complex (algebraic) geometry of  $X$  (B-side) is equivalent to the symplectic geometry of  $X^\vee$  (A-side) and vice versa [Hor+03][Asp+09]. The most general formulation of mirror symmetry known as the homological mirror symmetry conjecture was proposed by M. Kontsevich [Kon95]. The derived category of coherent sheaves on  $X$ ,  $D^b\text{Coh}(X)$ , is equivalent to the (derived) Fukaya category of  $X^\vee$ ,  $\text{Fuk}(X^\vee)$  [Sei12][Fuk+09]. On the other hand, the most basic form of mirror symmetry is the symmetry of Hodge numbers:  $h^{p,q}(X) = h^{d-p,q}(X^\vee)$  for all  $p, q$ . This simple

relation becomes more involved when we attempt to extend mirror symmetry to non-compact Calabi-Yau's  $U$  and  $U^\vee$ . In this case, Hodge data are refined by two filtrations: the weight filtration  $W_\bullet$  and the perverse Leray filtration  $P_\bullet$  associated to the canonical affinization map. Incorporating these refinements, we define a perverse-mixed Hodge polynomial of  $U$  as

$$PW_U(u, t, w, p) := \sum_{a,b,r,s} (\dim \mathrm{Gr}_F^a \mathrm{Gr}_{s+b}^W \mathrm{Gr}_{s+r}^P (H^s(U, \mathbb{C}))) u^a t^s w^b p^r.$$

where  $F^\bullet$  is the Hodge filtration (Definition 1.2.13).

**Conjecture 1.1.1.** (*Mirror  $P=W$  Conjecture*) *Assume that two  $n$ -dimensional log Calabi-Yau varieties  $U$  and  $U^\vee$  are mirror to each other. Then, we have the following polynomial identity.*

$$PW_U(u^{-1}t^{-2}, t, p, w) u^n t^n = PW_{U^\vee}(u, t, w, p) \quad (1.1.1)$$

We only focus on the  $w = 1$  specialization of the mirror  $P=W$  conjecture in the equation (1.1.1). This specialization gives the simpler relation between the weight filtration on  $H^\bullet(U)$  (B-side) and the perverse Leray filtration on  $H^\bullet(U^\vee)$  (A-side). More precisely, we have

$$\dim \mathrm{Gr}_F^q \mathrm{Gr}_{p+q+r}^W H^{p+q}(U) = \dim \mathrm{Gr}_F^{n-q} \mathrm{Gr}_{n+p-q+r}^P H^{n+p-q}(U^\vee) \quad (1.1.2)$$

for  $p, q, r$ .

To get a good control on non-compact spaces, we first assume  $U$  admits compactification  $(X, D_{sm})$  where  $X$  is a smooth  $n$ -dimensional Fano manifold and

$D_{sm} = X - U$  is a smooth anti-canonical divisor. Note that mirror symmetry of such Fano pairs has been studied extensively under Fano/Landau-Ginzburg (Fano/LG) correspondence [KKP08][KKP17] which we will review in Section 1.3. The mirror dual of  $(X, D_{sm})$  is a Landau-Ginzburg (LG) model, a pair  $(Y, \mathbf{w} : Y \rightarrow \mathbb{C})$  where  $Y$  is a  $n$ -dimensional Calabi-Yau and  $\mathbf{w}$  is a proper map called Landau-Ginzburg potential. The Fano/LG correspondence also incorporates the mirror symmetry between two compact Calabi-Yau varieties  $D_{sm}$  and  $Y_{sm}$ , a generic fiber of  $\mathbf{w} : Y \rightarrow \mathbb{C}$ . The category associated to  $X$  (resp.  $D_{sm}$ ) are the bounded derived category of coherent sheaves  $D^b\text{Coh}(X)$  (resp.  $D^b\text{Coh}(D_{sm})$ ). On the A-side, we consider the Fukaya-Seidel category  $\text{FS}(Y, \mathbf{w})$  and Fukaya category  $\text{Fuk}(Y_{sm})$ , which we will review in Section 2.1.4.

A various aspects of the mirror symmetry between the Fano pair  $(X, D_{sm})$  and the LG model  $(Y, \mathbf{w} : Y \rightarrow \mathbb{C})$  is summarized in Table 1.

	Geometry	Hodge numbers	Category
B-side	$(X, D_{sm})$	$(h^{p,q}(X), h^{p,q}(D_{sm}))$	$(D^b\text{Coh}(X), D^b\text{Coh}(D_{sm}))$
A-side	$(\mathbf{w} : Y \rightarrow \mathbb{C}, Y_{sm})$	$(h^{d-p,q}(Y, Y_{sm}), h^{d-1-p,q}(Y_{sm}))$	$(\text{FS}(Y, \mathbf{w}), \text{Fuk}(Y_{sm}))$

(Table 1. Fano/LG correspondence)

**Conjecture 1.1.2.** *(Relative HMS for the Fano pair  $(X, D_{sm})$ ). Let  $(X, D_{sm})$  be a Fano pair and  $(Y, \mathbf{w} : Y \rightarrow \mathbb{C})$  be a mirror LG model. Then, there is a commutative*

diagram of categories

$$\begin{array}{ccc}
\mathrm{D}^b\mathrm{Coh}(D_{sm}) & \begin{array}{c} \xleftarrow{i_*} \\ \xrightarrow{i^*} \end{array} & \mathrm{D}^b\mathrm{Coh}(X) \\
\downarrow \cong_{HMS} & & \downarrow \cong_{HMS} \\
\mathrm{Fuk}(Y_{sm}) & \begin{array}{c} \xleftarrow{\cup} \\ \xrightarrow{\cap} \end{array} & \mathrm{FS}(Y, w)
\end{array} \tag{1.1.3}$$

where  $(i_*, i^*)$  are induced functors from the inclusion  $i : D_{sm} \hookrightarrow X$  and  $(\cup, \cap)$  are restriction functor and Orlov's functor. Moreover, the diagram is expected to commute with the action of Serre functors.

Taking Hochschild homology  $HH_a$  in the upper horizontal sequence of diagram (1.1.3) yields a part of the spectral sequence associated to the weight filtration on  $H^\bullet(U)$ . Interestingly, the corresponding bottom sequence is the  $E_1$ -page of the spectral sequence associated to the perverse Leray filtration [CM10] on  $H^\bullet(Y)$ .

**Theorem 1.1.3.** *Given a Fano mirror pair  $\{(X, D \in |-K_X|), ((Y, \omega), \mathbf{w} : Y \rightarrow \mathbb{C})\}$ , the conjectural relative HMS (1.1.3) gives rise to the following homological (Hodge theoretic, topological) correspondence*

$$\bigoplus_{p-q=a} Gr_F^q Gr_{p+q+i}^W H^{p+q}(U) \cong Gr_{n+a+i}^P H^{n+a}(Y) \quad \text{for all } i = 0, 1 \tag{1.1.4}$$

Moreover, one can recover direct summands by taking associated graded pieces of monodromy weight filtration associated to the Serre functors. Then we have

$$Gr_F^q Gr_{p+q+i}^W H^{p+q}(U) \cong Gr_{2(n-q)}^W Gr_{n+p-q+i}^P H^{n+p-q}(Y) \quad \text{for all } i = 0, 1$$

In particular, if the canonical mixed Hodge structure on  $H^k(Y)$  is Hodge-Tate for all  $k$ , then we have the mirror  $P=W$  conjecture;

$$Gr_F^q Gr_{p+q+i}^W H^{p+q}(U) \cong Gr_{(n-q)}^F Gr_{n+p-q+i}^P H^{n+p-q}(Y) \quad \text{for all } i = 0, 1$$

The Hodge-Tate condition on the canonical mixed Hodge structure on the cohomology groups of  $Y$  essentially comes from the conjecture of Kontsevich-Katzarkov-Pantev [KKP17]. In *loc.cit.*, the authors studied flat deformations of ordinary Landau-Ginzburg models  $w : Y \rightarrow \mathbb{C}$ . The idea is to tamely compactify the LG model to  $f : Z \rightarrow \mathbb{P}^1$  with boundary divisor  $D_\infty := f^{-1}(\infty)$  and study the deformation of  $f$  anchored at  $\infty \in \mathbb{P}^1$ . It is controlled by the  $L_\infty$ -algebra associated to the  $f$ -adapted logarithmic deRham complex  $(\Omega_Z^\bullet(\log D_\infty, f), d)$ , the subcomplex of logarithmic deRham complex preserved by wedge product with  $df$ . Moreover, it gives rise to new Hodge numbers, the so-called  $f$ -adapted Hodge numbers  $f^{p,q}(Y, w) := \dim \mathbb{H}^q(Z, \Omega_Z^p(\log D_\infty, f))$ , which are conjectured to be equal to monodromy weight numbers  $h^{p,q}(Y, w) := \dim \mathrm{Gr}_p^{W(N)} H^{p+q}(Y, w)$  associated to the monodromy  $N$  around infinity of  $w : Y \rightarrow \mathbb{C}$ .

**Conjecture 1.1.4.** (*KKP Conjecture [KKP17]*) *Let  $((Y, \omega_Y), w : Y \rightarrow \mathbb{C})$  be a LG model mirror to a Fano variety  $X$ <sup>1</sup>. There is the identity of two Hodge numbers;*

$$f^{p,q}(Y, w) = h^{p,q}(Y, w)$$

for all  $p, q \geq 0$ .

In [Sha18], it is shown that Conjecture 1.1.4 is equivalent to the Hodge-Tate condition on the canonical mixed Hodge structures on the cohomology groups  $H^\bullet(Y)$ .

In the thesis, we will study the analogue of the various aspects of the Fano/LG correspondence introduced above in case where  $D$  is not smooth, but a simple

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<sup>1</sup>A choice of a anti-canonical divisor is not assumed

normal crossing with a certain positivity assumption. In particular, we will give answers to the following questions.

- Question 1.1.5.** 1. *How can we extend the Fano/LG correspondence to pairs  $(X, D)$  where  $D$  is a simple normal crossing divisor? In this case, can we still deduce the mirror  $P=W$  conjecture (Conjecture 1.1.1) from the relative homological mirror symmetry?*
2. *How do we control the deformation theory of the relevant LG models? What is the generalization of the KKP conjecture?*

For simplicity, we mainly consider the case where a smooth Fano manifold  $X$  has a simple normal crossing anti-canonical divisor  $D = D_1 \cup D_2$  with the intersection  $D_{12}$  where  $(D_1, D_{12})$  and  $(D_2, D_{12})$  are again smooth Fano pairs. In this case, Strominger-Yau-Zaslow (SYZ) mirror construction yields a pair of potentials  $h = (h_1, h_2) : Y \rightarrow \mathbb{C}^2$  associated to counting disks touching each irreducible component  $D_i$  [SYZ96][Aur07]. It induces the mirror of the Fano pair  $(D_1, D_{12})$  (resp.  $(D_2, D_{12})$ ) which is an ordinary LG model  $(Y_1, h_2|_{Y_1})$  (resp.  $Y_2, h_1|_{Y_2}$ ) where  $Y_i$  is a generic fiber of  $h_i$  for  $i = 1, 2$ . One can also obtain an ordinary LG potential  $w := h_1 + h_2 : Y \rightarrow \mathbb{C}$ , which is now non-proper, by composing with the summation map  $\Sigma : \mathbb{C}^2 \rightarrow \mathbb{C}$  since it corresponds to adding up two different counting. These mirror relations are summarized in Table 2.

B-side	$(X, D)$	$(D_1, D_{12})$	$(D_2, D_{12})$	$D_{12}$
A-side	$(w : Y \rightarrow \mathbb{C}, Y_{sm})$	$(Y_1, h_2 _{Y_1} : Y_1 \rightarrow \mathbb{C})$	$(Y_2, h_1 _{Y_2} : Y_2 \rightarrow \mathbb{C})$	$Y_{12}$

(Table 2. Extended Fano/LG correspondence)

Similar to the case where  $D$  is smooth, the relative version of homological mirror symmetry is given as an equivalence of categories associated to the pairs  $(X, D)$  and  $(Y, w : Y \rightarrow \mathbb{C})$ . However, we will refine this diagram by integrating all four mirror symmetry relations in the extended Fano/LG correspondence. This is motivated by the mirror  $P=W$  conjecture which suggests that the diagram of categories in both B and A sides be categorical shadows of the weight and perverse filtration associated to the hybrid LG model, respectively.

**Conjecture 1.1.6.** *(Relative HMS for Fano pair  $(X, D = D_1 \cup D_2)$ ) There is an equivalence of diagrams of categories*

$$\begin{array}{ccc}
 & \text{D}^b\text{Coh}(D_{12}) & \\
 \nearrow^{i_1^*} & & \nwarrow_{i_2^*} \\
 \text{D}^b\text{Coh}(D_1) & & \text{D}^b\text{Coh}(D_2) \\
 \nwarrow_{i_{1,X}^*} & & \nearrow^{i_{2,X}^*} \\
 & \text{D}^b\text{Coh}(X) &
 \end{array}
 \quad
 \begin{array}{ccc}
 & \text{Fuk}(Y_{12}) & \\
 \nearrow^{\cap_1} & & \nwarrow_{\cup_2} \\
 \text{FS}(Y_1, h_2|_{Y_1}) & & \text{FS}(Y_2, h_1|_{Y_2}) \\
 \nwarrow_{\cap_{1,Y}} & & \nearrow^{\cap_{2,Y}} \\
 & \text{FS}^{\text{wr}}(Y, w) &
 \end{array}
 \tag{1.1.5}$$

One needs to verify that all the functors, especially in the A-side diagram, are well-defined. This is achieved by verifying a gluing property of a generic fibration  $h|_{Y_{sm}} : Y_{sm} \rightarrow \mathbb{C}$ . In other words, the mirror of the anti-canonical  $D$  is expected to be glued from mirror LG models associated to the Fano pairs  $(D_1, D_{12})$  and  $(D_2, D_{12})$ .

**Lemma 1.1.7.** *Let  $(Y, h : Y \rightarrow \mathbb{C}^2)$  be a hybrid LG model which is mirror to a Fano complete intersection  $X$  with simple normal crossing anti-canonical divisor  $D = D_1 \cup D_2$ . Then a generic fibration  $h|_{Y_{sm}} : Y_{sm} \rightarrow \mathbb{C}$  is glued from  $h_2|_{Y_1} : Y_1 \rightarrow \mathbb{C}$  and  $h_1|_{Y_2} : Y \rightarrow \mathbb{C}$ .*

The proof of Lemma 1.1.7 relies on the construction of mirror hybrid LG models. We use the Hori-Vafa construction and take a suitable compactification of the (open) hybrid LG model. The key idea is to study the singular locus of the hybrid LG model.

Lemma 1.1.7 implies that the Fukaya-Siedel category  $FS(Y_{sm}, h|_{Y_{sm}})$  has semi-orthogonal decomposition  $\langle FS(Y_1, w_2), FS(Y_2, w_1) \rangle$ , which enables one to construct relevant functors on the A-side diagram in (1.1.5). In addition, the equivalence of two diagrams is compatible with relevant Serre functors so that we have the following main theorem.

**Theorem 1.1.8.** *Given a Fano mirror pair  $(X, D = D_1 \cup D_2), (Y, h : Y \rightarrow \mathbb{C}^2)$ , the conjectural relative HMS (Conjecture 1.4.8) and Proposition (1.4.5) gives rise to the following homological (Hodge theoretic, topological) correspondence.*

$$\bigoplus_{p-q=a} Gr_F^p Gr_{p+q+i}^W H^{p+q}(U) \cong Gr_{n+a+i}^P H^{n+a}(Y) \quad \text{for all } i = 0, 1, 2$$

Moreover, one can recover direct summands by taking associated graded pieces of monodromy weight filtration from the Serre functors. Then we have

$$Gr_F^q Gr_{p+q+i}^W H^{p+q}(U) \cong Gr_{2(n-q)}^W Gr_{n+a+i}^P H^{n+a}(Y) \quad \text{for all } i = 0, 1, 2$$



In particular, if the canonical mixed Hodge structure on  $H^k(Y)$  is Hodge-Tate for all  $k$ , then we have the mirror  $P=W$  conjecture;

$$Gr_F^q Gr_{p+q+i}^W H^{p+q}(U) \cong Gr_{(n-q)}^F Gr_{n+p-q+i}^P H^{n+p-q}(Y) \quad \text{for all } i = 0, 1, 2.$$

Next, we study the deformation theory of hybrid LG models and extend the KKP conjecture (Conjecture 1.1.4) in the hybrid setting. The Hodge-Tate condition on the canonical mixed Hodge structure on  $H^k(Y)$  for all  $k$  turns out to be equivalent to the extend version of the KKP conjecture (Conjecture 1.1.10).

Regarding the deformation theory of the hybrid LG model  $(Y, h : Y \rightarrow \mathbb{C}^2)$ , one can extend the previous story by taking an appropriate tame compactification  $f = (f_1, f_2) : Z \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  with  $D_Z := f^{-1}(\mathbf{L})$  where  $\mathbf{L} := \{\infty\} \times \mathbb{P}^1 \cup \mathbb{P}^1 \times \{\infty\}$  is the complement of  $\mathbb{C}^2$  in  $\mathbb{P}^1 \times \mathbb{P}^1$ .

**Theorem 1.1.9.** *Let  $f = (f_1, f_2) : Z \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  be a tame compactification of the hybrid LG model  $h : Y \rightarrow \mathbb{C}^2$ . Then the deformation theory of  $f$  anchored at the boundary  $\mathbf{L}$  is unobstructed.*

We introduce a  $f$ -adapted complex  $(\Omega_Z^\bullet(\log D_Z, f), d)$ , the subcomplex of  $(\Omega_Z^\bullet(\log D_Z), d)$  preserved by either  $df_1 \wedge$  or  $df_2 \wedge$ . It admits a stupid filtration, called Hodge filtration, whose spectral sequence degenerates at  $E_1$ -page (Proposition 1.4.17). Therefore, one can define  $f$ -adapted Hodge number in the hybrid setting, which turns out to be independent of the choice of the tame compactification. On the other hand, associated to the hybrid LG model, there are three

different monodromy weight filtrations  $N_1, N_2$ , and  $c_1N_1 + c_2N_2$  for  $c_1, c_2 > 0$  where  $N_1$  and  $N_2$  correspond to monodromies around each irreducible component of  $\Gamma$ . It allows to define three monodromy weight numbers of the cohomology  $H^a(Y, Y_{12})$ . This cohomology group is recovered from the hypercohomology of  $f$ -adapted de Rham complex. Now we can make conjectural relations between the associated Hodge numbers of  $\mathbb{H}^a(\Omega_Z^\bullet(\log D_\Gamma, f_1 \cup f_2), d)$  and monodromy weight numbers of  $H^a(Y, Y_{12})$ . It is expected that three monodromy filtrations give rise to the same monodromy weight number. In summary, we have the following conjecture;

**Conjecture 1.1.10.** (*Extended KKP Conjecture*) *Let  $f = (f_1, f_2) : Z \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  be a tame compactification of the hybrid LG model  $h : Y \rightarrow \mathbb{C}^2$ . For any  $p, q, c_1, c_2 \geq 0$ , we have the identification of Hodge numbers*

$$\dim \mathbb{H}^q(Z, (\Omega_Z^p(\log D_\Gamma, f), d)) = \dim Gr_{2(n-p)}^{W(c_1N_1+c_2N_2)} H^{p+q}(Y, Y_{12})$$

In general, when  $D$  has more than two components, one can obtain the similar results under the assumption that each irreducible component of the anti-canonical divisor is Fano itself. This is necessary condition for the relative HMS hold. We state relevant theorems and conjectures in Section 2.4.

## 1.2 Preliminaries

### 1.2.1 Weight filtration

We review the Deligne's construction of the weight filtration and mixed Hodge structure on the cohomology of a quasi-projective variety by following logistics in [PS08].

Let  $U$  be a quasi-projective variety over  $\mathbb{C}$  and assume that we have a good compactification <sup>2</sup>  $(X, D)$ . Recall that a pair  $(X, D)$  where  $X$  is a smooth and compact variety with a simple normal crossing divisor  $D$  is called a good compactification of  $U$  if  $U = X \setminus D$ .

Let  $j : U \rightarrow X$  be a natural inclusion. Consider a logarithmic de Rham complex

$$\Omega_X^\bullet(\log D) \subset j_*\Omega_U^\bullet$$

Locally at  $p \in D$  with an open neighborhood  $V \subset X$  with coordinates  $(z_1, \dots, z_n)$

in which  $D$  is given by  $z_1 \cdots z_k = 0$ , one can see

$$\Omega_X^1(\log D)_p = \mathcal{O}_{X,p} \frac{dz_1}{z_1} \oplus \cdots \oplus \mathcal{O}_{X,p} \frac{dz_k}{z_k} \oplus \mathcal{O}_{X,p} dz_{k+1} \oplus \cdots \oplus \mathcal{O}_{X,p} dz_n$$

$$\Omega_X^r(\log D)_p = \bigwedge^r \Omega_X^1(\log D)_p$$

There are two natural filtrations on  $\Omega_X^\bullet(\log D)$ .

**Definition 1.2.1.** 1. The decreasing filtration  $F^\bullet$  on  $\Omega_X^\bullet(\log D)$  is defined by

$$F^p \Omega_X^r(\log D) := \Omega_X^{\geq p}(\log D)$$

---

<sup>2</sup>Assumption on the compactification  $(X, D)$  is not essential. For example, it is allowed to have mild singularities(quotient, canonical, etc).

2. The increasing filtration  $W_\bullet$  on  $\Omega_X^\bullet(\log D)$  is defined by

$$W_m \Omega_X^r(\log D) := \begin{cases} 0 & m < 0 \\ \Omega_X^r(\log D) & m \geq r \\ \Omega_X^{r-m} \wedge \Omega_X^m(\log D) & 0 \leq m \leq r \end{cases}$$

**Theorem 1.2.2.** 1. The logarithmic de Rham complex  $\Omega_X^\bullet(\log D)$  is quasi-isomorphic to  $j_* \Omega_U^\bullet$ . i.e.

$$H^k(U; \mathbb{C}) = \mathbb{H}^k(X, \Omega_X^\bullet(\log D))$$

2. The decreasing filtration  $F^\bullet$  on  $\Omega_X^\bullet(\log D)$  induces the filtration in cohomology

$$F^p H^k(U; \mathbb{C}) = \text{Im}(\mathbb{H}^k(X, F^p \Omega_X^\bullet(\log D)) \rightarrow H^k(U; \mathbb{C}))$$

which is called **the Hodge filtration** on  $H^\bullet(U)$ .

Similarly, the increasing filtration  $W_\bullet$  on  $\Omega_X^\bullet(\log D)$  induces the filtration in cohomology

$$W_m H^k(U; \mathbb{C}) = \text{Im}(\mathbb{H}^k(X, W_{m-k} \Omega_X^\bullet(\log D)) \rightarrow H^k(U; \mathbb{C}))$$

which is called **the weight filtration** on  $H^\bullet(U)$ . In particular, the weight filtration can be defined over the field of rational numbers  $\mathbb{Q}$  and we denote it by  $W_\bullet^\mathbb{Q}$ .

3. The package  $(\Omega_X^\bullet(\log D), W_\bullet^\mathbb{Q}, F^\bullet)$  gives a rational mixed Hodge structure on  $H^k(U; \mathbb{C})$ .

The key properties of these two filtrations are the degeneration of the associated spectral sequences. More precisely, we have

**Proposition 1.2.3.** 1. *The spectral sequence for  $(\mathbb{H}(X, \Omega_X^\bullet(\log D), F^\bullet)$  whose*

*$E_1$ -page is given by*

$$E_1^{p,q} = \mathbb{H}^{p+q}(X, Gr_F^p \Omega_X^\bullet(\log D))$$

*degenerates at  $E_1$ -page. Thus,*

$$Gr_F^p \mathbb{H}^{p+q}(X, \Omega_X^\bullet(\log D)) = \mathbb{H}^{p+q}(X, Gr_F^p \Omega_X^\bullet(\log D))$$

2. *The spectral sequence for  $(\mathbb{H}(X, \Omega_X^\bullet(\log D), W_\bullet)$  whose  $E_1$ -page is given by*

$$E_1^{-m,k+m} = \mathbb{H}^k(X, Gr_m^W \Omega_X^\bullet(\log D))$$

*degenerates at  $E_2$ -page and the differential  $d_1 : E_1^{-m,k+m} \rightarrow E_1^{-m+1,k+m}$  is strictly compatible with the filtration  $F_\bullet$ . In other words,*

$$E_2^{-m,k+m} = E_\infty^{-m,k+m} = Gr_{m+k}^W \mathbb{H}^k(X, \Omega_X^\bullet(\log D))$$

In order to compute a mixed Hodge structure, we should describe the spectral sequences more explicitly. For a given normal crossing divisor  $D$ , let's denote  $D_i$  an irreducible component of  $D$ . We set  $D(k)$  to be a disjoint union of  $k$ -th intersection.

$$D(0) := X$$

$$D(1) := D_1 \cup \cdots \cup D_n$$

$$D(k) := \coprod_I D_I, \quad |I| = k$$

Also, for  $I = (i_1, \dots, i_m)$  and  $J = (i_1, \dots, \hat{i}_j, \dots, i_m)$ , there are inclusion maps

$$\rho_J^I : D_I \hookrightarrow D_J$$

$$\rho_j^m = \bigoplus_{|I|=m} \rho_J^I : D(m) \hookrightarrow D(m-1)$$

which induces a canonical Gysin map on the level of cohomology. Therefore, we have

$$\gamma_m = \bigoplus_{j=1}^m (-1)^{j-1} (\rho_j^m)_! : H^{k-m}(D(m))(-m) \rightarrow H^{k-m+2}(D(m-1))(-m+1)$$

where  $(-)_!$  is a Gysin map. Under the residue map, this gives a geometric description of the differential  $d_1 : E_1^{-m, k+m} \rightarrow E_1^{-m+1, k+m}$  as follows;

**Proposition 1.2.4.** *The following diagram is commutative.*

$$\begin{array}{ccc} E_1^{-m, k+m} & \xrightarrow{\text{res}_m} & H^{k-m}(D(m); \mathbb{C})(-m) \\ \downarrow d_1 & & \downarrow -r_m \\ E_1^{-m+1, k+m} & \xrightarrow{\text{res}_m} & H^{k-m+2}(D(m-1); \mathbb{C})(-m-1) \end{array} \quad (1.2.1)$$

where  $\text{res}_m$  is the residue map.

Note that all morphisms in the diagram (1.2.1) are compatible with Hodge filtration  $F^\bullet$ . This description provides several functorial properties of mixed Hodge structures under geometric morphisms, which provides computational tools. We refer more details to [PS08]. We introduce one more terminology, which will be used later.

**Definition 1.2.5.** Let  $U$  be a quasi-projective variety. A mixed Hodge structure on  $H^k(U)$  is called **Hodge-Tate** if the weight  $2l$ -Hodge structure on the associated graded pieces  $\text{Gr}_{2l} H^k(U)$  is concentrated at  $(l, l)$  for  $l \geq 0$ .

*Example 1.2.6.* Let  $U$  be a  $n$ -dimension torus  $(\mathbb{C}^*)^n$ . It admits a good compactification  $(\mathbb{C}\mathbb{P}^n, D)$  where  $D$  is a toric anti-canonical divisor. From the spectral sequence argument, it is easy to see that a mixed Hodge structure on  $H^k(U)$  is given by both long exact sequence of the pair  $(X_n, E)$  and Poincaré duality;

$$\begin{aligned} \cdots \subset 0 = W_{2k-1}^{\mathbb{Q}} \subset W_{2k}^{\mathbb{Q}} = H^k(U; \mathbb{Q}) \\ \cdots \subset 0 = F^{k+1} \subset F^k = H^k(U; \mathbb{C}) \end{aligned}$$

In other words, the only non-trivial associated graded piece is  $\mathrm{Gr}_k^F \mathrm{Gr}_W^{2k} H^k(U; \mathbb{C}) = \mathbb{C} \binom{n}{k}$  for all  $k$ . Using the notion of Tate twist, one can write down the mixed Hodge structure as follows;

$$H^k(U; \mathbb{Q}) \cong \mathbb{Q}(-k) \binom{n}{k}$$

and it is clearly of Hodge-Tate type.

*Example 1.2.7.* Let  $X_n$  be a Delpazzo surface of degree  $9 - n$  which is a blow up of  $\mathbb{C}\mathbb{P}^2$  at generic  $n(\leq 8)$  points. Note that  $X_n$  is Fano so that there is a smooth anti-canonical divisor  $E \subset X_n$ . By adjunction the complement  $U_n = X_n \setminus E$  is Calabi-Yau. The mixed Hodge structure on  $H^k(U)$  is given by

1.  $H^0(U; \mathbb{Q}) \cong \mathbb{Q}(0)$
2.  $H^2(U; \mathbb{Q})$  sits in a short exact sequence

$$0 \rightarrow \mathbb{Q}(-1)^{n-1} \rightarrow H^2(U; \mathbb{Q}) \rightarrow H^1(E; \mathbb{Q})(-1) \rightarrow 0$$

3. Otherwise,  $H^i(U; \mathbb{Q}) = 0$ .

It implies that the mixed Hodge structure on  $H^2(U; \mathbb{Q})$  is of Hodge-Tate type with non-trivial associated graded pieces of the weight filtration  $\text{Gr}_2^W$  and  $\text{Gr}_4^W$ .

### 1.2.2 Perverse filtration

In algebraic geometry, the notion of perversity was invented by Mark Goresky and Robert MacPherson [GM80] [GM83] to capture the singular behavior of an algebraic variety or sheaves via cohomology theories. This can be used to understand topology of algebraic maps. Let  $X$  be an algebraic variety or scheme. In case  $X$  is singular, a sheaf  $F$  on  $X$  behaves unexpectedly over singular locus which makes it difficult to understand the cohomology ring structure. To resolve this issue, instead of studying sheaves on  $X$ , one can introduce the notion of constructible sheaves which becomes locally constant over each singular strata. It forms a well-defined triangulated category,  $D_c^b(X)$ , called the (bounded) derived category of constructible sheaves.

**Definition 1.2.8.** Let  $X$  be an algebraic variety (or scheme) with  $D_c^b(X)$  a derived category of constructible sheaves on  $X$ . An object  $K^\bullet \in D_c^b(X)$  is called a **perverse sheaf** if it satisfies following two dual conditions.

- (Support Condition)  $\dim \text{supp}(\mathcal{H}^i(K^\bullet)) \leq -i$
- (Cosupport Condition)  $\dim \text{supp}(\mathcal{H}^i(\mathbb{D}K^\bullet)) \leq -i$  where  $\mathbb{D} : D_c^b(X) \rightarrow D_c^b(X)$  is a dualizing functor.

Recall that the dualizing functor  $\mathbb{D} = \text{Hom}_{\mathcal{O}_X}(-, p^!(\mathbb{C}_{pt}))$  where  $p : X \rightarrow pt$  is a trivial map. We call  $p^!(\mathbb{C}_{pt})$  a dualizing complex of  $X$ , and denoted by  $\omega_X$ .



In particular, if  $X$  is non-singular of (complex) dimension  $n$ ,  $\omega_X = \mathbb{C}_X[2n]$ . Note that the subcategory of perverse sheaves on  $X$ ,  $\mathcal{P}(X)$  forms an abelian category. Also, the support and cosupport condition induces a so-called a perverse  $t$ -structure  $({}^p D_c^{b, \geq 0}(X), {}^p D_c^{b, \leq 0}(X))$  on  $D_c^b(X)$  whose heart is  $\mathcal{P}(X)$ . Explicitly, it is given by

- $K^\bullet \in {}^p D_c^{b, \leq 0}(X)$  if and only if  $K$  satisfies the support condition. Also,  ${}^p D_c^{b, \leq n}(X) := {}^p D_c^{b, \leq 0}(X)[-n]$
- $K^\bullet \in {}^p D_c^{b, \geq 0}(X)$  if and only if  $K$  satisfies the cosupport condition. Also,  ${}^p D_c^{b, \geq n}(X) := {}^p D_c^{b, \geq 0}(X)[-n]$

We denote  ${}^p \tau_{\leq n} : D^b(X) \rightarrow {}^p D_c^{b, \leq n}(X)$  (resp.  ${}^p \tau_{\geq n} : D^b(X) \rightarrow {}^p D_c^{b, \geq n}(X)$ ) a canonical truncation functor. Now we can define perverse filtration associated to the algebraic map.

**Definition 1.2.9.** Let  $f : X \rightarrow Y$  be a map between algebraic varieties. For a complex of sheaves  $K^\bullet$ , the perverse (Leray) filtration on  $H^*(X; K^\bullet)$  is given by

$$P_b H^*(X, K^\bullet) := \text{Im}(H^*(Y, {}^p \tau_{\leq b} Rf_* K^\bullet) \rightarrow H^*(Y, Rf_* K^\bullet) = H^*(X, K^\bullet))$$

where  ${}^p \tau$  is a perverse truncation on  $D_c^b(Y)$ .

Due to the Decomposition Theorem [BBD82], the spectral sequence associated to the perverse filtration, (special case of Grothendieck's spectral sequence) degenerates at  $E_2$  page. Thus,

$$Gr_p^P H^{p+q}(X, K^\bullet) = E_\infty^{q,p} = E_2^{q,p} = H^q(Y, {}^p \mathcal{H}^p(Rf_* K^\bullet)) \quad (1.2.2)$$

The following theorem provides a geometric description of the perverse Leray filtration.

**Theorem 1.2.10.** *Let  $f : X \rightarrow Y$  be a map of varieties with  $Y$  being affine. For each  $K^\bullet \in D^b(X)$ , there is a generic flag  $Y_\bullet \subset Y$ , with pre-image flag  $X_\bullet \subset X$ , such that*

$$P_b H^*(X, K^\bullet) = \text{Ker}\{(H^*(X, K^\bullet) \rightarrow H^*(X_{b-*+1}, K^\bullet|_{X_{b-*+1}}))\}$$

where  $Y_k \subset Y$  has codimension  $k$ .

Note that the length of filtration depends on dimension of  $Y$ . One can show that the length is  $\dim(Y) + 1$

$$0 = P_{k-1} \subset P_k \subset \cdots \subset P_{\dim(Y)+k-1} \subset P_{\dim(Y)+k} = H^k(X, K)$$

The affinity condition in Theorem 1.2.10 is not very restrictive because for any quasi-projective variety  $U$ , there is a canonical affinization map

$$\text{Aff} : U \rightarrow \text{Spec } H^0(U, \mathcal{O}_U)$$

so that one can define the canonical perverse filtration associated to the variety itself.

*Example 1.2.11.* Let  $U$  be a  $n$ -dimensional affine torus  $(\mathbb{C}^*)^n$ , then the affinization map is given by the natural inclusion  $: U \rightarrow \mathbb{C}^n$ . By iteratively applying Lefschetz hyperplane theorem, one can see that the only non-trivial associated graded pieces of the canonical perverse Leray filtration is  $\text{Gr}_{n-k} H^k(U; \mathbb{C}) = H^k(U; \mathbb{C})$ .

*Example 1.2.12.* Let  $X$  be a smooth projective variety with smooth ample divisor  $D$ . Then the complement  $U := X \setminus D$  is always affine. Then we can iteratively apply Lefschetz hyperplane theorem to argue that  $\mathrm{Gr}_i^P H^k(U; \mathbb{C})$  is non-trivial only when  $i = \dim(U) - k$ .

Moreover, since the canonical perverse Leray filtration is compatible with the canonical mixed Hodge structure, we can define **perverse-mixed Hodge polynomial** which encodes refined Hodge numbers.

**Definition 1.2.13.** For any (non-singular) quasi-projective variety  $U$ , we define a **perverse-mixed Hodge polynomial** as follows

$$PW_U(u, v, w, p) = \sum_{a,b,r,s} (\dim \mathrm{Gr}_F^a \mathrm{Gr}_{s+b}^W \mathrm{Gr}_{s+r}^P (H^s(U, \mathbb{C}))) u^a v^s w^b p^r$$

*Example 1.2.14.* Combining previous examples, we get

1.  $PW_{(\mathbb{C}^*)^d}(u, v, w, p) = (uvw + p)^d$
2.  $X_n$  Delpezzo surface of degree  $9 - n$ , and  $E \in |K_{X_n}^{-1}|$  smooth anti-canonical divisor.  $Y_n := X_n \setminus E$

$$PW_{Y_n}(u, v, w, p) = p + uw^2p^2 + uv^2w^2 + (9 - n)uw^2$$

This is one of the main topics we will discuss further.

### 1.2.3 Categories on B-side

Let  $X$  be a quasi-projective algebraic variety. We consider two categories which encodes essential geometric information of  $X$ ;

- A derived category of perfect complexes on  $X$ ,  $\text{Perf}(X)$ , whose objects are bounded complexes of vector bundles of finite rank.
- A bounded derived category of coherent sheaves on  $X$ ,  $D^b\text{Coh}(X)$  whose objects are bounded complexes of coherent sheaves

A natural transformation  $I : \text{Perf}(X) \hookrightarrow D^b\text{Coh}(X)$  becomes an isomorphism when  $X$  is smooth. Therefore, the categorical localization of the natural transformation  $I$  detects singularities of  $X$  so is called bounded derived category of singularities,  $D_{sg}^b(X)$ .

Now assume that  $X$  is a smooth Fano variety of dimension  $n$  and choose a non-trivial anti-canonical divisor  $D = \{s = 0\}$  where  $s \in H^0(X, K_X^{-1})$ . By the adjunction formula,  $D$  is Calabi-Yau hypersurface  $i : D \hookrightarrow X$ . The complement  $j : U := X \setminus D \hookrightarrow X$  is also Calabi-Yau since the inverse of the section  $s$  induces a nowhere-vanishing holomorphic  $n$ -form. Note that both inclusions  $i$  and  $j$  induces natural transformation which allows to compare three (bounded) derived categories of coherent sheaves  $D^b\text{Coh}$  for  $X, D$  and  $U$ . In particular, the composition of natural transformations

$$D^b\text{Coh}(D) \xrightarrow{i_*} D^b\text{Coh}(X) \xrightarrow{j^*} D^b\text{Coh}(U)$$

is trivial. Moreover, we claim that the above sequence is a categorical localization [KS06] [Sei08]. To see this, we consider the following triangle of natural transfor-

mations;

$$\begin{array}{ccc}
 id_X & \xrightarrow{\quad\quad\quad} & i_*i^* \\
 & \swarrow \quad \searrow & \\
 & S_X[-n+1] &
 \end{array}$$

where  $S_X$  is the Serre functor of  $D^b\text{Coh}(X)$  which is given by  $- \otimes K_X[n]$  and

- $S_X[-n+1] \rightarrow id$  is given by the choice of  $s : \mathcal{O}_X \rightarrow K_X^{-1}$  and the shift functor  $[1]$ ;
- $id_X \rightarrow i_*i^*$  is the unit map of the adjunction  $i^* \dashv i_*$ ;
- $i_*i^* \rightarrow S_X$  is determined by the composition

$$\begin{aligned}
 \text{Hom}_{D^b(D)}(i^*E, i^*F[n-1]) &\rightarrow \text{Hom}_{D^b(D)}(i^*F, i^*E)^\vee \\
 &\rightarrow \text{Hom}_{D^b(X)}(F, E)^\vee \rightarrow \text{Hom}_{D^b(X)}(E, S_X(F))
 \end{aligned}$$

where the first and the last map come from Serre duality.

Since morphisms between coherent sheaves on  $U$  can be calculated by considering morphisms on  $X$  which may have poles along  $D$ . In other words, we have

$$\text{Hom}_{D^b(U)}(j^*E, j^*F) \cong \lim_p \text{Hom}_{D^b(X)}(E, F \otimes (K_X^{-1})^{\otimes p})$$

This shows that the category  $D^b\text{Coh}(U)$  is equivalent to the categorical localization of  $D^b\text{Coh}(X)$  at the natural transformation  $S_X \rightarrow id$  determined by  $s$ . Finally, the above triangle implies that the localizing subcategory is isomorphic to  $D^b\text{Coh}(D)$ .

$$\begin{array}{ccc}
 \overline{D^b\text{Coh}(D)} & \xleftrightarrow{i_*} & \overline{D^b\text{Coh}(X)} \\
 \overline{D^b\text{Coh}(D)} & \xleftarrow{i^*} & \overline{D^b\text{Coh}(X)}
 \end{array}
 \longrightarrow D^b\text{Coh}(U) \quad (1.2.3)$$

Finally, note that the bounded derived categories has a natural enhancement to differential-graded (dg) categories and the categorical localization in (1.2.3) can be understood as that of dg categories as well.

## 1.2.4 Categories on A-side

Next, we introduce the A-side analogue of diagram. Let  $(Y, \omega)$  be a (possibly non-compact) symplectic manifold with a symplectic form  $\omega \in \Omega^2(Y)$ . If  $(Y, \omega)$  is non-compact, we assume that it is Liouville manifold with contact boundary  $\partial_\infty Y$ .

We associate two  $A_\infty$ -categories which encodes geometric information  $Y$ .

- A Fukaya category of  $Y$ ,  $\text{Fuk}(Y)$ , whose objects are compact Lagrangian branes and morphisms are Floer chain complexes.
- A wrapped Fukaya category of  $Y$ ,  $\text{Fuk}^{wr}(Y)$ , whose objects are Lagrangians with conical at  $\infty$  and morsphims are wrapped Floer complex.

*Remark 1.* To define Fukaya categories properly, one needs to add more decorations on geometric structures on  $Y$  and specify brane structures. As this is not the main topic we will study, we simplify the definition and deliver the smallest amount of technicality. We refer to [Fuk+09][Aur13] for more details.

Moreover, we introduce another category, called Fukaya-Seidel category  $\text{FS}(Y, \mathbf{w})$ , associated to a Lefschetz fibration  $\mathbf{w} : Y \rightarrow \mathbb{C}$ . Recall that away from a compact region  $K_{cpt} \subset \mathbb{C}$ ,  $\mathbf{w}|_{\mathbb{C} \setminus K_{cpt}} : Y \setminus \mathbf{w}^{-1}(K_{cpt}) \rightarrow \mathbb{C}$  is a genuine fibration whose fiber is a Lefschetz submanifold.

**Definition 1.2.15.** An admissible Lagrangian associated to the Lefschetz fibration  $w : (Y, \omega) \rightarrow \mathbb{C}$  is a (possibly non-compact) Lagrangian  $L$  in  $Y$  such that  $w(L)$  is contained in a union of a compact subset and (possibly multiple) radical rays away from negative real axis.

To an admissible Lagrangian  $L$ , one can associate a subset  $D_L \subset (-\pi, \pi)$  of direction of  $L$  near  $\infty$ . In order to define Floer theory of admissible Lagrangians, we allow non-compact Hamiltonian perturbations as well as choose "counterclockwise" direction near  $\infty$ . Then for admissible Lagrangians  $K$  and  $L$ , we say  $K > L$  if  $\theta_k > \theta_L$  for any  $\theta_k \in D_K$  and  $\theta_L \in D_L$ .

Consider  $\mathfrak{A}$  be a directed  $A_\infty$  category whose

- Objects are admissible Lagrangian branes of  $((Y, \omega), w)$ ;
- Morphism spaces are

$$\mathrm{Hom}_{\mathfrak{A}}(K, L) = \begin{cases} CF^\bullet(K, L) & K > L \\ \mathbb{C} \langle e_L^+ \rangle & K = L \\ 0 & o.w \end{cases}$$

with  $A_\infty$  relations defined by counting  $J$ -holomorphic discs and  $e_L^+$  being a strict unit.

This is directed in the sense that  $ob\mathfrak{A}$  is a poset with  $\mathrm{hom}(K, L) = 0$  unless  $k > L$ . Moreover, in this category,  $L$  is not quasi-isomorphic to its perturbation  $\phi_\epsilon L$  because  $\mathrm{Hom}_{\mathfrak{A}}(\phi_\epsilon L, L) \neq L$  while  $\mathrm{hom}_{\mathfrak{A}}(L, \phi_\epsilon L) = 0$ . One achieve the quasi-isomorphism

between  $L$  and  $\phi_\epsilon(L)$  by inverting all quasi-units in the cohomology of morphism spaces. Let  $Z \subset H^0(\mathfrak{A})$  be a collection of all quasi-units.

**Definition 1.2.16** (Abouzaid-Seidel). The Fukaya-Seidel category  $FS(Y, \mathbf{w})$  is defined to be a categorical localization of  $\mathfrak{A}$  at  $Z$ .

*Remark 2.* There is another equivalent definition of Fukaya-Seidel category whose objects are Lagrangian thimbles, coming from vanishing cycles of a generic fiber. This is more intuitive, but rather difficult to handle technical issues.

We first construct relevant functors between the Fukaya-Seidel category  $FS(Y, \mathbf{w})$  and Fukaya category  $\text{Fuk}(Y_{sm})$  where  $Y_{sm}$  is a general fiber of  $\mathbf{w} : Y \rightarrow \mathbb{C}$ ;

$$\mu \circlearrowleft \text{Fuk}(Y_{sm}) \begin{array}{c} \xrightarrow{\cup} \\ \xleftarrow{\cap} \end{array} \text{FS}(Y, \mathbf{w}) \circlearrowright \phi_{2\pi}$$

where

- $\cap : FS(Y, \mathbf{w}) \rightarrow \text{Fuk}(Y_{sm})$  is called a cap functor, given by intersection of an admissible Lagrangian with the general fiber  $Y_{sm}$ .
- $\cup : \text{Fuk}(Y_{sm}) \rightarrow FS(Y, \mathbf{w})$  is called Orlov's functor or cup functor, given by the trajectory of a parallel transport along a U-shaped curve bounding all critical values.
- $\mu$  is the global monodromy induced by the parallel transport along a large enough loop and  $\phi_{2\pi}$  is defined to be counter-clockwise wrapping once.



*Remark 3.* The image of the cap functor lands in twisted Fukaya category of  $TwFuk(Y_{sm})$ . There is another version of construction of Fukaya category which fits into this discussion done by Abouzaid. The idea is to consider directed category to allow collection of Lagrangians and localize it at the collection of "quasi-units". It turns out to be quasi-equivalent to the ordinary construction of Fukaya category.

It is easy to see that  $\phi_{-2\pi} = \phi_{2\pi}^{-1}$  is a Serre functor (up to shift) of  $FS(Y, \mathbf{w})$ . For any admissible Lagrangian  $L$ , a choice of an element in  $\text{Hom}_{FS(Y, \mathbf{w})}(\phi_{2\pi}L, L)$  induces a natural transformation  $id \rightarrow \phi_{2\pi}$  which fits into the following exact triangle

$$\begin{array}{ccc} id & \xrightarrow{\quad} & \phi_{2\pi} \\ & \swarrow & \searrow \\ & \cup \cap & \end{array}$$

Now consider the localization of the Fukaya-Seidel category  $FS(Y, \mathbf{w})$  with respects to the natural transformation  $id \rightarrow \phi_{2\pi}$ , denoted by  $W(Y)$ . By theorem of Abouzaid and Seidel,  $W(Y)$  is isomorphic to the subcategory of the wrapped Fukaya category with objects in  $FS(Y, \mathbf{w})$ . Indeed, it is expected that  $W(Y)$  is  $A_\infty$ -equivalent to  $Fuk^{wr}(Y)$ . Now, we have the following diagram of  $A_\infty$  categorical localization [Sei08].

$$\begin{array}{ccc} \begin{array}{c} \curvearrowright \mu \\ \swarrow \end{array} & \begin{array}{c} \xrightarrow{\cup} \\ \xleftarrow{\cap} \end{array} & \begin{array}{c} \curvearrowright \phi_{2\pi} \\ \swarrow \end{array} \\ Fuk(Y_{sm}) & \longrightarrow & FS(Y, \mathbf{w}) \longrightarrow Fuk^{wr}(Y) \end{array}$$

### 1.2.5 Hochschild (Co)Homology

Hochschild homology (and cohomology) is a homology theory for associative algebras over rings. This notion can be generalized to homology theory for categories whose morphism spaces admit associative algebra structures up to homotopies. In this subsection, we review Hochschild homology and cohomology of algebras and categories.

Let  $A$  be an associative algebra over  $k$  and  $A^e := A \otimes A^{op}$  be an enveloping algebra of  $A$  where  $A^{op}$  is an opposite algebra. Consider a free  $A$ -bimodule (equivalently  $A^e$ -module) resolution of  $A$ , called the bar complex  $B_\bullet A$  of  $A$  such that  $B_n A := A^{\otimes(n+2)}$  with the differential  $b : B_n A \rightarrow B_{n-1} A$  given by

$$b(a_0 \otimes \cdots \otimes a_{n+1}) := \sum_{i=0}^n (-1)^i (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n)$$

with the canonical multiplication map  $\mu : A^{\otimes 2} \rightarrow A$ .

**Definition 1.2.17.** For any associative algebra  $A$  over  $k$  and an  $A$ -module  $M$ , we define Hochschild homology  $HH_\bullet(A, M)$  and Hochschild cohomology  $HH^\bullet(A, M)$  of  $(A, M)$  as follows;

$$HH_\bullet(A, M) := Tor_\bullet^{A^e}(A, M)$$

$$HH^\bullet(A, M) := Ext_{A^e}^\bullet(A, M)$$

In particular, if  $A = M$ , then we denote Hochschild homology (resp. cohomology) of  $(A, A)$  by  $HH_\bullet(A)$  (resp.  $HH^\bullet(A)$ )

We focus on geometric interpretation of Hochschild homology and cohomology by regarding  $A$  as the space of global regular functions on  $X := \text{Spec}(A)$ . We denote

$\Omega^n(A) := \wedge^n \Omega^1(A)$  the space of algebraic differential  $n$ -forms where  $\Omega^1(A)$  is the space of Kähler differentials on  $A$ . There is a natural morphism of  $A$ -modules

$$\pi_n : HH_n(A) \rightarrow \Omega^n(A)$$

$$a_0 \otimes \cdots \otimes a_n \mapsto a_0 da_1 \wedge \cdots \wedge da_n$$

for all  $n \geq 0$ . The following theorem tells that the morphism  $\pi_n$  becomes an isomorphism when  $X := \text{Spec}(A)$  is smooth and projective.

**Theorem 1.2.18.** *[HKR62] (Hochschild-Kostant-Rosenberg) Let  $A$  be a finitely presented, smooth, commutative algebra over  $k$ . Then there is an isomorphism of graded  $k$ -algebras*

$$HH_\bullet(A) \cong \Omega^\bullet(A/k)$$

*Dually, the Hochschild cohomology is identified with space of  $k$ -derivations of  $A$ .*

Theorem 1.2.18 can be generalized in various directions. First, let  $X$  be a (not-necessarily affine) smooth and proper scheme over  $k$ . The Hochschild homology and cohomology of  $X$  is defined to be that of the ring of global functions  $\mathcal{O}(X)$ . In this case, the analogue of Theorem 1.2.18 still holds.

More abstractly, one can define Hochschild homology and cohomology for a category  $\mathcal{C}$  whose morphism spaces are associative algebras<sup>3</sup>. Even though the body of Hochschild (co)chain complex seems too huge to deal with, there are many interesting cases one can compute Hochschild invariants explicitly. We introduce

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<sup>3</sup> $(\infty, 1)$  – categories

two main examples which will be used later. Such isomorphisms called the HKR isomorphism.

*Example 1.2.19.* Let  $X$  be quasi-separated scheme over  $\mathbb{C}$ . The Hochschild invariant of the (dg) category perfect complexes  $\text{Perf}(X)$  on  $X$  is isomorphic to that of  $X$  [Kel99]. If  $X$  is smooth and proper, combining with the HKR isomorphism, we have

$$HH_n(D^b\text{Coh}(X)) = HH_n(\text{Perf}(X)) \cong HH_n(X) \cong \bigoplus_{p+q=n} H^q(X, \Omega_X^p)$$

for all  $n$ .

*Example 1.2.20.* (1) Let  $(Y, \omega)$  be a compact (exact)  $2n$ -dimensional symplectic manifold with a symplectic structure  $\omega \in \Omega^2(Y)$ . The Hochschild invariant of Fukaya category  $\text{Fuk}(Y)$  is expected to be isomorphic to the quantum cohomology of  $Y$  under the open-closed maps [Fuk+09]

$$HH_\bullet(\text{Fuk}(Y)) \cong QH^{n+\bullet}(\text{Fuk}(Y))$$

$$HH^\bullet(\text{Fuk}(Y)) \cong QH^\bullet(\text{Fuk}(Y))$$

More generally, when  $Y$  becomes non-compact and a Liouville manifold, the Hochschild invariant of the wrapped Fukaya category  $\text{Fuk}^{wr}(Y)$  is expected to be isomorphic to symplectic cohomologies [Gan13].

(2) Let  $((Y, \omega), \mathbf{w} : Y \rightarrow \mathbb{C})$  be a Lefschetz fibration where  $(Y, \omega)$  is  $2n$ -dimensional symplectic manifold. To this data, one can associate the Fukaya-Seidel category

$\text{FS}(Y, \mathbf{w})$ . The Hochschild homology is given by

$$HH_{\bullet}(\text{FS}(Y, \mathbf{w})) \cong H^{n+\bullet}(Y, Y_{\infty})$$

where  $H^{\bullet}(Y, Y_{\infty})$  is the cohomology of vanishing cycles [AS].

## 1.3 Smooth Case

### 1.3.1 Fano/LG Correspondence

Let  $(X, D)$  be a Fano pair where  $X$  is  $n$ -dimensional complex Fano manifold and  $D$  is a smooth anti-canonical divisor. Denote the complement by  $U := X \setminus D$ . A mirror of this pair is given by the Landau-Ginzburg (LG) model  $((Y, \omega), \mathbf{w} : Y \rightarrow \mathbb{C})$  where  $(Y, \omega)$  is  $2n$ -dimensional Calabi-Yau symplectic manifold with a symplectic (kähler) form  $\omega \in \Omega^2(Y)$  is a holomorphic volume form. The LG potential  $\mathbf{w} : Y \rightarrow \mathbb{C}$  is a Lefschetz fibration. A general fiber of  $\mathbf{w}$ , denoted by  $Y_{sm}$ , is a compact Calabi-Yau manifold.

*Remark 4.* In [KKP17], the choice of defining anti-canonical section  $s_X \in |K_X^{-1}|$  as well as a holomorphic volume form on  $Y$ ,  $\text{vol}_Y$ , are considered in the Fano/LG correspondence. As such choices are not crucial for our discussion, we just abuse our notation of mirror Fano/LG correspondence.

The conjectural mirror symmetry relations of such pairs (B to A) are given as

follows;

$$X \iff (\mathbf{w} : Y \rightarrow \mathbb{C})$$

$$D \iff Y_{sm}$$

$$U \iff Y$$

These abstract correspondences can be made explicit once relevant mathematical objects to each pair are specified. In particular, the homological mirror symmetry (HMS) conjecture is stated as an equivalence of two categorial localizations (1.2.3) and (1.2.3).

**Conjecture 1.3.1. (*Relative HMS for the Fano pair*  $(X, D_{sm})$ )** *Let  $(X, D_{sm})$  be a Fano pair and  $((Y, \omega), \mathbf{w} : Y \rightarrow \mathbb{C})$  be a mirror LG model. There is an equivalence of sequences of  $\mathbb{C}$ -linear  $\mathbb{Z}$ -graded idempotent complete  $A_\infty$  categories*

$$\begin{array}{ccccc}
 D^b\text{Coh}(D_{sm}) & \xrightleftharpoons[i^*]{i_*} & D^b\text{Coh}(X) & \xrightarrow{j_*} & D^b\text{Coh}(U) \\
 \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
 \text{Fuk}(Y_{sm}) & \xrightleftharpoons[\cap]{\cup} & \text{FS}((Y, \omega), w) & \xrightarrow{\iota} & \text{Fuk}^{wr}(Y)
 \end{array} \tag{1.3.1}$$

where

- the upper (resp. lower) horizontal sequence is the categorial localization described in (1.2.3) (resp. (1.2.4));
- The vertical isomorphisms are compatible with Serre functors.

*Example 1.3.2.* Let  $(X, D)$  be  $(\mathbb{CP}^1, \{0\} \cup \{\infty\})$ . The mirror Landau-Ginzburg model  $(Y, \mathbf{w})$  is given by Laurant polynomial  $\mathbf{w}(z) = z + \frac{1}{z}$ . This is a Lefschetz

fibration with only one singular locus at  $0 \in \mathbb{C}$ . A generic fiber is two points which gives rise to objects (thimbles)  $\Delta_0, \Delta_1$  in Fukaya-Seidel category  $FS(Y, \mathbf{w})$ . Note that by perturbing  $\Delta_0$ , one can see that it intersects with  $\Delta_1$  at one point. In other words  $HF^\bullet(\Delta_0, \Delta_1) = \mathbb{C}$  and this implies that both thimbles  $\Delta_0$  and  $\Delta_1$  correspond to  $\mathcal{O}_X$  and  $\mathcal{O}_X(1)$ , respectively.

Moreover, a skyscraper sheaf  $\mathcal{O}_p$  for  $p \in X \setminus \{0\} \cup \{\infty\}$  corresponds to compact circle which is cotangent fiber when we view  $\mathbb{C}^*$  as  $T^*S^1$ . For  $p = 0, \infty$  it corresponds to  $U$ -shaped admissible Lagrangians.

*Example 1.3.3.* Let  $X$  be a Delpezzo surface of degree  $0 \leq d \leq 9$  and  $D$  be a smooth anti-canonical divisor. The Landau-Ginzburg model is an elliptic fibration  $f : Z \rightarrow \mathbb{P}^1$  over  $\mathbb{P}^1$  with  $3 + d$  singularities near the origin and the wheel of  $9 - d$  lines at the infinity. By removing the fiber at infinity  $f^{-1}(\infty)$ , we have a genuine LG model  $((Y, \omega), \mathbf{w} : Y \rightarrow \mathbb{C})$ . In [AKO06], the authors essentially proves the relative HMS.

*Remark 5.* The mirror symmetry of Del-pezzo surfaces with a smooth anti-canonical fiber is recently generalized to log-Calabi Yau surfaces with toric boundaries by [Gro+18]. A mirror LG potential  $\mathbf{w} : Y \rightarrow \mathbb{C}$  becomes non-proper, whose generic fiber is a complement of some divisors in elliptic curve. There is an interesting hidden behind the non-properness of the fibration. We will come back to this point in Section 2.3.

**Proposition 1.3.4.** *By taking Hochschild homology  $HH_a$  on the diagram in (1.3.1),*

we have an isomorphism of long exact sequences of cohomology groups;

$$\begin{array}{ccccccc}
\oplus_{p-q=a} H^q(D, \Omega_D^p) & \xrightarrow{i_*} & \oplus_{p-q=a} H^q(X, \Omega_X^p) & \xrightarrow{j^*} & \oplus_{p-q=a} H^q(U, \Omega_U^p) & \longrightarrow & \dots \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
H^{a+n-1}(Y_{sm}) & \xrightarrow{conn} & H^{a+n}(Y, Y_{sm}) & \xrightarrow{\iota^*} & H^{a+n}(Y) & \longrightarrow & \dots
\end{array} \tag{1.3.2}$$

*Proof.* The top horizontal sequence comes from HKR isomorphism and  $\mathbb{A}^1$ -homotopy theory. This is a classical Gysin sequence associated to the pair  $(X, D)$ . For the bottom sequence, note that the induced morphism of cap functor  $HH_a(\cap)$  on the level of Hochschild homology is a canonical map from cohomology of nearby cycles to that of vanishing cycles. Since a generic fiber of the LG potential  $w : Y \rightarrow \mathbb{C}$  is all diffeomorphic, the morphism  $HH_a(\cap)$  becomes the connecting homomorphism of the long exact sequence associated to the pair  $(Y, Y_{sm})$ .  $\square$

As two horizontal sequences are a part of spectral sequences associated to weight filtration on  $H^\bullet(U)$  and perverse Leray filtration on  $H^\bullet(Y)$ , we have the following theorem.

**Theorem 1.3.5.** *Given a Fano mirror pair  $\{(X, D \in |-K_X|), ((Y, \omega), w : Y \rightarrow \mathbb{C})$ , the conjectural relative HMS (1.3.1) gives rise to the following homological (Hodge theoretic, topological) correspondence*

$$\oplus_{p-q=a} Gr_F^q Gr_{p+q+i}^W H^{p+q}(U) \cong Gr_{n+a+i}^P H^{n+a}(Y) \quad \text{for all } i = 0, 1 \tag{1.3.3}$$

Moreover, one can recover direct summands by taking associated graded pieces of



monodromy weight filtration associated to the Serre functors. Then we have

$$Gr_F^q Gr_{p+q+i}^W H^{p+q}(U) \cong Gr_{2(n-q)}^W Gr_{n+p-q+i}^P H^{n+p-q}(Y) \quad \text{for all } i = 0, 1$$

*Proof.* Note that the left square of the commutative diagram (1.3.2) involves  $E_1$ -page of the spectral sequence of the weight filtration on  $H^\bullet(U)$  (the top row) and that of the perverse Leray filtration associated to  $w$  on  $H^\bullet(Y)$  (the bottom row). Moreover, by considering the Serre functors, we have the following commutative diagram

$$\begin{array}{ccc} D^b\text{Coh}(D) & \xrightarrow{i_*} & D^b\text{Coh}(X) \\ \downarrow S_{X|D} & & \downarrow S_X \\ D^b\text{Coh}(D) & \xrightarrow{i_*} & D^b\text{Coh}(X) \end{array}$$

where  $S_X(-) = - \otimes \omega_X[n]$  and  $S_{X|D}(-) = - \otimes i^* \omega_X[n]$ . The logarithm of such functors (up to sign) on Hochschild homologies is given by cup product of the first Chern class  $c_1(\omega_X)$  and  $c_1(\omega_{X|D})$ . As these actions are both nilpotent, the functor  $i_* : D^b\text{Coh}(D) \rightarrow D^b\text{Coh}(X)$  induces a morphism of filtered vector spaces as desired.

Similarly, the logarithm of the Serre functors induces the monodromy weight filtration  $W(\mathbf{w})$  on  $H^\bullet(Y_{sm})$  and  $H^\bullet(Y, Y_{sm})$  associated to  $\mathbf{w} : Y \rightarrow \mathbb{C}$  and they are compatible with the connecting homomorphism. Also, it fits into the long exact sequence of mixed Hodge structure

$$\cdots \rightarrow H^{a-1}(Y_{sm,\infty}) \rightarrow H^a(Y, Y_{sm,\infty}) \rightarrow H^a(Y) \rightarrow H^a(Y_{sm,\infty}) \rightarrow \cdots \quad (1.3.4)$$

where  $H^a(Y)$  admits a canonical mixed Hodge structure for all  $a$  and  $Y_{sm,\infty}$  is used to distinguish a limiting mixed Hodge structure from the canonical one. Therefore,

the associated graded pieces of such filtration gives isomorphisms

$$\mathrm{Gr}_F^q \mathrm{Gr}_{p+q+i}^W H^{p+q}(U) \cong \mathrm{Gr}_{2(n-q)}^W \mathrm{Gr}_{n+p-q+i}^P H^{n+p-q}(Y)$$

for all  $i = 0, 1$ . □

In order to complete the one side of the P=W conjecture, we need to understand the mixed Hodge structure on  $H^{n+p-q}(Y)$  and see how the associated graded pieces with respects to weight and Hodge filtrations are related [KPH19].

**Corollary 1.3.6.** *If  $H^k(Y)$  admits a Deligne's mixed Hodge structure of Hodge-Tate type for all  $k$ , then the conjectural relative HMS (1.3.1) implies the P=W conjecture[Sha18] ( Conjecture 1.1.1).*

*Remark 6.* The Hodge-Tate condition of the mixed Hodge structure on  $H^\bullet(Y)$  is not an unreasonable assumption and closely related to the conjecture of Katzarkov-Kontsevich-Pantev we will introduce (See Conjecture 1.1.4)

### 1.3.2 Deformation Theory and Hodge Numbers

In this subsection, we review the work of Kontsevich-Katzarkov-Pantev about the deformation theory of Landau-Ginzburg models and relevant Hodge numbers [KKP17].

Let  $(Y, w : Y \rightarrow \mathbb{C})$  be a (proper) Landau-Ginzburg model. A naive geometric deformation of the pair  $(Y, w : Y \rightarrow \mathbb{C})$  turns out to be non-flat because singular fibers could run to infinity. To remedy this issue and study flat deformations of the

LG model, one choose a tame compactification  $((Z, D_Z), f : Z \rightarrow \mathbb{P}^1)$  of  $\mathbf{w} : Y \rightarrow \mathbb{C}$ ;

$$\begin{array}{ccc} Y & \hookrightarrow & Z \\ \downarrow \mathbf{w} & & \downarrow f \\ \mathbb{C} & \hookrightarrow & \mathbb{P}^1 \end{array}$$

such that

- $Z$  is smooth projective variety and the morphism  $f : Z \rightarrow \mathbb{P}^1$  is flat;
- The complement of  $Y$  in  $Z$  is an anti-canonical divisor  $D_Z = D^h \cup D^v$  where  $D^h$  and  $D^v$  are horizontal and vertical divisors, respectively. ( $D^v = f^{-1}(\infty)$ );
- the critical locus  $\text{crit}(f)$  does not intersect with the horizontal divisors  $D^h$ ;
- $\text{vol}_Z \in H^0(Z, K(*D_Z))$  has nowhere vanishing meromorphic volume form with poles at most at  $D_Z$ .

We call the pair  $((Z, D_Z), \text{vol}_Z, f : Z \rightarrow \mathbb{P}^1)$  (or simply  $((Z, D_Z), f : Z \rightarrow \mathbb{P}^1)$ ) a compactified Landau-Ginzburg model of  $(Y, \mathbf{w} : Y \rightarrow \mathbb{C})$ .

We consider the deformation of the pair  $(Z, f)$  preserving the boundary divisor  $D_Z$ . We denote it by  $(Z, f)_{D_Z}$  where the subscript indicates the fixed part of the deformation. This is controlled by the sheaf of differential graded Lie algebras

$$\mathfrak{g}^\bullet := \left[ T_{X, D_Z} \xrightarrow{df} f^* T_{\mathbb{P}^1, \infty} \right] \quad (1.3.5)$$

where  $T_{M, N}$  is the relative tangent sheaf of  $M$  respects to  $N$  for the pair  $N \subset M$ .

*Remark 7.* Taking account the choice of volume form in the deformation  $(Z, f)_{D_Z}$  provides a  $\mathbb{C}^*$  bundle over the versal deformation space of  $(Z, f)_{D_Z}$ . Therefore, from now on, we focus on the deformation theory of the pair  $((Z, D_Z), f)$ .

**Theorem 1.3.7.** [KKP17] Let  $(Z, D_Z)$  be the compactified LG model of  $(Y, \mathbf{w} : Y \rightarrow \mathbb{C})$ . Then the  $L_\infty$ -algebra

$$R\Gamma(Z, \mathfrak{g}^\bullet) = R\Gamma\left(Z, \left[T_{X, D_Z} \xrightarrow{df} f^*T_{\mathbb{P}^1, \infty}\right]\right) \quad (1.3.6)$$

is homotopy abelian.

To prove Theorem 1.3.7, we first introduce the notion of  $f$ -adpated de Rham complex. Let  $(\Omega_Z^\bullet(\log D_Z), d)$  be a logarithmic de Rham complex which is quasi-isomorphic to the de Rham complex of  $Y$ ,  $(\Omega_Y^\bullet, d)$ . We define the subcomplex of  $(\Omega_Z^\bullet(\log D_Z), d)$  which is preserved by  $\wedge df$ ;

$$\Omega_Z^a(\log D_Z, f) := \{u \in \Omega_Z^a(\log D_Z) \mid u \wedge df \in \Omega_Z^{a+1}(\log D_Z)\}$$

for all  $a \geq 0$ .

*Example 1.3.8.* Let  $Z = \mathbb{C}^2$  and  $f(z_1, z_2) = \frac{1}{z_1 z_2}$ . Then we have

$$\begin{aligned} \Omega_Z^\bullet(\log D_Z) &= \mathcal{O}_Z \oplus \mathcal{O}_Z \frac{dz_1}{z_1} \oplus \mathcal{O}_Z \frac{dz_2}{z_2} \oplus \mathcal{O}_Z \frac{dz_1 dz_2}{z_1 z_2} \\ \Omega_Z^a(\log D_Z, f) &= (z_1 z_2) \mathcal{O}_Z \oplus (z_1 z_2) \mathcal{O}_Z \frac{dz_1}{z_1} \oplus (z_1 z_2) \mathcal{O}_Z \frac{dz_2}{z_2} \oplus \mathcal{O}_Z \frac{dz_1 dz_2}{z_1 z_2} \end{aligned}$$

**Lemma 1.3.9.** The  $f$ -adpated deRham complex  $\Omega_Z^a(\log D_Z, f)$  is locally free of rank  $\binom{n}{a}$  for all  $a \geq 0$ . Explicitly,

$$\Omega_Z^a(\log D_Z, f) = \bigoplus_{p=0}^a \left[ \frac{1}{f} \wedge^p W \oplus d \log f \wedge (\wedge^{p-1} W) \right] \otimes \wedge^{a-p} R \quad (1.3.7)$$

where  $W$  is spanned by logarithmic 1-forms of the vertical part of  $f : Z \rightarrow \mathbb{P}^1$  and  $R$  is spanned by holomorphic 1-forms on  $Y$  and logarithmic 1-forms associated to horizontal part of  $f : Z \rightarrow \mathbb{P}^1$ .

*Proof.* See [KKP17, Lemma 2.12] □

The most interesting feature of the  $f$ -adapted de Rham complex is that it admits two compatible differentials  $d$  and  $\wedge df$ . The following proposition is crucial to Theorem 1.3.7.

**Proposition 1.3.10.** (*Double Degeneration Property*) *Let  $((Z, D_Z), f : Z \rightarrow \mathbb{P}^1, D_Z)$  be a tame compactification of the given LG model  $(Y, \mathbf{w} : Y \rightarrow \mathbb{C})$ . For each  $a \geq 0$ , the dimension of cohomology groups*

$$\dim_{\mathbb{C}} \mathbb{H}^a(Z, (\Omega_Z^\bullet(\log D_Z, f), c_1 d + c_2 df \wedge))$$

*is independent of  $(c_1, c_2) \in \mathbb{C}^2$ .*

*Proof.* (Sketch) We only give a sketch of the proof and refer to [KKP17] for more details. We will come back to generalization of some of below arguments in the hybrid setting ( Section 2.3). It is enough to show that the dimension of the cohomology groups is constant for fixed two lines passing through the origin  $(0, 0) \in \mathbb{C}^2$ .

- $(c_2 = 0)$  The independency of the dimension of cohomology  $\mathbb{H}^a(\mathbb{Z}, (\Omega_Z^\bullet(\log D_Z), c_1 d))$  follows from the fact that the spectral sequence associated to the stupid filtration on  $(\Omega_Z^\bullet(\log D_Z), d)$  degenerates at  $E_1$ -page.
- $(c_1 = c_2)$  Note that the  $f$ -adapted de Rham complex with the differential  $d + df \wedge$  is quasi-isomorphic to algebraic deRham complex  $(\Omega_Y^\bullet, d + dw)$ . Moreover,

we have

$$\dim_{\mathbb{C}} H^a(Y, Y_{sm}; \mathbb{C}) = \dim_{\mathbb{C}} \mathbb{H}^a(Y_{zar}, (\Omega_Y^\bullet, dw \wedge)) = \dim_{\mathbb{C}} \mathbb{H}^a(Y_{zar}, (\Omega_Y^\bullet, d + dw \wedge))$$

- Finally, we have

$$\dim_{\mathbb{C}} H^a(Y, Y_{sm}; \mathbb{C}) = \dim_{\mathbb{C}} \mathbb{H}^a(Z, (\Omega_Z^\bullet(\log D_Z, f), d))$$

because Lemma 1.3.9 implies that the  $f$ -adapted deRham complex  $(\Omega_Z^\bullet(\log D_Z, f), d)$  is a limit of  $(\Omega_Z^\bullet(\log D_Z, \text{rel} f^{-1}(\rho)), d)$  as  $\rho \rightarrow -\infty$ . In particular, the Gauss-Manin parallel transport along  $\rho \in \mathbb{R}_{<0}$  identifies  $H^a(Y, Y_{sm}; \mathbb{C})$  with  $\mathbb{H}^a(Z, (\Omega_Z^\bullet(\log D_Z, f), d))$ .

□

Due to Proposition 1.3.10, one can associate two different kinds of Hodge numbers to the LG model  $(Y, w : Y \rightarrow \mathbb{C})$  with a tame compactification  $((Z, D_Z), f : Z \rightarrow \mathbb{P}^1)$ . First, consider the  $p$ -th hypercohomology of  $\Omega_Z^q(\log D_Z)$ . Note that the dimension is independent of the choice of a tame compactification because it is the same with  $(p, q)$ -piece of the relative cohomology of the pair  $(Y, Y_{sm})$ . Second, the monodromy of the general fiber  $Y_{sm}$  around the infinity gives rise to monodromy weight filtration  $W(w)_\bullet$  on the cohomology  $H^a(Y, Y_{sm}; \mathbb{C})$ . Note that the monodromy action is expected to be unipotent from the categorical viewpoint ([KKP17, Remark 2.5])

**Definition 1.3.11.** Let  $(Y, \mathbf{w} : Y \rightarrow \mathbb{C})$  be a LG model with a tame compactification  $((Z, D_Z), f : Z \rightarrow \mathbb{P}^1)$ . We define two Hodge numbers

$$f^{p,q}(Y, \mathbf{w}) := \dim_{\mathbb{C}} \mathbb{H}^p(Z, \Omega_Z^q(\log D_Z, f))$$

$$h^{p,q}(Y, \mathbf{w}) := \dim_{\mathbb{C}} \mathrm{Gr}_{2p}^{W(\mathbf{w})} H^{p+q}(Y, Y_{sm}; \mathbb{C})$$

for all  $p, q \geq 0$ .

**Conjecture 1.3.12.** *[KKP17](KKP Conjecture) The two Hodge numbers are the same for all  $(p, q)$ .*

$$f^{p,q}(Y, \mathbf{w}) = h^{p,q}(Y, \mathbf{w})$$

*Remark 8.* There is a geometric interpretation of Conjecture 1.3.12 via rescaling structures. [Sha18]

## 1.4 Simple Normal Crossing Case - Two components

### 1.4.1 Extended Fano/LG Correspondence

In case where  $D$  becomes singular, especially reducible, a corresponding Landau-Ginzburg model is expected to be non-proper. One way to see this phenomenon is the SYZ mirror construction, which gives rise to multi-potentials by counting holomorphic disks in  $X$  close to each irreducible component [SYZ96][Aur07]

We first study the case where an anti-canonical divisor  $D$  is a union of two smooth

irreducible components  $D_1$  and  $D_2$  whose intersection  $D_{12} := D_1 \cap D_2$  is smooth and connected. We also assume that the pairs  $(D_1, D_{12})$  and  $(D_2, D_{12})$  are both Fano pairs.

**Definition 1.4.1.** A **hybrid Landau-Ginzburg model** (mirror to  $(X, D)$ ) is a triple  $(Y, \omega, h = (\mathbf{w}_1, \mathbf{w}_2) : Y \rightarrow \mathbb{C}^2)$  where

- $(Y, \omega)$  is  $2n$ -dimensional complex Kähler Calabi-Yau manifold (orbifold) with Kähler form  $\omega \in \Omega^2(Y)$ .
- $h := (\mathbf{w}_1, \mathbf{w}_2) : Y \rightarrow \mathbb{C}^2$  is a proper morphism (Lefschetz fibration) such that
  - (1) A generic fiber of  $\mathbf{w}_1$  (resp.  $\mathbf{w}_2$ ), denoted by  $Y_1$  (resp.  $Y_2$ ) with  $h|_{Y_1} = \mathbf{w}_2 : Y_1 \rightarrow \mathbb{C}$  (resp.  $h|_{Y_2} = \mathbf{w}_1 : Y_2 \rightarrow \mathbb{C}$ ) is mirror to  $(D_1, D_{12})$  (resp.  $(D_2, D_{12})$ )
  - (2) A generic fiber of  $h$ , denoted by  $Y_{12}$ , is mirror to  $D_{12}$ .
  - (3) By composing with the summation map  $\Sigma : \mathbb{C}^2 \rightarrow \mathbb{C}$ , we get an ordinary LG model

$$\mathbf{w} := \Sigma \circ h : Y \rightarrow \mathbb{C}$$

which is mirror to  $X$  in the sense of the usual Fano/LG correspondence.



A hybrid LG model encodes four different mirror symmetries

$$X \iff ((Y, \omega), w : Y \rightarrow \mathbb{C})$$

$$(D_1, D_{12}) \iff ((Y_1, \omega), w_2 : Y_1 \rightarrow \mathbb{C})$$

$$(D_2, D_{12}) \iff ((Y_2, \omega), w_1 : Y_2 \rightarrow \mathbb{C})$$

$$D_{12} \iff Y_{12}$$

*Example 1.4.2.* Consider a Fano pair with  $(\mathbb{P}^2, D)$  where  $D = Q \cup L$  is the union of conic and line. By Hori-Vafa construction, one can get a hybrid LG model  $((Y, \omega), h : Y \rightarrow \mathbb{C}^2)$  such that

- An original LG potential  $w = \Sigma \circ h : Y \rightarrow \mathbb{C}$  is obtained by removing two horizontal divisor in the elliptic fibration over  $\mathbb{P}^1$  which is mirror to  $(\mathbb{P}^2, D_{sm})$
- 1.3.3
- A hybrid LG model  $h : Y \rightarrow \mathbb{C}^2$  is a branched double cover of  $\mathbb{C}^2$  whose discriminant locus is

$$\{a^2b = 4\} \cup \{b = 0\} \tag{1.4.1}$$

- More explicitly,

$$\begin{array}{c} Y := \{x + y = az, z^2 = bxy\} \subset \mathbb{P}_{x,y,z}^2 \times \mathbb{C}_a \times \mathbb{C}_b \\ \downarrow \\ \mathbb{C}_a \times \mathbb{C}_b \end{array}$$

whose ordinary LG model is the same with one introduced in [Aur07].

*Example 1.4.3.* Consider the Fano pair  $(\mathbb{P}^3, D)$  where  $D = C \cup L$  is the union of cubic  $C$  and linear hypersurface  $H$ . By Hori-Vafa construction, a hybrid LG model  $(Y, h : Y \rightarrow \mathbb{C}^2)$  is an elliptic fibration over  $\mathbb{C}^2$  whose discriminant locus is

$$\{a^3b = 27\} \cup \{b = 0\}$$

and singular fibers are of type  $A_2$ . Note that over a generic coordinate line, the restriction of the hybrid LG potential  $h : Y \rightarrow \mathbb{C}^2$  induces mirror LG models associated to Fano surfaces  $C$  and  $H$ .

*Remark 9.* If one views a smooth cubic surface as del Pezzo surface of degree 3, a mirror LG model is expected to be the elliptic fibration  $f : Y \rightarrow \mathbb{P}^1$  over  $\mathbb{P}^1$  with 6 singularities away from  $\infty$  and the wheel of 3 projective lines  $I_3$  at  $\infty$ . This LG model could be deformed to the one introduced in Example 1.4.3.

*Example 1.4.4.* Consider the Fano pair with  $(\mathbb{P}^3, D)$  where  $D$  is the union of two quadric surfaces. By Hori-Vafa construction, one get a hybrid LG model  $(Y, h : Y \rightarrow \mathbb{C}^2)$

- It satisfies the conditions in Definition 1.4.1.
- A singular locus of  $h : Y \rightarrow \mathbb{C}^2$  is given by

$$\{a^2b^2 = 16\} \cup \{ab = 0\} \tag{1.4.2}$$

### **Construction of Hybrid LG models**

Here we introduce a general scheme of constructing hybrid LG models for the pair  $(\mathbb{P}^n, D)$  where  $D = D_1 \cup D_2$  and each irreducible component is smooth with

$\deg(D_i) = n_i$  and  $n_1 + n_2 = n + 1$ . The idea is to decompose the Hori-Vafa potential with respects to the choice of anti-canonical divisor  $D$ . In general, this construction works for smooth complete intersections as well.

First recall that the Hori-Vafa mirror for  $\mathbb{P}^n$  with a smooth anti-canonical divisor is given by

$$\begin{aligned} w : (\mathbb{C}^*)^n &\rightarrow \mathbb{C} \\ (x_1, \dots, x_n) &\mapsto x_1 + x_2 + \dots + x_n + \frac{1}{x_1 \cdots x_n} \end{aligned} \tag{1.4.3}$$

Depending on the choice of the divisor  $D = D_1 \cup D_2$ , it will be modified to

$$\begin{aligned} h = (h_1, h_2) : (\mathbb{C}^*)^n &\rightarrow \mathbb{C}^2 \\ (x_1, \dots, x_n) &\mapsto \left( x_1 + x_2 + \dots + x_{n_1}, x_{n_1+1} + \dots + x_n + \frac{1}{x_1 \cdots x_n} \right) \end{aligned} \tag{1.4.4}$$

*Remark 10.* The choice of decomposing the Hori-Vafa potential amounts to choosing nef partition of polytope associated to  $\mathbb{P}^n$ . We do not cover a general scheme for smooth toric Fano variety here [KPH19].

Now, the task is to find a suitable fiberwise compactification of  $h$  to satisfy the conditions in Definition 1.4.1. We take a naive compactification rather than using machinery of toric geometry which is done in [KPH19]. The advantage of our approach allows us to compute singular locus of hybrid LG models explicitly.

1. (**Compactification**) We compactify  $(\mathbb{C}^*)^n$  to  $\mathbb{P}_{x_1, \dots, x_{n_1}, t}^{n_1} \times \mathbb{P}_{x_{n_1+1}, \dots, x_n, s}^{n_2-1}$ . Consider  $u_0 := h_1$  (resp.  $v_0 := h_2$ ) as a section of  $\mathcal{O}(1, 0)$  (resp.  $\mathcal{O}(n_1, n_2)$ ). By choosing sections at infinity  $u_\infty := t$  (resp.  $v_\infty := sx_1 \cdots x_n$ ), we have two

pencils of hypersurfaces;

$$\begin{aligned}\phi_u &:= \mathbb{P}^{n_1} \times \mathbb{P}^{n_2-1} \dashrightarrow \mathbb{P}^1[u_0 : u_\infty] \\ \phi_v &:= \mathbb{P}^{n_1} \times \mathbb{P}^{n_2-1} \dashrightarrow \mathbb{P}^1[v_0 : v_\infty]\end{aligned}\tag{1.4.5}$$

with base locus of  $\phi_u$  and  $\phi_v$  given by

$$\begin{aligned}B(\phi_u) &= \{x_1 + \cdots + x_{n_1} = 0\} \cap \{t = 0\} \cong \mathbb{P}^{n_1-2} \times \mathbb{P}^{n_2-1} \\ B(\phi_v) &= \{(x_{n_1+1} + \cdots + x_n)x_1 \cdots x_n + t^{n_1}s^{n_2} = 0\} \cap \{sx_1 \cdots x_n = 0\} \\ &\cong \{s = 0, x_i = 0\} \cup \{s = 0, x_{n_1+1} + \cdots + x_n = 0\} \\ &\cup \{t = 0, x_i = 0\} \cup \{t = 0, x_{n_1+1} + \cdots + x_n = 0\}\end{aligned}$$

To obtain compactification of open hybrid LG model  $h : (\mathbb{C}^*)^n \rightarrow \mathbb{C}^2$ , we consider two pencils together by forming a rational map

$$\phi := (\phi_u, \phi_v) : \mathbb{P}^{n_1} \times \mathbb{P}^{n_2-1} \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1\tag{1.4.6}$$

The base locus of  $\phi$ , denoted by  $B(\phi)$ , is the union of base loci of  $\phi_u$  and  $\phi_v$ . Then we can extend  $\phi$  to be a morphism by blowing up  $B(\phi)$ . Note that it is equivalent to blowing up  $B(\phi_u)$  first and successively blow up total transformation of  $B(\phi_v)$  and vice versa. Therefore we have a genuine morphism

$$\bar{\phi} : \text{Bl}_{B(\phi)} \mathbb{P}^{n_1} \times \mathbb{P}^{n_2-1} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1\tag{1.4.7}$$

By removing the boundary  $L := \{\infty\} \times \mathbb{P}^1 \cup \mathbb{P}^1 \times \{\infty\} \subset \mathbb{P}^1 \times \mathbb{P}^1$ , we have a

family of intersections

$$\begin{aligned} \mathcal{Y} &:= \{ \sum_{i=1}^{n_1} x_i = at, (\sum_{j=n_1+1}^n x_j)x_1 \cdots x_n + t^{n_1} s^{n_2} = bsx_1 \cdots x_n \} \\ &\subset \mathbb{P}^{n_1} \times \mathbb{P}^{n_2-1} \times \mathbb{C}_a \times \mathbb{C}_b \end{aligned} \quad (1.4.8)$$

$$\begin{array}{c} \downarrow \bar{\phi} \\ \mathbb{C}_a \times \mathbb{C}_b \end{array}$$

*Remark 11.* Note that the boundary  $L$  is the base locus of rational map, a compactification of the summation map,

$$\begin{aligned} \bar{\Sigma} : \mathbb{P}^1 \times \mathbb{P}^1 &\dashrightarrow \mathbb{P}^1 \\ [u_0, u_\infty], [v_0, v_\infty] &\mapsto [u_0v_\infty + v_0u_\infty, u_\inftyv_\infty] \end{aligned}$$

By removing the boundary  $\Gamma$ , it induces an ordinary LG model

$$w := \Sigma \circ \bar{\phi} : \mathcal{Y} \rightarrow \mathbb{C}$$

One can think that  $\phi^{-1}(\Gamma)$  is a vertical divisor of compactification of  $w$ .

2. **(Crepanant Resolution)** Let's assume that  $n_1, n_2 > 1$ . The idea is to take crepanant resolution of  $\mathcal{Y}$  to make a generic fiber to be smooth. We first describe a singular locus of generic fiber by Jacobi criterion.

- (A singular locus of generic fiber of  $\pi$ ) We compute a singular locus of generic fiber of  $\bar{\phi}$ .

(a) If  $s = 0$ , then  $\prod_{i=1}^n x_i = 0$ . Moreover, we have  $\prod_{i=1}^n x_i (\sum_{j=n_1+1}^{n_2} x_j) = 0$  which vanishes further.

(b) If  $t = 0$ , again  $\prod_{i=1}^n x_i = 0$  same as above.

(c) Therefore, we have the following singular locus for generic fiber is

$$\mathcal{S} := \text{Sing}(\{(\sum_{j=n_1+1}^n x_j)x_1 \cdots x_n = 0\}) \cap \{st = 0\} \cap \mathcal{Y} \quad (1.4.9)$$

(d) If  $st \neq 0$ , then over  $(a, b)$  satisfying  $a^{n_1}b^{n_2} = (n_1)^{n_1}(n_2)^{n_2}$ , a fiber has singularity at  $[1 : 1 : \cdots : 1 : \frac{n_1}{a}] \times [1 : 1 : \cdots : 1 : \frac{n_2}{b}]$ .

(e) In case  $n_1 = 1$ (resp.  $n_2 = 1$ ), we have additional singular locus over  $a = 0$ (resp.  $b = 0$ ).

- We blow up the locus  $\mathcal{S}$  in  $\mathcal{Y}$ , by following the algorithm in [PS15]. It guarantees that we have LG models which are mirror to the irreducible components  $D_1$  and  $D_2$  as Fano manifolds. Moreover, since  $\text{Bl}_{\mathcal{S}} \mathcal{Y}$  is still singular whose local equation is given by  $z^k = ab$ . One can also take the crepant resolution to get non-singular body of hybrid LG model and we still denote it by  $\text{Bl}_{\mathcal{S}} \mathcal{Y}$ . (Todo: Verify this this...)
- By composing with summation map  $\Sigma : \mathbb{C}^2 \rightarrow \mathbb{C}$ , we obtain an ordinary LG model  $w : \text{Bl}_{\mathcal{S}} \mathcal{Y} \rightarrow \mathbb{C}$  whose generic fiber is smooth and non-proper. As blow-up locus  $\mathcal{S}$  sits in horizontal divisor of  $\Sigma \circ \bar{\phi} : \mathcal{Y} \rightarrow \mathbb{C}$ , we still have same Hodge data.

## Gluing Structure

The advantage of the above construction is that we can describe a singular locus of the hybrid LG potential  $h : Y \rightarrow \mathbb{C}^2$  explicitly. In particular, It allows us to

examine a fibration structure of  $Y_{sm}$  given by the restriction of the hybrid potential,  $h|_{Y_{sm}} : Y_{sm} \rightarrow \mathbb{C}$ . Denote  $w_k : Y_k \rightarrow \mathbb{C}$  be a restriction of  $h$  over  $Y_k$  for  $k = 1, 2$ .

**Proposition 1.4.5.** *A fibration on  $Y_{sm}$ ,  $h|_{Y_{sm}} : Y_{sm} \rightarrow \mathbb{C}$  is topologically glued by two fibrations  $w_1 := h|_{Y_1} : Y_1 \rightarrow \mathbb{C}$  and  $w_2 := h|_{Y_2} : Y_2 \rightarrow \mathbb{C}$ .*

*Proof.* The locus of singular fiber of  $h$  is

$$\Delta(h) := \begin{cases} \{a^{n_1}b^{n_2} = n_1^{n_1}n_2^{n_2}\} \cup \{ab = 0\} & \text{if}(n_1, n_2 > 1) \\ \{ab^{n_2} = n_2^{n_2}\} \cup \{a = 0\} & \text{if}(n_1 = 1, n_2 > 1) \\ \{a^{n_1}b = n_1^{n_1}\} \cup \{b = 0\} & \text{if}(n_1 > 1, n_2 = 1) \end{cases} \quad (1.4.10)$$

It intersects with a generic anti-diagonal line  $L$  in  $\mathbb{C}^2$  with  $n_1 + n_2$  distinct points where  $h|_{Y_{sm}}$  has singular fibers. We ignore the intersection with coordinate lines for simplicity. Note that generic coordinate lines  $H_a = \{b = const\}$ ,  $H_b = \{a = const\}$  intersect with  $\Delta(h)$  at  $n_1$  and  $n_2$  distinct points, respectively.

One can take open subsets  $U_1, U_2$  of the line  $L$ , each contains  $n_1$  and  $n_2$  points, respectively. Then, the restriction of  $h|_{Y_{sm}}$  over  $U_1$  (resp.  $U_2$ ) isomorphic to  $w_1$  (resp.  $w_2$ ). The idea is to find a (linear) degeneration of the anti-diagonal line  $L$  to  $H_a$  such that  $h^{-1}(U_1)$  is diffeomorphic to  $Y_1$ . First note that as the line  $L$  is deformed to  $H_a$ , only  $n_1$  singular points go to  $Y_1$  while the rest  $n_2$  points go to infinity. In order to show that  $h^{-1}(U_1)$  and  $Y_1$  are diffeomorphic, we need to choose a nice degeneration such that there exists one parameter family involving these two manifolds without collapse of  $n_1$  singular points. The existence of such degeneration

comes from the equation of singular loci. Let  $b = ka + l$  be such an one parameter family starting from  $(k, l) = (-1, l_0)$  and  $(k, l) = (0, l_1)$ . Then we can show that for any  $-1 < k < 0$ , there are finitely many  $l$  where collapses occur. Then by genericity arguments, we can find a path from  $l_0$  to  $l_1$  without touching the loci of collision.

More invariantly, one can make more canonical choice of  $n_1$  and  $n_2$  points depending on the choice of generic coordinate hyperplanes  $H_a$  and  $H_b$  and (linear) deformation of  $L$  to them. In other word, by fixing  $H_a, H_b$  and  $L$ ,  $L$  can be (linearly) deformed to  $H_a$  and  $H_b$  keeping the number of intersection points in  $L \cap H_a$  and  $L \cap H_b$ . Then any point in  $L \cap \Delta(h)$  goes to either  $H_a$  or  $H_b$ , not both of them. □

**Lemma 1.4.6.** *Consider a pair of topological spaces  $(Y, S)$  where  $Y = Y_1 \cup Y_2$  with simply connected intersection and  $S = S_1 \cup S_2$  where  $S_i \subset Y_i$  for  $i = 1, 2$ . Then we have a Mayer-Vietoris sequence of relative cohomology groups*

$$\cdots \rightarrow H^i(Y, S) \rightarrow H^i(Y_1, S_1) \oplus H^i(Y_2, S_2) \rightarrow H^i(Y_1 \cap Y_2, S_1 \cap S_2) \rightarrow \cdots$$

Applying Lemma 1.4.6 to our case, we have the following corollary.

**Corollary 1.4.7.** *Let  $((Y, \omega), h : Y \rightarrow \mathbb{C}^2)$  be a hybrid LG model. Then we have an isomorphism of cohomology groups*

$$H^i(Y_1, Y_{12}) \oplus H^i(Y_2, Y_{12}) \cong H^i(Y_{sm}, Y_{12})$$

for all  $i \geq 0$ .



## 1.4.2 Relative HMS and the Mirror P=W Conjecture

In the hybrid setting, we also expect that the relative homological mirror symmetry conjecture is a categorical shadow of the mirror P=W conjecture (Theorem 1.1.3).

To see this, we first look at the relative HMS of boundary Fano pairs  $(D_1, D_{12})$  and  $(D_2, D_{12})$ . By Proposition 1.3.4 and the gluing structure of  $Y_{sm}$  (Proposition 1.4.5),

we have the following commutative diagrams of cohomology groups

$$\begin{array}{ccc}
\oplus_{p-q=a} H^q(D_{12}, \Omega_{D_{12}}^p) & \xrightarrow{(i_{1*}, -i_{2*})} & \oplus_{p-q=a} H^q(D_1, \Omega_{D_1}^p) \oplus \oplus_{p-q=a} H^q(D_2, \Omega_{D_2}^p) \\
\downarrow \cong & & \downarrow \cong \\
H^{a+n-2}(Y_{12}) & \xrightarrow{(conn_1, -conn_2)} & H^{a+n-1}(Y_1, Y_{12}) \oplus H^{a+n-1}(Y_2, Y_{12}) \\
\downarrow id & & \downarrow (r_1, -r_2) \\
H^{a+n-2}(Y_{12}) & \xrightarrow{\rho_1=conn} & H^{a+n-1}(Y_{sm}, Y_{12})
\end{array} \tag{1.4.11}$$

This diagram gives the P=W statement between  $D$  and  $Y_{sm}$ . In order to extend it to the P=W statement between  $U$  and  $Y$ , one needs to introduce the relative HMS of the Fano pair  $(X, D = D_1 \cup D_2)$ .

**Conjecture 1.4.8. (Relative HMS for the Fano pair  $(X, D = D_1 \cup D_2)$ )** For  $k = 1, 2$ , there is an equivalence of diagrams of categories;

$$\begin{array}{ccc}
D^b\text{Coh}(D_k) & \xrightarrow{i_{k,X*}} & D^b\text{Coh}(X) \\
\downarrow \cong, HMS & & \downarrow \cong, HMS \\
\text{FS}(Y_k, \mathbf{w}_k) & \xrightarrow{\Phi_k} & \text{FS}^{wr}(Y, \mathbf{w})
\end{array} \tag{1.4.12}$$

where vertical functors come from HMS and the bottom functor  $\Phi_k : \text{FS}(Y_k, \mathbf{w}_k) \rightarrow \text{FS}^{wr}(Y, \mathbf{w})$  is given by composition;

$$\text{FS}(Y_k, \mathbf{w}_k) \xrightarrow{\iota_k} \text{FS}(Y_{sm}, h_{Y_{sm}}) \xrightarrow{\iota} \text{Fuk}^{wr}(Y_{sm}) \xrightarrow{\cup} \text{FS}^{wr}(Y, \mathbf{w}) \tag{1.4.13}$$

where  $\iota_k$  comes from the semi-orthogonal decomposition of  $\text{FS}(Y_{sm}, h_{Y_{sm}})$ ;

$$\text{FS}(Y_{sm}, h_{Y_{sm}}) = \langle [\text{FS}(Y_1, \mathbf{w}_1)], [\text{FS}(Y_2, \mathbf{w}_2)] \rangle$$

Combining the relative HMS of the boundary data, we have an equivalence of diagrams of categories and we call it **relative HMS for the Fano pair**  $(X, D)$ .

$$\begin{array}{ccc}
 \text{D}^b\text{Coh}(D_{12}) & & \text{Fuk}(Y_{12}) \\
 \begin{array}{c} \nearrow i_1^* \\ \searrow i_{1*} \end{array} & & \begin{array}{c} \nearrow \cap_1 \\ \searrow \cup_1 \end{array} \\
 \text{D}^b\text{Coh}(D_1) & & \text{FS}(Y_1, \mathbf{w}_1) \\
 \begin{array}{c} \nearrow i_{1,X^*} \\ \searrow i_{1,X}^* \end{array} & & \begin{array}{c} \nearrow \cup_{1,Y} \\ \searrow \cap_{1,Y} \end{array} \\
 \text{D}^b\text{Coh}(X) & & \text{FS}^{\text{wr}}(Y, \mathbf{w}) \\
 \begin{array}{c} \nearrow i_{2,X^*} \\ \searrow i_{2,X}^* \end{array} & & \begin{array}{c} \nearrow \cup_{2,Y} \\ \searrow \cap_{2,Y} \end{array} \\
 \text{D}^b\text{Coh}(D_2) & & \text{FS}(Y_2, \mathbf{w}_2) \\
 \begin{array}{c} \nearrow i_2^* \\ \searrow i_{2*} \end{array} & & \begin{array}{c} \nearrow \cup_2 \\ \searrow \cap_2 \end{array} \\
 \text{D}^b\text{Coh}(D_{12}) & & \text{Fuk}(Y_{12})
 \end{array}
 \tag{1.4.14}$$

*Remark 12.* One can consider the categories  $D^b\text{Coh}(D)$  and  $\text{Perf}(D)$  and state the relative HMS conjecture similar to the smooth case (Theorem 1.3.1). However, there are some technical difficulties to handle especially when  $D$  has a dimension greater than 1. For example, it's hard to compute Hochschild invariants and understand semi-orthogonal decomposition. This is why we avoid mentioning categories directly associated to  $D$  and only look at the mirror symmetry of irreducible components of  $D$ .

By taking Hochschild homology  $HH_a$ , we get the following diagram of cohomology groups; For  $k = 1, 2$ ,

$$\begin{array}{ccc}
 \oplus_{p-q=a} H^q(D_k, \Omega_{D_k}^p) & \xrightarrow{i_{k,X^*}} & \oplus_{p-q=a} H^q(X, \Omega_X^p) \\
 \downarrow \cong & & \downarrow \cong \\
 H^{a+n-1}(Y_k, Y_{12}) & \xrightarrow{\phi_k} & H^{a+n}(Y, Y_{sm})
 \end{array}
 \tag{1.4.15}$$

Combining with the upper part of the diagram we have

$$\begin{array}{ccccc}
\bigoplus_{p-q=a} H^q(D_{12}, \Omega_{D_{12}}^p) & \xrightarrow{(i_{1*}, -i_{2*})} & \bigoplus_i \bigoplus_{p-q=a} H^q(D_i, \Omega_{D_i}^p) & \xrightarrow{i_{1, X_*+i_2, X_*}} & \bigoplus_{p-q=a} H^q(X, \Omega_X^p) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
H^{a+n-2}(Y_{12}) & \xrightarrow{(conn_1, -conn_2)} & \bigoplus_i H^{a+n-1}(Y_i, Y_{12}) & \xrightarrow{\phi_1+\phi_2} & H^{a+n}(Y, Y_{sm})
\end{array} \tag{1.4.16}$$

Also, the  $E_1$  term of the spectral sequence of the perverse Leray filtration associated to the hybrid potential  $h : Y \rightarrow \mathbb{C}^2$  on  $H^{a+n}(Y)$  should be identified with the following;

$$\begin{array}{ccccc}
H^{a+n-2}(Y_{12}) & \xrightarrow{(conn_1, -conn_2)} & H^{a+n-1}(Y_1, Y_{12}) \oplus H^{a+n-1}(Y_1, Y_{12}) & \xrightarrow{\phi_1-\phi_2} & H^{a+n}(Y, Y_{sm}) \\
\downarrow id & & \downarrow (r_1, -r_2) & & \downarrow id \\
H^{a+n-2}(Y_{12}) & \xrightarrow{\rho_1} & H^{a+n-1}(Y_{sm}, Y_{12}) & \xrightarrow{\rho_2} & H^{a+n}(Y, Y_{sm})
\end{array} \tag{1.4.17}$$

Unfortunately, due to the alternating term, we cannot simply combine these two diagrams of categories and extend the lower part of the diagram (1.4.11). This can be resolved by the following lemma.

**Lemma 1.4.9.** *Let  $A, B_1, B_2$  and  $C$  be finite dimensional vector spaces over  $\mathbb{C}$ .*

*Assume that we have two complexes of vector spaces*

$$\begin{array}{ccc}
A & \xrightarrow{d_0=(f_1, f_2)} & B_1 \oplus B_2 \xrightarrow{d_1=(g_1+g_2)} C \\
A & \xrightarrow{d'_0=(f_1, f_2)} & B_1 \oplus B_2 \xrightarrow{d'_1=(g_1-g_2)} C
\end{array}$$

*Then, we have (non-canonical) isomorphisms*

$$\ker(d_0) \cong \ker(d'_0), \quad \text{Coker}(d_1) \cong \text{Coker}(d'_1), \quad \frac{\ker(d_1)}{\text{Im}(d_0)} \cong \frac{\ker(d'_1)}{\text{Im}(d'_0)} \tag{1.4.18}$$

Now, we are ready to state the main theorem.

**Theorem 1.4.10.** *Given a Fano mirror pair  $(X, D = D_1 \cup D_2), ((Y, \omega), h : Y \rightarrow \mathbb{C}^2)$ , the conjectural relative HMS (Conjecture 1.4.8) gives rise to the following homological (Hodge theoretic, topological) correspondence.*

$$\bigoplus_{p-q=a} Gr_F^p Gr_{p+q+i}^W H^{p+q}(U) \cong Gr_{n+a+i}^P H^{n+a}(Y) \quad \text{for all } i = 0, 1, 2$$

Moreover, one can recover direct summands by taking associated graded pieces of monodromy weight filtration from the Serre functors. Then we have

$$Gr_F^q Gr_{p+q+i}^W H^{p+q}(U) \cong Gr_{2(n-q)}^W Gr_{n+a+i}^P H^{n+a}(Y) \quad \text{for all } i = 0, 1, 2$$

In this case, we need to put extra care on studying the action of Serre functors. This is because two monodromy weight filtrations  $W(\mathbf{w})$  and  $W(h|_{Y_{sm}})$  on  $H^\bullet(Y, Y_{sm})$  and  $H^\bullet(Y_{sm}, Y_{12})$ , respectively, are not compatible in the sense that the connecting homomorphism  $\rho_2$  is trivial as a filtered homomorphism.

Note that the differential  $\rho_2 : H^{a+n-1}(Y_{sm}, Y_{12}) \rightarrow H^{a+n}(Y, Y_{sm})$  factors through

$$d_1 : H^{a+n-1}(Y_{sm}, Y_{12}) \xrightarrow{\iota} H^{a+n-1}(Y_{sm}) \xrightarrow{\rho} H^{a+n}(Y, Y_{sm})$$

where  $\iota$  is induced by natural inclusions and  $\rho = \rho_2$  is the connecting homomorphism. The middle cohomology group  $H^{a+n-1}(Y_{sm})$  have two different filtrations;

- (1) The weight filtration  $W_\bullet(\mathbf{w})$  associated to the monodromy around infinity as the fiber of  $\mathbf{w} : Y \rightarrow \mathbb{C}$ . Then the connecting homomorphism  $\rho$  is indeed a filtered homomorphism

$$\rho : (H^{a+n-1}(Y_{sm}), W_\bullet(\mathbf{w})) \rightarrow (H^{a+n}(Y, Y_{sm}), W_\bullet(\mathbf{w}))$$

(2) The other one is the Deligne's canonical weight filtration  $W_\bullet$ . Then the connecting homomorphism  $\iota$  is indeed a filtered homomorphism

$$\iota : (H^{a+n-1}(Y_{sm}, Y_{12}), W_\bullet(h|_{Y_{sm}})) \rightarrow (H^{a+n-1}(Y_{sm}), W_\bullet)$$

Since these filtrations except the Deligne's canonical weight filtration are induced by the logarithm of the action of Serre functors, we refine mirror P=W correspondence in Theorem 1.4.10.

**Corollary 1.4.11.** *If  $H^k(Y)$  admits a mixed Hodge structure of Hodge-Tate type for all  $k$ , then the conjectural relative HMS (1.4.8) implies the mirror P=W conjecture (Conjecture 1.1.1).*

*Proof.* •  $\mathrm{Gr}_k^W \mathrm{Gr}_{a+n}^P H^{a+n}(Y) \cong \mathrm{Gr}_k^{W(N)} \mathrm{Coker}(\rho_2)$ .

Consider the long exact sequence of mixed Hodge structures

$$\cdots \rightarrow H^{a+n-1}(Y, Y_{sm}) \rightarrow H^{a+n-1}(Y) \rightarrow H^{a+n-1}(Y_{sm}) \xrightarrow{\rho} H^{a+n}(Y, Y_{sm}) \rightarrow \cdots$$

where  $H^\bullet(Y, Y_{sm})$  and  $H^\bullet(Y_{sm})$  admit the limiting mixed Hodge structure and  $H^\bullet(Y)$  admits a canonical mixed Hodge structure. The cokernel of  $\rho$  admits an induced mixed Hodge structures so that we have

$$\begin{aligned} \mathrm{Gr}_k^{W(N)} \mathrm{Coker}(\rho) &\cong \mathrm{Coker}(\rho : \mathrm{Gr}_k^{W(N)} H^{a+n-1}(Y_{sm}) \rightarrow \mathrm{Gr}_k^{W(N)} H^{a+n}(Y, Y_{sm})) \\ &\cong \mathrm{Gr}_k^W \mathrm{Gr}_{a+n}^P H^{a+n}(Y) \end{aligned}$$

Moreover, it is clear that  $\mathrm{Coker}(\rho) \cong \mathrm{Coker}(\rho_2)$ .

- $\mathrm{Gr}_k^W \mathrm{Gr}_{a+n+1}^P H^{a+n}(Y) \cong \mathrm{Gr}_k^{W(h|_{Y_{sm}})}(\ker(\rho_2)/\mathrm{Im}(\rho_1))$ .

Consider the long exact sequence of mixed Hodge structures

$$\cdots \rightarrow H^{a+n-2}(Y_{sm}) \rightarrow H^{a+n-2}(Y_{12}) \xrightarrow{\rho_1} H^{a+n-1}(Y_{sm}, Y_{12}) \xrightarrow{\iota} \cdots$$

where  $H^\bullet(Y_{sm}, Y_{12})$  and  $H^\bullet(Y_{12})$  admit the limiting mixed Hodge structure and  $H^\bullet(Y_{sm})$  admits the canonical mixed Hodge structure. Then we have an isomorphism of the mixed Hodge structures.

$$\mathrm{Coker}(\rho_1) \cong \ker(H^{a+n-1}(Y_{sm}) \rightarrow H^{a+n-1}(Y_{12}))$$

Also, the kernel  $\ker(\rho_2)$  is isomorphic to  $\iota^{-1}(\ker(\rho))$ . Therefore, we have an isomorphism of mixed Hodge structures

$$\mathrm{Gr}_k^W \mathrm{Gr}_{a+n+1}^P H^{a+n}(Y) \cong \mathrm{Gr}_k^{W(h|_{Y_{sm}})}(\ker(\rho) \cap \ker(H^{a+n-1}(Y_{sm}) \rightarrow H^{a+n-1}(Y_{12})))$$

- $\mathrm{Gr}_k^W \mathrm{Gr}_{a+n+2}^P H^{a+n}(Y) \cong \mathrm{Gr}_k^{W(h|_{Y_{sm}})} \mathrm{Ker}(\rho_1)$ . Recall that  $\mathrm{Gr}_{a+n+2}^P H^{a+n}(Y) \cong \mathrm{Gr}_{a+n+1}^P H^{a+n}(Y_{sm})$  which is compatible with the canonical weight filtrations.

□

### 1.4.3 Deformation Theory and Hodge Numbers

In this subsection, we study the deformation theory of hybrid Landau-Ginzburg models by following the strategy in [KKP17] (See Section 2.2.2). Let's fix a hybrid LG model  $(Y, h : Y \rightarrow \mathbb{C}^2)$  which is mirror to a Fano pair  $(X, D = D_1 \cup D_2)$ . Similar to the case where the anti-canonical divisor of  $X$  is smooth, we introduce the tame

compactification of the LG model and study the relevant deformation theory which controls the behavior at the boundary.

**Definition 1.4.12.** A compactified hybrid LG model is datum  $((Z, f), D_Z, \text{vol}_Z)$  where:

- $Z$  is a smooth projective variety and  $f = (f_1, f_2) : Z \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  is a flat projective morphism.
- $D_Z := D_1 \cup D_2$  is a reduced normal crossings divisor such that  $D_i := (f_i^{-1}(\infty))$  is simple normal crossings divisor for all  $i = 1, 2$ .
- $\text{vol}_Z$  is a meromorphic section of  $K_Z$  with poles at most at  $D_Z$  and no zeros.

Note that there is no horizontal divisor since we assume that the hybrid model  $h : Y \rightarrow \mathbb{C}^2$  is proper. Let  $\mathbf{L} = \{\infty\} \times \mathbb{P}^1 \cup \mathbb{P}^1 \times \{\infty\}$  be the complement of  $\mathbb{C}^2$  in  $\mathbb{P}^1 \times \mathbb{P}^1$ . By definition,  $(Z, f_1)$  is a compactified LG model whose open part is  $(Y, h_1)$ . Moreover, the horizontal divisor of  $(Z, f_1)$  is given by the reduced part of  $f_2^{-1}(\infty)$ .

The deformation theory of  $(Z, f)$  preserving the boundary  $D_Z$ , denoted by  $(Z, f)_{D_Z}$ , is computed by the following sheaf of differential graded (dg) Lie algebra

$$\mathfrak{g}^\bullet := \left[ T_{Z, D_Z} \xrightarrow{df=(df_1, df_2)} f^* T_{\mathbb{P}^1 \times \mathbb{P}^1, L} \right] \quad (1.4.19)$$

**Lemma 1.4.13.** *There is an isomorphism between two sheaves over the base  $\mathbb{P}^1 \times \mathbb{P}^1$*

$$T_{\mathbb{P}^1 \times \mathbb{P}^1, L} \cong T_{\mathbb{P}^1, \infty} \times T_{\mathbb{P}^1, \infty} \quad (1.4.20)$$

*Proof.* Trivial by definition. □

It implies that the sheaf of dg Lie algebra  $\mathfrak{g}^\bullet$  can be identified with the following form.

$$\mathfrak{g}^\bullet = \left[ T_{Z, D_Z} \xrightarrow{(df_1, df_2)} f_1^* T_{\mathbb{P}^1, \infty} \oplus f_2^* T_{\mathbb{P}^1, \infty} \right]$$

For  $i = 1, 2$ , we consider two sheaves of dg Lie algebra

$$\mathfrak{g}_i^\bullet := \left[ T_{Z, D_Z} \xrightarrow{df_i} f_i^* T_{\mathbb{P}^1, \infty} \right]$$

controlling the deformations of compactified LG model  $(Z, f_i)_{D_Z}$ . Note that there exists an injective map of complexes from  $\mathfrak{g}^\bullet$  to  $\mathfrak{g}_1^\bullet \oplus \mathfrak{g}_2^\bullet$  as follows;

$$\begin{array}{ccc} \mathfrak{g}^\bullet & T_{Z, D_Z} & \xrightarrow{df} & f_1^* T_{\mathbb{P}^1, \infty} \oplus f_2^* T_{\mathbb{P}^1, \infty} \\ \downarrow \Delta & \downarrow \Delta & & \downarrow id \\ \mathfrak{g}_1^\bullet \oplus \mathfrak{g}_2^\bullet & T_{Z, D_Z} \oplus T_{Z, D_Z} & \xrightarrow{df} & f_1^* T_{\mathbb{P}^1, \infty} \oplus f_2^* T_{\mathbb{P}^1, \infty} \end{array}$$

where  $\Delta$  is the diagonal map. We claim that it induces an injective morphism on the level of hypercohomology.

**Lemma 1.4.14.** *The induced morphism of  $\Delta : \mathfrak{g}^\bullet \rightarrow \mathfrak{g}_1^\bullet \oplus \mathfrak{g}_2^\bullet$  on the hypercohomology*

$$\mathbb{H}^a(\Delta) : \mathbb{H}^a(\mathfrak{g}^\bullet) \rightarrow \mathbb{H}^a(\mathfrak{g}_1^\bullet \oplus \mathfrak{g}_2^\bullet)$$

*is injective for all  $a$ .*

*Proof.* Since the morphism  $\Delta : \mathfrak{g}^\bullet \rightarrow \mathfrak{g}_1^\bullet \oplus \mathfrak{g}_2^\bullet$  is injective, we have a short exact sequence of complexes;

$$0 \rightarrow \mathfrak{g}^\bullet \xrightarrow{\Delta} \mathfrak{g}_1^\bullet \oplus \mathfrak{g}_2^\bullet \xrightarrow{\alpha} \text{Coker}(\Delta) \rightarrow 0$$



where  $\text{Coker}(\Delta) = [T_{Z,D_Z} \xrightarrow{0} 0]$  and  $\alpha$  is canonical anti-diagonal map. It also induces the long exact sequence on the level of hyper cohomology as follows;

$$\cdots \rightarrow \mathbb{H}^a(\mathfrak{g}^\bullet) \xrightarrow{\mathbb{H}^i(\Delta)} \mathbb{H}^a(\mathfrak{g}_1^\bullet \oplus \mathfrak{g}_2^\bullet) \xrightarrow{\mathbb{H}^a(\alpha)} \mathbb{H}^a(\text{Coker}(\Delta)) \rightarrow \mathbb{H}^{a+1}(\mathfrak{g}^\bullet) \rightarrow \cdots \quad (1.4.21)$$

It suffices to show that  $\mathbb{H}^a(\mathfrak{g}_1^\bullet \oplus \mathfrak{g}_2^\bullet) \xrightarrow{\mathbb{H}^a(\alpha)} \mathbb{H}^a(\text{Coker}(\Delta))$  is surjective for all  $a$ . Note that  $\mathbb{H}^a(\mathfrak{g}_1^\bullet \oplus \mathfrak{g}_2^\bullet) \cong \mathbb{H}^a(\mathfrak{g}_1^\bullet) \oplus \mathbb{H}^a(\mathfrak{g}_2^\bullet)$  and the morphism  $\alpha$  is an alternating map of the projection maps  $pr_i : T_{Z,D_Z} \otimes T_{Z,D_Z} \rightarrow T_{Z,D_Z}$ . Therefore the surjectivity of  $\mathbb{H}^a(\alpha)$  follows from the surjectivity of  $\mathbb{H}^a(\pi_i)$  where

$$\pi_i : \mathfrak{g}_i^\bullet \rightarrow [T_{Z,D_Z} \rightarrow 0]$$

This can be computed by the following short exact sequence of complexes:

$$\begin{array}{ccc} 0 & \longrightarrow & f_i^* T_{\mathbb{P}^1, \infty} \\ \downarrow & & \downarrow id \\ T_{Z,D_Z} & \xrightarrow{df_i} & f_i^* T_{\mathbb{P}^1, \infty} \\ \downarrow id & & \downarrow \\ T_{Z,D_Z} & \longrightarrow & 0 \end{array}$$

Since  $f_i$ 's are flat by assumption, the hypercohomology

$$\mathbb{H}^a(Z, f_i^* T_{\mathbb{P}^1, \infty}) \cong \mathbb{H}^a(\mathbb{P}^1, T_{\mathbb{P}^1, \infty}) \cong H^a(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = 0$$

are zero for all  $a \geq 1$ . It implies that the surjectivity of  $\mathbb{H}^a(\pi_i)$  for  $a \geq 1$ . Therefore, we have

$$0 \rightarrow \mathbb{H}^0(\mathfrak{g}_i^\bullet) \xrightarrow{\mathbb{H}^0(\pi_i)} \mathbb{H}^0(T_{Z,D_Z}) \rightarrow \mathbb{H}^1(f_i^* T_{\mathbb{P}^1, \infty}[1]) \rightarrow \mathbb{H}^1(\mathfrak{g}_i^\bullet) \xrightarrow{\mathbb{H}^1(\pi_i)} \mathbb{H}^1(T_{Z,D_Z}) \rightarrow 0 \quad (1.4.22)$$

To show  $\mathbb{H}^0(\pi_i)$  is surjective, we consider the right inverse of  $\pi_i$ , denoted by  $\iota_i$ . In other words,  $\pi_i \circ \iota_i = Id$  so that we have

$$\mathbb{H}^0(T_{Z,D_Z}) \xrightarrow{\mathbb{H}^0(\iota_i)} \mathbb{H}^0(\mathfrak{g}_i^\bullet) \xrightarrow{\mathbb{H}^0(\pi_i)} \mathbb{H}^0(T_{Z,D_Z})$$

where the composition is identity. It implies that  $\mathbb{H}^0(\pi_i)$  is surjective.  $\square$

Since both dg Lie algebra  $R\Gamma(Z, \mathfrak{g}^\bullet)$  are homotopy abelian, so is their product. The lemma implies that  $R\Gamma(\Delta) : R\Gamma(Z, \mathfrak{g}^\bullet) \rightarrow R\Gamma(\mathfrak{g}_1^\bullet \oplus \mathfrak{g}_2^\bullet)$  induces the injective map on cohomology. Therefore,  $R\Gamma(Z, \mathfrak{g}^\bullet)$  is homotopy abelian.

**Corollary 1.4.15.** *The deformation theory of  $(Z, f)_{D_Z}$  is unobstructed.*

Note that the unobstructness of the deformation theory  $(Z, f)_{D_Z}$  depends on the degeneration property of the  $f_i$ -adapted de Rham complexes for  $i = 1, 2$ .

*Remark 13.* The versal deformation space of triple  $(Z, \text{vol}_Z, f)_{D_Z}$  is a  $\mathbb{C}^*$ -bundle over the versal deformation space of  $(Z, f)_{D_Z}$ . Hence, it is also unobstructed (Remark 7).

*Remark 14.* In the non-hybrid setting, the deformation complex  $\mathfrak{g}^\bullet$  is quasi-isomorphic to truncated sheaf of  $f$ -adapted tangent complex  $T_Z^\bullet(-\log D_Z, f)[-1]$  which is dual to  $f$ -adapted de Rham complex. Therefore, it is reasonable to expect this to happen in the hybrid setting by introducing relevant  $f$ -adapted de Rham complex or its dual. However, it turns out that the natural candidate of  $f$ -adapted de Rham complex does not provide a dg Lie algebra, quasi-isomorphic to the deformation complex  $\mathfrak{g}^\bullet$ .

Next, we will study relevant de Rham complexes associated to the (compactified) hybrid LG model and Hodge numbers.

### **$f$ -adapted Hodge Numbers**

To the compactified LG model  $((Z, D_Z), f : Z \rightarrow \mathbb{P}^1 \times \mathbb{P}^1)$ , we can associate two different subcomplexes of logarithmic de Rham complex  $\Omega_Z^\bullet(\log D_Z)$ . First, consider the subcomplex which are preserved by wedge product of both  $df_1$  and  $df_2$ . We denote it by  $\Omega_Z^\bullet(\log D_Z, f_1, f_2)$  which is indeed a pullback of the following diagram;

$$\begin{array}{ccc} \Omega_Z^\bullet(\log D_Z, f_1, f_2) & \xrightarrow{j_1} & \Omega_Z^\bullet(\log D_Z, f_1) \\ \downarrow j_2 & & \downarrow i_1 \\ \Omega_Z^\bullet(\log D_Z, f_2) & \xrightarrow{i_2} & \Omega_Z^\bullet(\log D_Z) \end{array}$$

where  $i_1$  and  $i_2$  are natural inclusions. By definition, it admits three differentials  $d$ ,  $\wedge df_1$  and  $\wedge df_2$ . The other subcomplex is the pushout of  $j_1$  and  $j_2$ , denoted by  $\Omega_Z^\bullet(\log D_Z, f)$ , which we call  **$f$ -adapted deRham complex**. Unlike the complex  $\Omega_Z^\bullet(\log D_Z, f_1, f_2)$ , it only admits a standard de Rham differential.

We perform a local computation of two deRham complexes  $(\Omega_Z^\bullet(\log D_Z, f_1, f_2), d)$  and  $(\Omega_Z^\bullet(\log D_Z, f), d)$ . Recall that the complement of  $Y$  in  $Z$  is given by  $D_Z = D_1 \cup D_2$  where  $D_i$  is a vertical boundary divisor of  $f_i : Z \rightarrow \mathbb{P}^1$ . Denote  $D_{12}$  the inverse image  $f^{-1}(\{\infty\} \times \{\infty\})$ . For  $p \in D_{12}$ , we can find local analytic coordinates  $z_1, \dots, z_n$  centered at  $p$  such that in a neighborhood of  $p$ :

- the divisor  $D_1$  is given by  $\prod_{i=1}^k z_i = 0$  and the potential  $f_1$  is given by

$$f(z_1, \dots, z_n) = \frac{1}{z_1^{m_1} \dots z_k^{m_k}}$$

for some  $m_i \geq 1$ .

- the divisor  $D_2$  is given by  $\prod_{j=k+1}^{n-k} z_j = 0$  and the potential  $f_2$  is given by

$$f(z_1, \dots, z_n) = \frac{1}{z_{k+1}^{m_{k+1}} \cdots z_n^{m_n}}$$

for some  $m_j \geq 1$ .

**Lemma 1.4.16.** *For  $i = 1, 2$ , the  $f_i$ -adapted deRham complex  $\Omega_Z^a(\log D_Z, f_i)$  is locally free for all  $a$ . Explicitly*

$$\Omega_Z^a(\log D_Z, f_i) = \bigoplus_{p=0}^a \left[ \frac{1}{f_i} \wedge^p W_i \oplus d \log f_i \wedge (\wedge^{p-1} W_i) \right] \otimes \wedge^{a-p} R_i \quad (1.4.23)$$

where  $W_i$  is spanned by logarithmic 1-forms associated to the vertical part of  $f_i : Z \rightarrow \mathbb{P}^1$  and  $R_i$  is spanned by holomorphic 1-forms on  $Y$  and logarithmic 1-forms associated to the horizontal part of  $f_i : Z \rightarrow \mathbb{P}^1$ .

*Proof.* See [KKP17, Lemma 2.12] □

The above local description allows one to describe  $\Omega_Z^a(\log D_Z, f_1, f_2)$  for all  $a$ .

Explicitly, we have

$$\begin{aligned} \Omega_Z^a(\log D_Z, f_1, f_2) &= \bigoplus_{p+q=0}^a \left[ \frac{1}{f_1} \wedge^p W_1 \oplus d \log f_1 \wedge (\wedge^{p-1} W_1) \right] \\ &\quad \otimes \left[ \frac{1}{f_2} \wedge^q W_2 \oplus d \log f_2 \wedge (\wedge^{q-1} W_2) \right] \\ &\quad \otimes \wedge^{a-p-q} R \end{aligned} \quad (1.4.24)$$

where  $R$  is spanned by holomorphic 1-forms on  $Y$ . Similarly, one can give a local description of the  $f$ -adapted complex  $\Omega_Z^\bullet(\log D_Z, f)$ . Since this is equivalent to the

subcomplex of logarithmic de Rham complex  $\Omega_Z^\bullet(\log D_Z)$  preserved by either  $df_1$  or  $df_2$ , we can see that it is generated by differential forms in both  $\Omega_Z^\bullet(\log D_Z, f_1)$  and  $\Omega_Z^\bullet(\log D_Z, f_2)$ .

$$\begin{aligned} \Omega_Z^a(\log D_Z, f_1, f_2) &= \bigoplus_{p=0}^a \left[ \frac{1}{f_1} \wedge^p W_1 \oplus d \log f_1 \wedge (\wedge^{p-1} W_1) \right] \\ &\quad + \left[ \frac{1}{f_2} \wedge^p W_2 \oplus d \log f_2 \wedge (\wedge^{p-1} W_2) \right] \\ &\quad \otimes \wedge^{a-p} R \end{aligned} \tag{1.4.25}$$

By taking the contraction with the meromorphic volume form  $\text{vol}_Z$ , the de Rham complex  $\Omega_Z^\bullet(\log D_Z, f)$  (resp.  $\Omega_Z^\bullet(\log D_Z, f_1 \cup f_2)$ ) induces a locally free subsheaf of polyvector fields  $\wedge^\bullet T_Z$ , denoted by  $(\wedge^\bullet T_Z)(-\log D_Z, f)$  (resp.  $(\wedge^\bullet T_Z)(-\log D_Z, f_1 \cup f_2)$ ). Moreover, both form a Batalin-Vilkovisky algebra, hence encodes geometric deformation associated to the pair  $((Z, D_Z), f : Z \rightarrow \mathbb{P}^1 \times \mathbb{P}^1)$ . However, it turns out that neither control the anchored deformation  $(Z, f)_{D_Z}$  (1.4.19). Nevertheless, both de Rham complexes encode interesting geometric information. We first study the Hodge filtrations on  $(\Omega_Z^\bullet(\log D_Z, f), d)$ .

**Proposition 1.4.17.** *The Hodge-to-de Rham spectral sequences of both complexes  $(\Omega_Z^\bullet(\log D_Z, f), d)$  and  $(\Omega_Z^\bullet(\log D_Z, f_1, f_2), d)$  degenerate at  $E_1$ -page.*

*Proof.* Recall that the strictness of Hodge filtrations on both  $R\Gamma(\Omega_Z^\bullet(\log D_Z, f_1), d)$  and  $R\Gamma(\Omega_Z^\bullet(\log D_Z, f_2), d)$  was done in [KKP17][ESY17] by applying the method of Deligne-Illusie [DI87]. The main idea is to reduce the problem to the field of positive characteristic  $p > 0$ ,  $\mathbb{K}$ , and show the formality by constructing the global

lifting of Frobenius morphism over  $W_2(\mathbb{K})$ , the ring of witt vectors of length 2 of  $\mathbb{K}$ . For the complex  $(\Omega_Z^\bullet(\log D_Z, f_1, f_2), d)$ , its local description 1.4.24 allows one to apply the same argument used in [KKP17][ESY17]. The only non-trivial part is to construct gluing morphism between two choices of local lifting. However, since horizontal part of boundary divisor with respects to  $f_1$  is the vertical part of boundary divisor with respects to  $f_2$  and vice versa, the choice of local lifting for the case of each  $\Omega_{Z/\mathbb{K}}^\bullet(\log D_Z, f_i), d$  are compatible. Therefore, the complex  $R\Gamma(\Omega_Z^\bullet(\log D_Z, f_1, f_2), d)$  is strict. Also note that the pushout diagram

$$\begin{array}{ccc} \Omega_Z^\bullet(\log D_Z, f_1, f_2) & \xrightarrow{j_1} & \Omega_Z^\bullet(\log D_Z, f_1) \\ \downarrow j_2 & & \downarrow i_1 \\ \Omega_Z^\bullet(\log D_Z, f_2) & \xrightarrow{i_2} & \Omega_Z^\bullet(\log D_Z, f) \end{array}$$

induces a short exact sequence of filtered complexes;

$$\begin{aligned} 0 \rightarrow (R\Gamma(\Omega_Z^\bullet(\log D_Z, f), d), F^\bullet) &\xrightarrow{(j_1, j_2)} \bigoplus_{i=1}^2 (R\Gamma(\Omega_Z^\bullet(\log D_Z, f_i), d), F^\bullet) \\ &\xrightarrow{i_1 - i_2} (R\Gamma(\Omega_Z^\bullet(\log D_Z, f), d), F^\bullet) \rightarrow 0 \end{aligned}$$

Since the first two terms are strict and both  $(j_1, f_2)$  and  $H(j_1, j_2)$  are strict morphisms, we conclude that the cone complex  $(R\Gamma(\Omega_Z^\bullet(\log D_Z, f), d), F^\bullet)$  is strict as well.  $\square$

Next, we look at the local description of relative de Rham complex. Let  $\epsilon =$

$(\epsilon_1, \epsilon_2)$  be a point near the infinity and  $Y_\epsilon$  be  $f^{-1}(\epsilon)$ . Here we get

$$\begin{aligned} \Omega_Z^a(\log D_Z, \text{rel} Y_\epsilon) &:= \ker(\Omega_Z^a(\log D_Z) \rightarrow i_{Y_\epsilon} \Omega_{Y_\epsilon}^a) \\ &= \bigoplus_{p=0}^a \left[ (z_1 \cdots z_k - \epsilon_1) \wedge^p W_1 + d \log f_1 \wedge (\wedge^{p-1} W_1) \right] \otimes \wedge^{a-p} R_1 \\ &\quad + \left[ (z_{k+1} \cdots z_n - \epsilon_2) \wedge^p W_2 + d \log f_2 \wedge (\wedge^{p-1} W_2) \right] \otimes \wedge^{a-p} R_2 \end{aligned}$$

so when  $\epsilon \rightarrow (0, 0)$ , the sheaf specializes to the  $f$ -adapted deRham form  $\Omega_Z^a(\log D_Z, f)$ .

In other words, we have

**Proposition 1.4.18.** *The complex  $(\Omega_Z^\bullet(\log D_Z, f), d)$  is a well-defined limit of the relative de Rham complex  $(\Omega_Z^\bullet(\log D_Z, f^{-1}(\rho_1, \rho_2)), d)$  as  $(\rho_1, \rho_2) \rightarrow (-\infty, -\infty)$ . In particular, the Gauss-Manin parallel transport has a well defined limit as  $\rho \rightarrow (-\infty, -\infty)$  which identifies  $H^a(Y, h^{-1}(\rho); \mathbb{C})$  with  $\mathbb{H}^a(Z, \Omega_Z^\bullet(\log D_Z, f))$ .*

*Proof.* Let  $\Delta^2 \subset \mathbb{P}^1 \times \mathbb{P}^1$  be a small polydisk centered at  $\{\infty\} \times \{\infty\} \in \mathbb{P}^1 \times \mathbb{P}^1$ .

Let  $p : \mathcal{Z} = Z \times \Delta^2 \rightarrow \Delta^2$  be a proper family. We consider the two relative divisors in  $\mathcal{Z}$ ;

$$D_{\mathcal{Z}} := D_Z \times \Delta^2$$

$$\Gamma := (p \times f)^{-1}(\text{graph} : \Delta \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1)$$

Note that  $D_{\mathcal{Z}}$  is a simple normal crossing divisor and  $\Gamma$  is smooth. We also denote the intersection of  $D_{\mathcal{Z}}$  and  $\Gamma$  by  $D_\Gamma$ .

Next, Recall that the sheaf of relative meromorphic differential forms,  $\Omega_{\mathcal{Z}/\Delta^2}^a(\log D_{\mathcal{Z}})$  is given by

$$\Omega_{\mathcal{Z}/\Delta^2}^a(\log D_{\mathcal{Z}}) := \ker(\Omega_{\mathcal{Z}}^a(\log D_{\mathcal{Z}}) \rightarrow p^* \Omega_{\Delta^2}^a(\log L))$$

where  $L$  is the boundary of  $\Delta^2$ . We consider the subsheaf of  $\Omega_{Z/\Delta^2}^a(\log D_Z)$  which vanishes along  $D_\Gamma$  as follows;

$$\Omega_{Z/\Delta^2}^a(\log D_Z, \text{rel}\Gamma) := \ker(\Omega_{Z/\Delta^2}^a(\log(D)) \rightarrow i_{\Gamma*}\Omega_{\Gamma/\Delta^2}^a(\log D_\Gamma))$$

This is locally free and the complex  $(\Omega_{Z/\Delta^2}^\bullet(\log D_Z, \text{rel}\Gamma), d)$  is preserved under the relative deRham differential. Let's denote this complex by  $\mathbf{E}_{Z/\Delta^2}^\bullet$ . Note that from the local computation, we have

$$(\mathbf{E}_{Z/\Delta^2}^\bullet|_{Z \times \{\epsilon \neq 0\}}) = (\Omega_Z^\bullet(\log D_Z, \text{rel}Y_\epsilon), d)$$

$$(\mathbf{E}_{Z/\Delta^2}^\bullet|_{Z \times \{\epsilon = 0\}}) = (\Omega_Z^\bullet(\log D_Z, f), d)$$

In other words, the complex  $\mathbf{E}_{Z/\Delta^2}^\bullet$  interpolates between the relative logarithmic forms vanishing on  $Y_\epsilon$  and relative  $f$ -adapted forms.

Geometrically, this interpolation can be refined as the Gauss-Manin parallel transport. Consider the local system  $\mathcal{E}^a$  of  $\mathbb{C}$ -vector spaces on  $(\Delta^\times)^2 \subset \Delta^2$  whose fiber at  $\epsilon = (\epsilon_1, \epsilon_2)$  is isomorphic to the Betti cohomology  $H^a(Y, Y_\epsilon; \mathbb{C})$ . The underlying coherent sheaf  $\mathcal{E}^a \otimes \mathcal{O}_{(\Delta^\times)^2}$  is identified with  $\mathbb{R}^a p_* \mathbf{E}_{Z^\times/(\Delta^\times)^2}^\bullet$ . Here  $(-)^{\times}$  implies the restriction over  $(\Delta^\times)^2$ . The Gauss-Manin connection is given by a  $\mathbb{C}$ -linear map of sheaves

$$\nabla^{\text{GM}} : \mathbb{R}^a p_* \mathbf{E}_{Z^\times/(\Delta^\times)^2}^\bullet \rightarrow \mathbb{R}^a p_* \mathbf{E}_{Z^\times/(\Delta^\times)^2}^\bullet \otimes_{(\Delta^\times)^2} \Omega_{(\Delta^\times)^2}^1$$

satisfying the Leibniz rule. One can identify the Gauss-Manin connection  $\nabla^{\text{GM}}$



with the connecting homomorphism of the long exact sequence;

$$0 \rightarrow \mathbf{E}_{\mathcal{Z}^\times/(\Delta^\times)^2}^\bullet[-1] \otimes_{p^{-1}\mathcal{O}_{(\Delta^\times)^2}} p^{-1}\Omega_{(\Delta^\times)^2}^1 \rightarrow \left( \frac{(\Omega_{\mathcal{Z}^\times}^\bullet(\log D_{\mathcal{Z}^\times}, \text{rel}\Gamma^\times))}{p^{-1}(\Omega_{(\Delta^\times)^2}^2) \wedge (\Omega_{\mathcal{Z}^\times}^{\bullet-2}(\log D_{\mathcal{Z}^\times}, \text{rel}\Gamma^\times))}, d \right) \\ \rightarrow \mathbf{E}_{\mathcal{Z}^\times/(\Delta^\times)^2}^\bullet \rightarrow 0$$

In order to show that the parallel transport with respects to the Gauss-Manin connection  $\nabla^{\text{GM}}$ , it is enough to show that the middle complex

$$(\Omega_{\mathcal{Z}^\times}^\bullet(\log D_{\mathcal{Z}^\times}, \text{rel}\Gamma^\times)/p^{-1}(\Omega_{(\Delta^\times)^2}^2 \wedge \Omega_{\mathcal{Z}^\times}^{\bullet-2}(\log D_{\mathcal{Z}^\times}, \text{rel}\Gamma^\times)), d)$$

extends to a well-defined subcomplex  $\mathbf{E}_{\mathcal{Z}}^\bullet$  in  $(\Omega_{\mathcal{Z}}^\bullet(\log D_{\mathcal{Z}}, d))$  on all  $\mathcal{Z}$ , which will fit into the short exact sequence of complex

$$0 \rightarrow \mathbf{E}_{\mathcal{Z}/\Delta^2}^\bullet[-1] \otimes_{p^{-1}\mathcal{O}_{\Delta^2}} p^{-1}\Omega_{\Delta^2}^1 \rightarrow \mathbf{E}_{\mathcal{Z}}^\bullet \rightarrow \mathbf{E}_{\mathcal{Z}/\Delta^2}^\bullet \rightarrow 0$$

Define the subcomplex  $\mathbf{E}_{\mathcal{Z}}^\bullet$  to be

$$\mathbf{E}_{\mathcal{Z}}^\bullet := \ker \left( \frac{\Omega_{\mathcal{Z}}^\bullet(\log D_{\mathcal{Z}} \cup f^{-1}(L))}{p^{-1}(\Omega_{(\Delta^\times)^2}^2) \wedge \Omega_{\mathcal{Z}}^{\bullet-2}(\log D_{\mathcal{Z}} \cup f^{-1}(L))} \rightarrow i_{\Gamma^*} \frac{\Omega_{\Gamma}^\bullet(\log D_{\Gamma})}{p^{-1}(\Omega_{(\Delta^\times)^2}^2) \wedge \Omega_{\Gamma}^{\bullet-2}(\log D_{\Gamma})} \right) \quad (1.4.26)$$

Recall that  $\mathbf{L} := \Delta^2 \setminus (\Delta^\times)^2$  and the above morphism is well-defined since the canonical morphism  $\Omega_{\mathcal{Z}}^\bullet(\log D_{\mathcal{Z}} \cup f^{-1}(\mathbf{L})) \rightarrow i_{\Gamma^*}\Omega_{\Gamma}^\bullet$  is compatible with Koszul filtration.

Then, one can check that the short exact sequence is defined by the kernel of the

following surjective morphism of the short exact sequences;

$$\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
\Omega_{Z/\Delta^2}^\bullet(\log D_Z, \text{rel}\Gamma)[-1] \otimes p^{-1}\Omega_{\Delta^2}^1 & \longrightarrow & i_{\Gamma*}\Omega_{\Gamma/\Delta^2}^\bullet(\log D_\Gamma)[-1] \otimes p^{-1}\Omega_{\Delta^2}^1 \\
\downarrow & & \downarrow \\
\frac{\Omega_Z^\bullet(\log D_Z \cup f^{-1}(L))}{p^{-1}(\Omega_{(\Delta \times)^2}^2) \wedge \Omega_Z^{\bullet-2}(\log D_Z \cup f^{-1}(L))} & \longrightarrow & i_{\Gamma*} \frac{\Omega_\Gamma^\bullet(\log D_\Gamma)}{p^{-1}(\Omega_{(\Delta \times)^2}^2) \wedge \Omega_\Gamma^{\bullet-2}(\log D_\Gamma)} \\
\downarrow & & \downarrow \\
\Omega_{Z/\Delta^2}^\bullet(\log D_Z, \text{rel}\Gamma) & \longrightarrow & i_{\Gamma*}\Omega_{\Gamma/\Delta^2}^\bullet(\log D_\Gamma) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & 0
\end{array} \tag{1.4.27}$$

□

*Remark 15.* One can get a similar result for the de Rham complex  $(\Omega_Z^\bullet(\log D_Z, f_1 \cup f_2), d)$ . In other words,  $(\Omega_Z^\bullet(\log D_Z, f_1 \cup f_2), d)$  is a well-defined limit of relative de Rham complex  $(\Omega^\bullet(\log D_Z, f_1^{-1}(\rho_1) \sqcup f_2^{-1}(\rho_2))$  as  $(\rho_1, \rho_2) \rightarrow (-\infty, -\infty)$  and its hypercohomology is identified with  $H^a(Y, Y_1 \sqcup Y_2; \mathbb{C})$ . However, we will not deal with this complex because the author does not fully understand monodromy weight filtration on the cohomology  $H^a(Y, Y_1 \sqcup Y_2; \mathbb{C})$ . For completeness, we only focus on the  $f$ -adapted de Rham complex  $(\Omega_Z^\bullet(\log D_Z, f), d)$ .

**Definition 1.4.19.** Let  $((Z, D_Z), f : Z \rightarrow \mathbb{P}^1 \times \mathbb{P}^1)$  be the compactified hybrid LG model of  $(Y, h : Y \rightarrow \mathbb{C}^2)$ . Then we define  $f$ -adapted Hodge number as follows. For  $p, q \geq 0$ ,

$$f^{p,q}(Y, h) := \dim_{\mathbb{C}} \mathbb{H}^p(Z, \Omega_Z^q(\log D_Z, f)).$$

Note that  $E_1$ -degeneration of the stupid filtration on  $(\Omega_Z^\bullet(\log D_Z, f), d)$  implies

that the sum of  $f$ -adapted Hodge numbers over  $p+q = a$  is  $\dim_{\mathbb{C}} \mathbb{H}^a(Z, \Omega_Z^q(\log D_Z, f))$ .

Consider the relative Hodge numbers of the pair  $(Y, Y_{12})$ ,

$$h^{p,q}(Y, Y_{12}) := \dim_{\mathbb{C}} H^p(Z, \Omega_Z^q(\log D_Z, \text{rel}f^{-1}(\rho)))$$

for a generic  $\rho \in \mathbb{C}^2$ . By applying Grauert's semicontinuity theorem to Proposition 1.4.18, we have the following argument, which implies that the  $f$ -adapted Hodge numbers are independent of the choice of tame compactification.

**Corollary 1.4.20.** *Let  $((Z, D_Z), f : Z \rightarrow \mathbb{P}^1 \times \mathbb{P}^1)$  be the compactified hybrid LG model of  $(Y, h : Y \rightarrow \mathbb{C}^2)$ . Then there is an equality between two Hodge numbers;*

$$h^{p,q}(Y, Y_{12}) = f^{p,q}(Y, h)$$

for all  $p, q \geq 0$ .

The next goal is to extend Conjecture 1.3.12 to the hybrid setting. Recall that in the non-hybrid case, one can associate the monodromy weight filtration to the LG potential  $w : Y \rightarrow \mathbb{C}$ , which comes from the monodromy around  $\infty$ . However, in the hybrid setting, we should consider two monodromy operators, denoted by  $N_1$  (resp.  $N_2$ ) around the infinity  $\{\infty\} \times \{\infty\}$  along the first (resp. second) coordinate axis. In addition, a linear combination of two operators  $N_1$  and  $N_2$  with positive coefficients gives rise to a third-kind weight filtration, which in general different from the ones associated to  $N_1$  and  $N_2$ . Therefore, we should deal with three monodromy weight filtrations  $W(N_1)$ ,  $W(N_2)$  and  $W(c_1N_1 + c_2N_2)$  for  $c_1, c_2 > 0$  that give rise to three different Hodge numbers.

**Definition 1.4.21.** Let  $(Y, h : Y \rightarrow \mathbb{C}^2)$  be a hybrid LG model. For  $c_1, c_2 \geq 0$ , define three different Hodge numbers by

$$h_{W(c_1N_1+c_2N_2)}^{p,q}(Y, h) := \dim_{\mathbb{C}} \mathrm{Gr}_{2p}^{W(c_1N_1+c_2N_2)} H^{p+q}(Y, Y_{12,\infty}; \mathbb{C})$$

where the additional subscript  $\infty$  in  $Y_{12,\infty}$  is used to distinguish the monodromy weight filtration from the Deligne's canonical weight filtration.

**Conjecture 1.4.22.** (*Extended KKP*) For  $p, q, c_1, c_2 \geq 0$ , there is an identification of Hodge numbers;

$$f^{p,q}(Y, h) = h_{W(c_1N_1+c_2N_2)}^{p,q}(Y, h)$$

*In other words, the associated graded pieces of three monodromy weight filtrations are the same and equal to the relevant  $f$ -adapted Hodge numbers.*

### Comparison between ordinary and hybrid LG models

In this subsection, we will compare two different deformation theories of hybrid and ordinary LG model and construct the morphism between deformation spaces. Fix a hybrid LG model  $(Y, h : Y \rightarrow \mathbb{C}^2)$  with an induced ordinary LG model  $w : Y \rightarrow \mathbb{C}$ . Choose a tame compactification of the hybrid LG model,  $((Z, D_Z), f = (f_1, f_2) : Z \rightarrow \mathbb{P}^1 \times \mathbb{P}^1)$ . Unfortunately, this doesn't directly produce a tame compactification of the ordinary LG model  $(Y, w : Y \rightarrow \mathbb{C})$  because the compactification of the

summation map  $\Sigma : \mathbb{C}^2 \rightarrow \mathbb{C}$  is given by

$$\begin{array}{ccc} \mathrm{Bl}_S(\mathbb{P}^1 \times \mathbb{P}^1) & \xrightarrow{\quad \bar{\Sigma} \quad} & \mathbb{P}^1 \\ & \searrow \pi & \nearrow [ad+bc:bd] \\ & \mathbb{P}_{[a:b]}^1 \times \mathbb{P}_{[c:d]}^1 & \end{array}$$

where the blow-up loci  $S$  are two points  $\{p_1 = (0, \infty), p_2 = (\infty, 0)\}$ . Therefore, in order to obtain a tame compactification of the ordinary LG model from that of the hybrid LG model, we need to blow up the base over the discriminant locus  $S$  and take the pullback of  $Z$ . We denote it by  $\bar{Z}$ . Then, we obtain a tame compactification  $((\bar{Z}, D_{\bar{Z}}), \bar{f} = \bar{\Sigma} \circ f_{\pi} : \bar{Z} \rightarrow \mathbb{P}^1)$  where  $D_{\bar{Z}}$  is the union of  $D_Z$  and  $E_i \times f^{-1}(p_i)$  for  $i = 1, 2$  where  $E_i$ 's are exceptional divisor in  $\mathrm{Bl}_S(\mathbb{P}^1 \times \mathbb{P}^1)$ . This is summarized as follows;

$$\begin{array}{ccccc} & & Z & \xleftarrow{\quad \bar{\pi} \quad} & \bar{Z} \\ & & \downarrow f & & \downarrow f_{\pi} \\ \mathbb{C}^2 & \hookrightarrow & \mathbb{P}_{[a:b]}^1 \times \mathbb{P}_{[c:d]}^1 & \xleftarrow{\quad \pi \quad} & \mathrm{Bl}_S(\mathbb{P}^1 \times \mathbb{P}^1) \\ \downarrow \Sigma & & \downarrow [ad+bc:bd] & & \swarrow \bar{\Sigma} \\ \mathbb{C} & \hookrightarrow & \mathbb{P}^1 & & \end{array}$$

Let  $\mathcal{M}_{(Z,f)_{D_Z}}$  be a (formal) versal deformation space of the hybrid LG model  $(Y, h : Y \rightarrow \mathbb{C}^2)$  with the chosen compactification  $((Z, D_Z), f : Z \rightarrow \mathbb{P}^1 \times \mathbb{P}^1)$  and  $\mathcal{M}_{(\bar{Z}, \bar{f})_{D_{\bar{Z}}}}$  be a (formal) versal deformation space of the ordinary LG model  $(Y, w : \Sigma \circ h : Y \rightarrow \mathbb{C})$  with the induced compactification  $((\bar{Z}, D_{\bar{Z}}), f : \bar{Z} \rightarrow \mathbb{P}^1)$ .

Then the above construction gives a map

$$\Phi : \mathcal{M}_{(Z,f)_{D_Z}} \rightarrow \mathcal{M}_{(\bar{Z}, \bar{f})_{D_{\bar{Z}}}}$$

**Proposition 1.4.23.** *The morphism  $\Phi : \mathcal{M}_{(Z,f)D_Z} \rightarrow \mathcal{M}_{(\bar{Z},\bar{f})D_{\bar{Z}}}$  constructed above is a submersion.*

*Proof.* Recall that the deformation complex of  $(Z, f)_{D_Z}$  is given by the sheaf of differential graded Lie algebras

$$\mathfrak{g}^\bullet := \left[ T_{Z,D_Z} \xrightarrow{df} f^* T_{\mathbb{P}^1 \times \mathbb{P}^1, L} \right]$$

Also, by definition, it is quasi-isomorphic to the sheaf of differential graded Lie algebras associated to the blow-up pair  $((\bar{Z}, D_{\bar{Z}}), f_\pi : \bar{Z} \rightarrow \text{Bl}_S(\mathbb{P}^1 \times \mathbb{P}^1))$

$$\bar{\mathfrak{g}}^\bullet := \left[ T_{\bar{Z}, D_{\bar{Z}}} \xrightarrow{df_\pi} f_\pi^* T_{\text{Bl}_S(\mathbb{P}^1 \times \mathbb{P}^1), L \sqcup E_1 \sqcup E_2} \right]$$

and it admits a map to the deformation complex associated to the tame compactification  $((\bar{Z}, D_{\bar{Z}}), \bar{f} : \bar{Z} \rightarrow \mathbb{P}^1)$

$$\begin{array}{ccc} T_{\bar{Z}, D_{\bar{Z}}} & \xrightarrow{df_\pi} & f_\pi^* T_{\text{Bl}_S(\mathbb{P}^1 \times \mathbb{P}^1), L \sqcup E_1 \sqcup E_2} \\ \downarrow = & & \downarrow d\bar{\Sigma} \\ T_{\bar{Z}, D_{\bar{Z}}} & \xrightarrow{d\bar{f}} & \bar{f}^* T_{\mathbb{P}^1, \infty} \end{array}$$

whose kernel is  $\ker(d\bar{\Sigma})[-1]$ . As  $\mathbb{H}^2(\ker(d\bar{\Sigma})[-1]) = H^1(\bar{Z}, \ker(d\bar{\Sigma})) = 0$ , the induced long exact sequence of hypercohomology gives a surjective map  $\mathbb{H}^1(\bar{\mathfrak{g}}^\bullet) \rightarrow \mathbb{H}^1(\bar{\mathfrak{g}}_1)$  whose kernel is given by

$$H^0(\bar{Z}, f_\pi^* \ker(d\bar{\Sigma})) \cong H^0(\text{Bl}_S(\mathbb{P}^1 \times \mathbb{P}^1), \ker(d\bar{\Sigma}))$$

□

## 1.5 Simple Normal Crossing Case - General

### 1.5.1 Extended Fano/LG Correspondence

A theory of hybrid Landau-Ginzburg models and related topics can be generalized to the case where the mirror Fano pair  $(X, D)$  has more than two components. In this section, we state main result whose proof are essentially the same with ones in the case of two components.

Let  $D$  be an effective anti-canonical divisor with  $k$  components,  $D_1 \cup D_2 \cup \cdots \cup D_k$ .

For any index set  $I = \{i_1, i_2, \cdots, i_m\} \subset I_k := \{1, 2, \cdots, k\}$ , we define

$$D_I := D_{i_1} \cap \cdots \cap D_{i_m}$$

$$D(I) := \Sigma_{j \notin I} D_I \cap D_j$$

For example, if  $I = \{1\}$ , then  $D_I = D_1$  and  $D(1) = (D_2 \cup \cdots \cup D_k) \cap D_1$ . We also assume that all quasi-Fano pairs  $(D_I, D(I))$  is indeed Fano. This positivity assumption is necessary condition for relative HMS conjecture hold for Fano pairs [Bal+13].

Similar to the two-components case, the SYZ mirror construction [SYZ96][Aur07] suggests that counting holomorphic disk which touches each boundary component provides a potential. Hence, we inductively define a hybrid LG potential,  $h : Y \rightarrow \mathbb{C}^k$ , where the target is  $k$ -dimensional affine space.

**Definition 1.5.1.** (See Definition 1.4.1) A **hybrid Landau-Ginzburg model**

(mirror to  $(X, D)$ ) is a triple  $(Y, \omega, h = (h_1, h_2, \dots, h_k) : Y \rightarrow \mathbb{C}^k)$  where

- $(Y, \omega)$  is  $2n$ -dimensional complex Kähler Calabi-Yau manifold (orbifold) with Kähler form  $\omega \in \Omega^2(Y)$ .
- $h := (h_1, h_2, \dots, h_k) : Y \rightarrow \mathbb{C}^k$  is a proper morphism (Lefschetz fibration) such that

- (1) A generic fiber of  $h_i$ , denoted by  $Y_i$  with  $h|_{Y_i} : Y_i \rightarrow \mathbb{C}^{k-1}$  is mirror to  $(D_i, D(i))$  for all  $i$ ;
- (2) By composing with the summation map  $\Sigma : \mathbb{C}^k \rightarrow \mathbb{C}$ , we get an ordinary LG model

$$w := \Sigma \circ h : Y \rightarrow \mathbb{C}$$

which is mirror to  $X$  in the sense of the usual Fano/LG correspondence.

A hybrid LG model encodes all different mirror symmetries associated to  $(X, D)$ .

*Example 1.5.2.* Consider a Fano pair with  $(\mathbb{P}^2, D)$  where  $D = L_1 \cup L_2 \cup L_3$  is the union of 3 lines, i.e toric divisor. By Hori-Vafa construction, one can get a hybrid LG model  $((Y, \omega), h : Y \rightarrow \mathbb{C}^3)$  such that

- An original LG potential  $w : \Sigma \circ h : Y \rightarrow \mathbb{C}$  is obtained by removing three horizontal divisor in the elliptic fibration over  $\mathbb{P}^1$  which is mirror to  $(\mathbb{P}^2, D_{sm})$ . (Example 1.3.3)



- A hybrid LG model  $h : Y \rightarrow \mathbb{C}^3$  is

$$h : (\mathbb{C}^*)^2 \rightarrow \mathbb{C}^3$$

$$(x, y) \mapsto (x/y, 1/y, y^2/x)$$

whose generic fiber is empty. This corresponds to the fact that  $D_{123} = \emptyset$ .

- A restriction of  $h : Y \rightarrow \mathbb{C}^3$  over each generic coordinate line recovers the hybrid LG model mirror to  $(\mathbb{P}^1, 2pt)$ .

The construction of hybrid LG models is similar to one introduced in Section 1.4.1. The only difference is that we should choose  $k$  sections each corresponding to a line bundle mirror to  $D_i$  for each  $i$ .

The gluing property (Proposition 1.4.5) is essential as well, but one should deal with gluing of hybrid LG models, not the ordinary ones. This is due to the inductive nature of hybrid LG models. For example, when  $k = 3$ , a hybrid LG model is  $(Y, h : Y \rightarrow \mathbb{C}^3)$ . Denote  $Y_{sm}$  a generic fiber of the induced ordinary LG potential  $w := \Sigma \circ h : Y \rightarrow \mathbb{C}$ . Associated to  $Y_{sm}$ , there is a canonical fibration, the restriction of the hybrid potential  $h|_{Y_{sm}} : Y_{sm} \rightarrow \mathbb{C}^2$ . Therefore, it is reasonable to glue hybrid LG models  $(Y_i, h|_{Y_i} \rightarrow \mathbb{C}^2)$ . mirror to  $(D_i, D(i))$  for each  $i$ .

**Claim 1.5.3.** *A fibration on  $Y_{sm}$ ,  $h|_{Y_{sm}} : Y_{sm} \rightarrow \mathbb{C}^{k-1}$  is topologically glued by  $k$  fibrations  $h|_{Y_i} : Y_i \rightarrow \mathbb{C}^{k-1}$  for all  $i = 1, \dots, k$ .*

The above claim can be verified for Hori-Vafa hybrid LG models, mirror to a Fano complete intersection  $X$ . The key is to describe discriminant loci of the hybrid

LG model and study (linear) deformation of a general anti-diagonal hypersurface, a base of the fibration  $h|_{Y_{sm}} : Y_{sm} \rightarrow \mathbb{C}^{k-1}$ .

### 1.5.2 Relative HMS and the Mirror P=W Conjecture

Next, we introduce the relative HMS conjecture for the Fano pair  $(X, D)$ .

**Conjecture 1.5.4.** (*Relative HMS for the Fano pair  $(X, D)$* ). *There is an equivalence of (Čech-type) diagrams of categories;*

$$\begin{array}{ccccccc}
D^b\mathrm{Coh}(D_{I_k}) & \xrightarrow{\phi_k} & \bigoplus_{|I|=k-1} D^b\mathrm{Coh}(D_I) & \xrightarrow{\phi_{k-1}} \cdots \xrightarrow{\phi_2} & \bigoplus_{|I|=1} D^b\mathrm{Coh}(D_I) & \xrightarrow{\phi_1} & D^b\mathrm{Coh}(X) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
\mathrm{Fuk}(Y_{I_k}) & \xrightarrow{\psi_k} & \bigoplus_{|I|=k-1} \mathrm{FS}^{wr}(Y_I, \mathbf{w}) & \xrightarrow{\psi_{k-1}} \cdots \xrightarrow{\psi_2} & \bigoplus_{|I|=1} \mathrm{FS}^{wr}(Y_I, \mathbf{w}) & \xrightarrow{\psi_1} & \mathrm{Fuk}^{wr}(Y, \mathbf{w})
\end{array} \tag{1.5.1}$$

where  $\phi_i$  (resp.  $\psi_i$ ) is an alternating sum of inclusion (resp. cup) functors.

By taking Hochschild homology  $HH_a$ , one can expect that the induced sequence represents the relation between the weight and perverse Leray filtration. Although this is clear on the B-side, one need to verify that the induced sequence of cohomology groups is equivalent to the spectral sequence of the perverse Leray filtration associated to the hybrid LG model  $(Y, h : Y \rightarrow \mathbb{C}^k)$ . For simplicity, let's assume that  $k = 3$ . The induced sequence is the following;

$$H^{a+n-3}(Y_{123}) \xrightarrow{\psi_3} \bigoplus_{i < j} H^{a+n-2}(Y_{ij}, Y_{123}) \xrightarrow{\psi_2} \bigoplus_{i=1}^3 H^{a+n-1}(Y_i, Y_{i,sm}) \xrightarrow{\psi_1} H^{a+n}(Y, Y_{sm}) \tag{1.5.2}$$

where  $Y_{i,sm}$  is a generic fiber of  $w : Y_i \rightarrow \mathbb{C}$ . The gluing property implies that  $Y_{sm} \cong Y_1 \cup Y_2 \cup Y_3$  and  $Y_{i,sm} \cong Y_{ij} \cup Y_{ik}$  for  $\{i, j, k\} = \{1, 2, 3\}$ . Denote  $Y_{sm,sm}$  a union  $Y_{12} \cup Y_{23} \cup Y_{13} \cong Y_{1,sm} \cup Y_{2,sm} \cup Y_{3,sm}$ . Then the above equation 1.5.2 is quasi-isomorphic to the following complex

$$H^{a+n-3}(Y_{123}) \xrightarrow{\rho_3} H^{a+n-2}(Y_{sm,sm}, Y_{123}) \xrightarrow{\rho_2} \bigoplus_{i=1}^3 H^{a+n-1}(Y_{sm}, Y_{sm,sm}) \xrightarrow{\rho_1} H^{a+n}(Y, Y_{sm}) \quad (1.5.3)$$

where  $\rho_i$ 's are connecting homomorphism. In order to obtain the  $E_1$ -page of the spectral sequence of the perverse Leray filtration associated to the hybrid LG potential  $h : Y \rightarrow \mathbb{C}^3$ , we need to verify that the flag

$$Y_{123} \subset Y_{sm,sm} \subset Y_{sm} \subset Y$$

forms a general linear flag [CM10]. In general, we have the following conjecture;

**Conjecture 1.5.5.** *For a given Fano mirror pair  $(X, D)$  and  $(Y, \omega, h : Y \rightarrow \mathbb{C}^k)$ , the relative HMS (Conjecture ) implies the mirror  $P=W$  conjecture*

$$\dim Gr_F^q Gr_{p+q+i}^W H^{p+q}(U) = \dim Gr_F^{n-q} Gr_{n+p-q+i}^P H^{n+p-q}(Y)$$

for all  $i \in I_k$ .

The main issue is to show that the induced A-side diagram is quasi-isomorphic to the spectral sequence of perverse Leray filtration associated to the hybrid LG potential  $h : Y \rightarrow \mathbb{C}^k$ .

### 1.5.3 Deformation Theory and Hodge Numbers

In Section 2.3, we studied the deformation theory of hybrid LG models  $(Y, h : Y \rightarrow \mathbb{C}^2)$  by looking at the tame compactification  $((Z, D_Z), f : Z \rightarrow \mathbb{P}^1 \times \mathbb{P}^1)$ . All the arguments and theorems can be easily extended to a general case. Let  $(Y, h : Y \rightarrow \mathbb{C}^k)$  be a hybrid LG model, which is mirror to the Fano pair  $(X, D)$  where  $D$  satisfies the positivity assumption.

**Definition 1.5.6.** A compactified hybrid LG model is datum  $((Z, f), D_Z, \text{vol}_Z)$  where:

- $Z$  is a smooth projective variety and  $f = (f_1, \dots, f_k) : Z \rightarrow (\mathbb{P}^1)^k$  is a flat projective morphism;
- $D_Z := D_1 \cup \dots \cup D_k$  is a reduced normal crossings divisor such that  $D_i := (f_i^{-1}(\infty))$  is simple normal crossings divisor for all  $i$ ;
- $\text{vol}_Z$  is a meromorphic section of  $K_Z$  with poles at most at  $D_Z$  and no zeros.

Let  $\mathbf{L} = \bigcup \mathbb{P}^1 \times \dots \times \{\infty\} \times \dots \times \mathbb{P}^1$  be the complement of  $\mathbb{C}^k$  in  $(\mathbb{P}^1)^k$ . Then the deformation theory of  $(Z, f)$  preserving the boundary  $D_Z$ , denoted by  $(Z, f)_{D_Z}$ , is computed by the following sheaf of differential graded (dg) Lie algebra

$$\mathfrak{g}^\bullet := \left[ T_{Z, D_Z} \xrightarrow{df=(df_1, \dots, df_k)} f^* T_{(\mathbb{P}^1)^k, \mathbf{L}} \right] \quad (1.5.4)$$

Applying the same argument in Lemma 1.4.14, we obtain the following theorem.

**Theorem 1.5.7.** *The deformation theory of  $(Z, f)_{D_Z}$  is unobstructed.*

One can also introduce the notion of  $f$ -adapted de Rham complex  $(\Omega_Z^\bullet(\log D_Z, f), d)$ , a subcomplex of the logarithmic de Rham complex  $(\Omega_Z^\bullet(\log D_Z), d)$  preserved by either  $\wedge df_i$ 's. More generally, there are variants of  $f$ -adapted de Rham complex depending on which differential  $df_i \wedge$  preserves the forms. For an index set  $I \subset \{1, \dots, k\}$ , we denote  $\Omega_Z^a(\log D_Z, f_I)$  a set of logarithmic differential  $a$ -forms which is preserved by either all  $df_i \wedge$ 's or  $df_j$ 's for  $i \in I, j \notin I$ . For example, when  $I = \{1, \dots, k\}$ , we have  $\Omega_Z^a(\log D_Z, f_I) = \Omega_Z^a(\log D_Z, f)$  for all  $a$ . Similar to Lemma 1.4.16, one can write down the local description of  $\Omega_Z^a(\log D_Z, f_I)$  which leads to the following propositions.

**Proposition 1.5.8.** *The Hodge-to-de Rham spectral sequences of both  $(\Omega_Z^\bullet(\log D_Z, f_I), d)$  degenerate at  $E_1$ -page.*

**Proposition 1.5.9.** *The complex  $(\Omega_Z^\bullet(\log D_Z, f), d)$  is a well-defined limit of the relative de Rham complex  $(\Omega_Z^\bullet(\log D_Z, f^{-1}(\rho)), d)$  as  $\rho \rightarrow (-\infty, \dots, -\infty)$ . In particular, the Gauss-Manin parallel transport has a well defined limit as  $\rho \rightarrow (-\infty, \dots, -\infty)$  which identifies  $H^a(Y, h^{-1}(\rho); \mathbb{C})$  with  $\mathbb{H}^a(Z, \Omega_Z^\bullet(\log D_Z, f))$ .*

The above two propositions allow one to define  $f$ -adapted Hodge numbers which is intrinsic to the hybrid LG model  $(Y, h : Y \rightarrow \mathbb{C}^k)$ .

**Definition 1.5.10.** Let  $((Z, D_Z), f : Z \rightarrow (\mathbb{P}^1)^k)$  be the compactified hybrid LG model of  $(Y, h : Y \rightarrow \mathbb{C}^k)$ . Then we define  $f$ -adapted Hodge number as follows. For  $p, q \geq 0$ ,

$$f^{p,q}(Y, h) := \dim_{\mathbb{C}} \mathbb{H}^p(Z, \Omega_Z^q(\log D_Z, f)).$$

Moreover, in this hybrid setting, there are  $k$  monodromy operators  $N_i$  ( $i = 1, \dots, k$ ) around the infinity  $\{\infty\}^k$  along each coordinate axis. Hence, there are  $2^k - 1$  monodromy weight filtration  $W(c_1N_1 + c_2N_2 + \dots + c_kN_k)$  depending on  $c_i \geq 0$ .

**Definition 1.5.11.** Let  $(Y, h : Y \rightarrow \mathbb{C}^k)$  be a hybrid LG model. For  $c_i \geq 0$ , define  $2^k - 1$  different Hodge numbers by

$$h_{W(c_1N_1+\dots+c_kN_k)}^{p,q}(Y, h) := \dim_{\mathbb{C}} \mathrm{Gr}_{2p}^{W(c_1N_1+\dots+c_kN_k)} H^{p+q}(Y, Y_{\rho,\infty}; \mathbb{C})$$

where the additional subscript  $\infty$  in  $Y_{\rho,\infty}$  is used to distinguish the monodromy weight filtration from the Deligne's canonical weight filtration.

**Conjecture 1.5.12.** (*Extended KKP*) (See Conjecture 1.4.22) For  $p, q, c_1, \dots, c_k \geq 0$ , there is an identification of Hodge numbers;

$$f^{p,q}(Y, h) = h_{W(c_1N_1+\dots+c_kN_k)}^{p,q}(Y, h)$$

*In other words, the associated graded pieces of  $2^k - 1$  monodromy weight filtrations are the same and equal to the relevant  $f$ -adapted Hodge numbers.*

Finally, we can also describe the relation between two versal deformation spaces associated to the hybrid LG model  $(Y, : Y \rightarrow \mathbb{C}^k)$  by extending the case when  $k = 2$ . From a tame compactification  $(Z, D_Z, f : Z \rightarrow (\mathbb{P}^1)^k)$  of the hybrid LG model, one can obtain the tame compactification of the ordinary LG model  $(Y, w : \Sigma \circ h : Y \rightarrow \mathbb{C})$ . This is achieved by compactifying  $\Sigma : \mathbb{C}^k \rightarrow \mathbb{C}$  to  $\bar{\Sigma} : \mathrm{Bl}_S(\mathbb{P}^1)^k \rightarrow \mathbb{P}^1$  where  $S$

is the base locus of rational map

$$\Sigma : (\mathbb{P}^1)^k \dashrightarrow \mathbb{P}^1$$

$$([x_1 : y_1], \dots, [x_k, y_k]) \mapsto \left[ \frac{x_1}{y_1} + \dots + \frac{x_k}{y_k} : 1 \right]$$

and take the pullback of  $Z$  under the canonical map  $\pi : \text{Bl}_S(\mathbb{P}^1)^k \rightarrow (\mathbb{P}^1)^k$ . We denote it by  $\bar{Z}$ . Then the tame compactification is given by  $((\bar{Z}, D_{\bar{Z}}), \bar{f} : \bar{Z} \rightarrow \mathbb{P}^1)$  where  $D_{\bar{Z}}$  is a total transformation of  $D_Z$ . This construction gives a morphism between two versal deformation spaces

$$\Phi : \mathcal{M}_{(Z,f)_{D_Z}} \rightarrow \mathcal{M}_{(\bar{Z},\bar{f})_{D_{\bar{Z}}}}$$

We claim that this morphism  $\Phi$  is submersion (Proposition 1.4.23

**Proposition 1.5.13.** *The morphism  $\Phi : \mathcal{M}_{(Z,f)_{D_Z}} \rightarrow \mathcal{M}_{(\bar{Z},\bar{f})_{D_{\bar{Z}}}}$  constructed above is a submersion.*

*Proof.* The proof is essentially the same with the proof of Proposition 1.4.23  $\square$

# Chapter 2

## Semi-polarized meromorphic

## Hitchin and Calabi-Yau integrable

## systems

### 2.1 Introduction

Since the seminal work of Hitchin [Hit87b][Hit87a], Higgs bundles and their moduli spaces have been studied extensively. There have been numerous deep results on the moduli space of Higgs bundles related to other areas of mathematics such as the  $P = W$  conjecture [CHM12][CMS19], the fundamental lemma in the Langlands program [Ngô06][Ngô10], the geometric Langlands conjecture [KW07] and mirror symmetry [HT03][DP12]. One of the striking properties of these moduli spaces is



that they admit a holomorphic symplectic form and the structure of an integrable system, called the *Hitchin system*. In particular, the generic fiber of an integrable system is an abelian variety which turns out to be the Jacobian or (generalized) Prym variety of an associated spectral or cameral curve. This picture generalizes to the meromorphic situation where we allow the Higgs field to have poles along some divisors. While the meromorphic Hitchin system is no longer symplectic, it is still Poisson and integrable with respect to the Poisson structure.

On the other hand, Donagi-Markman and Donagi-Diaconescu-Pantev (DDP) introduced in [DM96a][DM96b][Dia+06][DDP07] integrable systems coming from some families of projective or quasi-projective Calabi-Yau threefolds, called *Calabi-Yau integrable systems*. A generic fiber is a complex torus or an abelian variety [Dia+06][DDP07], now obtained as the intermediate Jacobian of a Calabi-Yau threefold in the family.

It is shown in [DDP07] that for adjoint groups  $G$  of type ADE, there is an isomorphism between  $G$ -Hitchin systems and suitable Calabi-Yau integrable systems, which we call the *DDP correspondence*. An interesting aspect of the construction in [DDP07] is that although the relevant Calabi-Yau threefold is non-compact, the (a priori mixed) Hodge structure on its third cohomology happened to be pure of weight one up to Tate twist. Because of this, the corresponding intermediate Jacobian is a compact torus (in fact an abelian variety). Since the data of a weight 1 Hodge structure is equivalent to the data of an abelian variety, this isomorp-

shism can be rephrased as an isomorphism between variations of weight 1 Hodge structures equipped with the abstract Seiberg-Witten differential, see for example [DDP07] [Bec20].

It is worth mentioning that the origin of this story comes from physics, specifically, large  $N$  duality [Dia+06]. Recently, the correspondence has also found its place in the study of T-branes in F-theory [AHK14][And+17].

The isomorphism between Hitchin and Calabi-Yau integrable systems has been generalized successfully to groups of type BCFG by the work of Beck et al. [Bec20][Bec19][BDW20] using the technique of foldings.

## Main results

The goal of this paper is to extend the DDP correspondence to the setting of meromorphic  $SL(n, \mathbb{C})$ -Hitchin system  $h : \mathcal{M}(n, D) \rightarrow B$  where  $D$  is a reduced divisor of the base curve. The best case scenario will be to construct a family of non-compact Calabi-Yau threefolds over the same base  $B$  and show that the associated Calabi-Yau integrable system is isomorphic to the meromorphic Hitchin system as Poisson integrable systems. However, since the deformation space of such non-compact Calabi-Yau's is strictly smaller than the base  $B$ , we do not expect to get a natural family which induces the Poisson integrable system (see [KS14]).

Instead, we consider the notion of *semi-polarized integrable systems* introduced by Kontsevich-Soibelman [KS14]. These are non-compact versions of symplectic

integrable systems whose fiber is a semi-abelian variety, an extension of an abelian variety by an affine torus. The main advantage is that they canonically induce the Poisson integrable systems as their compact quotients. In Section 2, we study this structure from the Hodge theoretic viewpoint. Since the data of a semi-polarized semi-abelian variety is equivalent to the data of a semi-polarized  $\mathbb{Z}$ -mixed Hodge structure of type  $\{(-1, -1), (-1, 0), (0, -1)\}$  (see Appendix), the semi-polarized integrable system can be described as an admissible variation of  $\mathbb{Z}$ -mixed Hodge structures of such type with an abstract Seiberg-Witten differential as in the classical case.

The main objects on the Hitchin side are the moduli space of diagonally framed Higgs bundles (resp. unordered), introduced by Biswas-Logares-Peón-Nieto [BLP19][BLP20]<sup>1</sup>, and we denote these moduli space by  $\overline{\mathcal{M}}^\Delta(n, D)$  (resp.  $\mathcal{M}^\Delta(n, D)$ ). The moduli space  $\overline{\mathcal{M}}^\Delta(n, D)$  is a subspace of the moduli space of framed Higgs bundles  $\mathcal{M}_F(n, D)$  whose object is a triple  $(E, \theta, \delta)$  where  $(E, \theta)$  is a  $SL(n, \mathbb{C})$ -Higgs bundle and  $\delta$  is a framing of  $E$  at  $D$ . As the name suggests, an object in  $\overline{\mathcal{M}}^\Delta(n, D)$  is a framed Higgs bundle such that the residue of its Higgs field is diagonal with respect to the framing  $\delta$ . The unordered version  $\mathcal{M}^\Delta(n, D)$  is obtained as the quotient of  $\overline{\mathcal{M}}^\Delta(n, D)$  by  $S_n^{|D|}$  where  $S_n^{|D|}$  is the product of symmetric groups  $S_n$  acting on the space of the framings by permuting the order of components. The following

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<sup>1</sup>In [BLP20], what we call "diagonally framed" is referred to as "relatively framed" in [21].

diagram summarizes the relation among the moduli spaces:

$$\begin{array}{ccc}
\overline{\mathcal{M}}^\Delta(n, D) & \hookrightarrow & \mathcal{M}_F(n, D) \\
\downarrow q & & \downarrow f_1 \\
\mathcal{M}^\Delta(n, D) & \xrightarrow{f_2} & \mathcal{M}(n, D) \\
& \searrow h_\Delta & \downarrow h \\
& & B
\end{array} \tag{2.1.1}$$

where  $q : \overline{\mathcal{M}}^\Delta(n, D) \rightarrow \mathcal{M}^\Delta(n, D)$  is the quotient map,  $f_1$  and  $f_2$  are the maps of forgetting the framings and  $h_\Delta := h \circ f_2 : \mathcal{M}^\Delta(n, D) \rightarrow B$  is the Hitchin map on the moduli space of unordered diagonally framed Higgs bundles that we will study. In this paper, we will mainly work over the locus  $B^{ur} \subset B$  of smooth cameral curves which are unramified over  $D$  and have simple ramifications. In particular, for a triple  $(E, \theta, \delta)$  over  $b \in B^{ur}$ , the residue of  $\theta$  over  $D$  has distinct eigenvalues. We shall write the restrictions as  $\overline{\mathcal{M}}^\Delta(n, D)^{ur} := (h_\Delta \circ q)^{-1}(B^{ur})$  and  $\mathcal{M}^\Delta(n, D)^{ur} := h_\Delta^{-1}(B^{ur})$ .

We will show that  $\overline{\mathcal{M}}^\Delta(n, D)^{ur}$  and  $\mathcal{M}^\Delta(n, D)^{ur}$  are symplectic using deformation theoretic arguments. They also carry a smooth semi-polarized integrable system structure over the locus  $B^{ur}$ . The following is the first result of the paper.

**Theorem 2.1.1.** *(Proposition 2.3.17, Corollary 2.3.20) The moduli space of unordered diagonally framed Higgs bundle  $\mathcal{M}^\Delta(n, D)$  is symplectic. The Hitchin fibration*

$$h_\Delta^{ur} : \mathcal{M}^\Delta(n, D)^{ur} \rightarrow B^{ur}$$

*forms a smooth semi-polarized integrable system whose fiber is a semi-abelian vari-*

ety.

In order to prove this, we study the fiber  $(h_{\Delta}^{ur})^{-1}(b)$  over each  $b \in B^{ur}$  via the spectral correspondence between unordered diagonally framed Higgs bundles on  $\Sigma$  and framed line bundles on the associated spectral cover  $\bar{p}_b : \bar{\Sigma}_b \rightarrow \Sigma$ . The framed line bundles on  $\bar{\Sigma}_b$  are then parametrized by the Prym variety  $Prym(\bar{\Sigma}_b^{\circ}, \Sigma^{\circ})$  associated to the restricted spectral cover  $\bar{p}_b^{\circ} := \bar{p}_b|_{\bar{\Sigma}_b^{\circ}} : \bar{\Sigma}_b^{\circ} \rightarrow \Sigma^{\circ}$  where  $\bar{\Sigma}_b^{\circ} := \bar{\Sigma}_b \setminus \bar{p}_b^{-1}(D)$  and  $\Sigma^{\circ} := \Sigma \setminus D$ . More precisely,  $Prym(\bar{\Sigma}_b^{\circ}, \Sigma^{\circ})$  is a semi-abelian variety defined as the kernel of the punctured norm map  $Nm^{\circ} : Jac(\bar{\Sigma}_b^{\circ}) \rightarrow Jac(\Sigma^{\circ})$ .

**Proposition 2.1.2.** *(Proposition 2.3.8, Spectral correspondence) A generic fiber  $h_{\Delta}^{-1}(b)$  is canonically isomorphic to the semi-abelian variety  $Prym(\bar{\Sigma}_b^{\circ}, \Sigma^{\circ})$ . In particular, the first homology  $H_1(Prym(\bar{\Sigma}_b^{\circ}, \Sigma^{\circ}))$  admits a  $\mathbb{Z}$ -mixed Hodge structure of type  $\{(-1, -1), (-1, 0), (0, -1)\}$ .*

On the Calabi-Yau side, we construct a family of Calabi-Yau threefolds  $\pi : \mathcal{X} \rightarrow B$  by using the elementary modification technique in [Smi15]. To produce the relevant Calabi-Yau integrable systems, we should restrict the family  $\pi : \mathcal{X} \rightarrow B$  to  $B^{ur}$ , denoted by  $\pi^{ur} : \mathcal{X}^{ur} \rightarrow B^{ur}$ , whose fiber is smooth and its third homology admits a  $\mathbb{Z}$ -mixed Hodge structures of type  $\{(-1, -1), (-1, 0), (0, -1)\}$  up to Tate twist. Now, by taking fiberwise intermediate Jacobians, we obtain a family of semi-abelian varieties  $\pi^{ur} : \mathcal{J}(\mathcal{X}^{ur}/B^{ur}) \rightarrow B^{ur}$ . The local period map induces an integrable system structure of this family.

The main result of the paper is to establish an isomorphism between the two semi-polarized integrable systems:

**Theorem 2.1.3.** *(Theorem 2.5.1) There is an isomorphism of smooth semi-polarized integrable systems*

$$\begin{array}{ccc}
 \mathcal{J}(\mathcal{X}^{ur}/B^{ur}) & \xrightarrow{\cong} & \mathcal{M}^\Delta(n, D)^{ur} \\
 & \searrow \pi^{ur} & \swarrow h_\Delta^{ur} \\
 & & B^{ur}
 \end{array} \tag{2.1.2}$$

The idea is to compare the admissible variations of  $\mathbb{Z}$ -mixed Hodge structures associated to the two semi-polarized integrable systems, by using the gluing techniques in [DDP07], [Bec20]. To complete the proof, we check that the comparison map intertwines the abstract Seiberg-Witten differentials on each side.

### Related work

The ideas of the spectral correspondence for unordered diagonally framed Higgs bundles and the infinitesimal study of their moduli spaces are drawn from [BLP19]. We follow their approach closely in Section 2.3.3. However, we provide an improvement of their result in order to show that  $\overline{\mathcal{M}}^\Delta(n, D)^{ur}$  and  $\mathcal{M}^\Delta(n, D)^{ur}$  are symplectic which was not proved before. We also focus more on the Hodge structures of the relevant Hitchin fibers to prove Theorem 2.1.3.

A general construction of the moduli space of unordered diagonally framed Higgs bundles  $\mathcal{M}^\Delta(n, D)$  comes from symplectic implosion [GJS02] associated to the level group action on  $\mathcal{M}_F(n, D)$ , viewed as the cotangent bundle of the moduli of framed

bundles [Mar94]. One can obtain the Hitchin fibration over the full base  $B$ , but it is a stratified space and very singular which makes it difficult to control. Indeed, as we only need the smooth part for our main result, we focus on Higgs fields that are diagonalizable over  $D$  throughout the paper.

Kontsevich-Soibelman proposed a different construction of the relevant Calabi-Yau integrable system as an affine conic bundle over a holomorphic symplectic surface containing a given spectral curve (see [KS14]). This can be done by blowing up intersections of spectral curves and the preimage of the divisor  $D$  in the total space of the twisted cotangent bundle  $K_{\Sigma}(D)$ . After removing the proper transform of the preimage, one gets the desired symplectic holomorphic surface. This model is birationally equivalent to the one we introduce in Section 4.

## Plan

We first recollect the basics of integrable systems and introduce the notion of a semi-polarized integrable system in Section 2. In Section 3, we study the integrable system structure of the moduli space of unordered diagonally framed Higgs bundle. Also, we give both the spectral and cameral descriptions for completeness. In Section 4, we construct the semi-polarized Calabi-Yau integrable systems by using the technique of elementary modification. It is then followed by a Hodge theoretic computation. Finally, in Section 5, we give a proof of Theorem 2.1.3.

## Notation

- $\Sigma$  - a non-singular curve of genus  $g$ .
- $D$  - an effective divisor of  $d$  reduced points.
- $\Sigma^\circ$  - the complement of the divisor  $D$  in  $\Sigma$ .
- $\mathcal{M}(n, D)$  - the moduli space of  $K_\Sigma(D)$ -twisted  $SL(n, \mathbb{C})$ -Higgs bundles.
- $\mathcal{M}_F(n, D)$  - the moduli space of framed Higgs bundles.
- $\overline{\mathcal{M}}^\Delta(n, D)$  - the moduli space of diagonally framed Higgs bundles.
- $\mathcal{M}^\Delta(n, D)$  - the moduli space of unordered diagonally framed Higgs bundles.
- $B = \bigoplus_{i=2}^n H^0(\Sigma, K_\Sigma(D)^{\otimes i})$  - the Hitchin base.
- $B^{ur} \subset B$  - the subset consists of smooth cameral curves which are unramified over  $D$  and have simple ramifications. Throughout the paper, we will always assume an element  $b \in B$  is sitting in  $B^{ur}$ .
- $\bar{p}_b : \bar{\Sigma}_b \rightarrow \Sigma$  - the spectral cover for  $b \in B$ .
- $\tilde{p}_b : \tilde{\Sigma}_b \rightarrow \Sigma$  - the cameral cover for  $b \in B$ .

## 2.2 Semi-polarized integrable systems

In this section, we recall the notion of a *semi-polarized integrable system*, originally introduced in [KS14]. This is a non-compact generalization of the notion of algebraic



integrable system [Hit87a] which provides a new way to view integrable systems in the Poisson setting. Similarly to the classical setting where algebraic integrable systems can be associated with variations of polarized weight one Hodge structures, we also have a Hodge-theoretic description of semi-polarized integrable systems. To make the paper self-contained, we shall begin reviewing basics of algebraic integrable systems by following [Bec20][Bec19].

### 2.2.1 Integrable systems and variations of Hodge structures

**Definition 2.2.1.** Let  $(M^{2n}, \omega)$  be a holomorphic symplectic manifold of dimension  $2n$  and  $B$  be a connected complex manifold of dimension  $n$ . A holomorphic map  $\pi : M \rightarrow B$  is called an *algebraic integrable system* if it satisfies the following conditions.

- (1)  $\pi$  is proper and surjective;
- (2) there exists a Zariski open dense subset  $B^\circ \subset B$  such that the restriction

$$\pi^\circ := \pi|_{M^\circ} : M^\circ \rightarrow B^\circ, \quad M^\circ := \pi^{-1}(B^\circ)$$

has smooth connected Lagrangian fibers and admits a relative polarization of index 0.

In particular, if  $B^\circ = B$ , then  $(M, \omega, \pi)$  is called a *smooth algebraic integrable system*.

The second condition that a generic fiber is Lagrangian puts rather restrictive constraints on the geometry of the fiber. To see this, first consider  $\ker(d\pi^\circ)$ , the sheaf of vector fields on  $M^\circ$  which are tangent to the fibers of  $\pi^\circ$ . Since the fibers of  $\pi^\circ$  are Lagrangians, the holomorphic symplectic form  $\omega$  induces an isomorphism  $\ker(d\pi^\circ) \cong (\pi^\circ)^*T^\vee B^\circ$  via  $v \mapsto \omega(v, -)$ . By taking pushforward to  $B^\circ$ , we have an isomorphism of coherent sheaves  $\pi_*^\circ \ker(d\pi^\circ) \cong \pi_*^\circ (\pi^\circ)^*T^\vee B^\circ$ . In fact, one can apply the projection formula and see  $\pi_*^\circ (\pi^\circ)^*T^\vee B^\circ \cong T^\vee B^\circ$  because the fibers of  $\pi^\circ$  are connected. Thus, the sheaf  $\pi_*^\circ \ker(d\pi^\circ)$  is isomorphic to  $T^\vee B^\circ$ , hence locally free. We denote it by  $\mathcal{V}$  and call it a vertical bundle of  $\pi^\circ$ .

Next, choose a sufficiently small open subset  $U \subset B^\circ$  and two local sections  $u, v : U \rightarrow \mathcal{V}$  such that they are Hamiltonian vector fields  $u = X_{(\pi^\circ)^*f}, v = X_{(\pi^\circ)^*g}$  for the functions  $f, g : U \rightarrow \mathbb{C}$ . As the fibers of  $\pi^\circ$  are Lagrangians, we have  $[u, v] = X_{\omega(u, v)} = 0$ . It implies that the Lie algebra  $(\mathcal{V}, [-, -])$  is abelian so that one can define a group action of  $\mathcal{V}$  on  $M^\circ$  via the fiberwise exponential map. In other words, the flows of the vector fields along the fibers of  $\pi^\circ$  corresponding to the sections of  $\mathcal{V}$  act on  $M^\circ$  while preserving the fibers of  $\pi^\circ$ .

The submanifold

$$\Gamma = \{v \in \mathcal{V} \mid \exists x \in M^\circ \text{ such that } v \cdot x = x\}$$

forms a full lattice in each fiber and induces a family of abelian varieties  $\mathcal{A}(\pi^\circ) := \mathcal{V}/\Gamma \rightarrow B^\circ$  which acts simply transitively on  $\pi^\circ : M^\circ \rightarrow B^\circ$ . Therefore, a generic fiber of  $\pi : M \rightarrow B$  is non-canonically isomorphic to an abelian variety.

From now on, we will focus on smooth integrable systems ( $B^\circ = B$ ). From the viewpoint of Hodge theory, a family of polarized abelian varieties can be obtained from a variation of weight 1 polarized  $\mathbb{Z}$ -Hodge structures  $\mathbf{V} = (V_{\mathbb{Z}}, F^\bullet V_{\mathcal{O}}, Q)$  over  $B$  where  $V_{\mathcal{O}} := V_{\mathbb{C}} \otimes \mathcal{O}_B$  and  $F^\bullet$  is the Hodge filtration. This is done by taking the relative Jacobian fibration so that we have the family

$$p : \mathcal{J}(\mathbf{V}) := \mathbf{Tot}(V_{\mathcal{O}}/(F^1 V_{\mathcal{O}} + V_{\mathbb{Z}})) \rightarrow B \quad (2.2.1)$$

whose vertical bundle is  $\mathcal{V} := V_{\mathcal{O}}/F^1 V_{\mathcal{O}} \rightarrow B$ .

A natural question is a condition for the family  $p : \mathcal{J}(\mathbf{V}) \rightarrow B$  being an integrable system. In other words, we need a symplectic form on  $\mathcal{J}(\mathbf{V})$  where fibers are connected Lagrangians. This can be achieved by the following theorem.

**Theorem 2.2.2.** *[Bec20] Let  $\mathbf{V} = (V_{\mathbb{Z}}, F^\bullet V_{\mathcal{O}}, Q)$  be a variation of weight 1 polarized  $\mathbb{Z}$ -Hodge structures over  $B$  and  $\nabla^{GM}$  be the Gauss-Manin connection on  $V_{\mathcal{O}}$ . Assume that there exists a global section  $\lambda \in H^0(B, V_{\mathcal{O}})$  such that*

$$\begin{aligned} \phi_\lambda : TB &\rightarrow F^1 V_{\mathcal{O}} \\ \mu &\mapsto \nabla_\mu^{GM} \lambda \end{aligned}$$

*is an isomorphism. Then the polarization  $Q$  induces a canonical symplectic form  $\omega_\lambda$  on  $\mathcal{J}(\mathbf{V})$  such that the induced zero section becomes Lagrangian. Moreover, the symplectic form is independent of the polarization  $Q$  up to symplectomorphisms.*

Consider the dual variation of Hodge structure of  $\mathbf{V}$ ,  $\mathbf{V}^\vee = \text{Hom}_{\text{VHS}}(\mathbf{V}, \mathbb{Z}_B)(-1)$  over  $B$ . The polarization  $Q$  identifies  $\mathcal{V} = V_{\mathcal{O}}/F^1 V_{\mathcal{O}}$  with  $F^1 V_{\mathcal{O}}^\vee$ . Consider the

compositions

$$\iota : \mathcal{V} \xrightarrow{\psi_Q} F^1 V_{\mathcal{O}}^{\vee} \xrightarrow{\phi_{\lambda}^{\vee}} T^{\vee} B. \quad (2.2.2)$$

where  $\psi_Q$  is the identification induced by the polarization  $Q$  and  $\phi_{\lambda}^{\vee}$  is dual of  $\phi_{\lambda}$ . Then the lattice  $V_{\mathbb{Z}}$  in  $\mathcal{V}$  embeds into  $T^{\vee} B$  as a Lagrangian submanifold. Therefore, we obtain a symplectic structure from the canonical one on  $T^{\vee} B$  by descending to  $\mathcal{J}(V_{\mathcal{O}}) \cong T^{\vee} B / \iota(V_{\mathbb{Z}})$ . We call such  $\lambda$  an *abstract Seiberg-Witten differential* [Bec20][Don97].

## 2.2.2 Semi-polarized integrable systems and variations of mixed Hodge structures

One can generalize the notion of an algebraic integrable system by allowing fibers to be non-proper. This is the main object of our study, first introduced in [KS14]. We recall the definition in a form convenient for our story.

**Definition 2.2.3.** Let  $(M^{2n+2k}, \omega)$  be a holomorphic symplectic manifold of dimension  $2n + 2k$  and  $B$  be a connected complex manifold of dimension  $n + k$ . A holomorphic map  $\pi : M \rightarrow B$  is called a *semi-polarized integrable system* if it satisfies the following conditions.

1.  $\pi$  is flat and surjective;
2. there exists a Zariski open dense subset  $B^{\circ} \subset B$  such that the restriction

$$\pi^{\circ} := \pi|_{M^{\circ}} : M^{\circ} \rightarrow B^{\circ}, \quad M^{\circ} := \pi^{-1}(B^{\circ})$$

has smooth connected Lagrangian fibers;

3. each fiber of  $\pi^\circ$  is a semi-abelian variety which is an extension of a  $n$ -dimensional polarized abelian variety by a  $k$ -dimensional affine torus.

In particular, if  $B^\circ = B$ , then  $(M, \omega, \pi)$  is called a *smooth* semi-polarized integrable system.

Similar to the classical case, the main example comes from an admissible variation of torsion-free  $\mathbb{Z}$ -mixed Hodge structures. Let  $\mathbf{V} = (V_{\mathbb{Z}}, W_{\bullet}V_{\mathbb{Z}}, F^{\bullet}V_{\mathcal{O}})$  be an admissible variation of  $\mathbb{Z}$ -mixed Hodge structures of type  $\{(-1, -1), (-1, 0), (0, -1)\}$  over  $B$  where  $V_{\mathcal{O}} := V_{\mathbb{C}} \otimes \mathcal{O}_B$  and  $\mathrm{Gr}_{-1}^W V_{\mathbb{C}}$  is polarizable. In other words, we have

- $0 = W_{-3} \subset W_{-2} \subset W_{-1} = V_{\mathbb{Z}}$
- $0 = F^1 \subset F^0 \subset F^{-1} = V_{\mathcal{O}}$

and can choose a relative polarization on  $\mathrm{Gr}_{-1}^W V_{\mathcal{O}}$ . Throughout this paper, we choose a semi-polarization on  $V_{\mathbb{Z}}$ , a degenerate bilinear form  $Q : V_{\mathbb{Z}} \times V_{\mathbb{Z}} \rightarrow \mathbb{Z}_B$  which yields the relative polarization on  $\mathrm{Gr}_{-1}^W V_{\mathcal{O}}$ . We call it a variation of semi-polarized  $\mathbb{Z}$ -mixed Hodge structures. Moreover, one can obtain a semi-abelian variety from a  $\mathbb{Z}$ -mixed Hodge structure of type  $\{(-1, -1), (-1, 0), (0, -1)\}$  by taking the Jacobian (see Appendix). Therefore, we have a family of semi-abelian varieties by taking the relative Jacobian fibration

$$p : \mathcal{J}(\mathbf{V}) := \mathbf{Tot}(V_{\mathcal{O}}/(F^0V_{\mathcal{O}} + V_{\mathbb{Z}})) \rightarrow B \quad (2.2.3)$$

with its compact quotient  $p_{\text{cpt}} : \mathcal{J}_{\text{cpt}}(\mathbf{V}) := \mathbf{Tot}(W_{-1}V_{\mathcal{O}}/(W_{-1}V_{\mathcal{O}} \cap F^0V_{\mathcal{O}} + V_{\mathbb{Z}})) \rightarrow B$ .

To define an abstract Seiberg-Witten differential, we consider the dual variation of  $\mathbb{Z}$ -mixed Hodge structures  $\mathbf{V}^{\vee} = (V_{\mathbb{Z}}^{\vee}, W_{\bullet}V_{\mathbb{Z}}^{\vee}, F^{\bullet}V_{\mathcal{O}}^{\vee}) := \text{Hom}_{\mathbf{VMHS}}(\mathbf{V}, \mathbb{Z}_B)$  of  $\mathbf{V}$ . Note that we don't take a Tate twist so that it is of type  $\{(0, 1), (1, 0), (1, 1)\}$ . Unlike the classical case, the Seiberg-Witten differential is defined as a global section of the dual vector bundle  $V_{\mathcal{O}}^{\vee}$ .

**Definition 2.2.4.** Let  $\mathbf{V} = (V_{\mathbb{Z}}, W_{\bullet}V_{\mathbb{Z}}, F^{\bullet}V_{\mathcal{O}}, Q)$  be an admissible variation of semi-polarized  $\mathbb{Z}$ -mixed Hodge structures of type  $\{(-1, -1), (-1, 0), (0, -1)\}$  over  $B$ , and  $\nabla^{GM}$  be the Gauss-Manin connection on  $V_{\mathcal{O}}$ . We define an abstract Seiberg-Witten differential as a global section of the dual bundle  $V_{\mathcal{O}}^{\vee}$ ,  $\lambda \in H^0(B, V_{\mathcal{O}}^{\vee})$ , such that the following morphism

$$\begin{aligned} \phi_{\lambda} : TB &\rightarrow F^1V_{\mathcal{O}}^{\vee} \\ \mu &\mapsto \nabla_{\mu}^{GM} \lambda \end{aligned} \tag{2.2.4}$$

is an isomorphism.

It is clear that the vertical bundle  $\mathcal{V}$  of  $\mathcal{J}(\mathbf{V}) \rightarrow B$  can be identified with  $(F^1V_{\mathcal{O}}^{\vee})^{\vee}$  via the canonical non-degenerate pairing,  $V_{\mathcal{O}}/F^0V_{\mathcal{O}} \otimes F^1V_{\mathcal{O}}^{\vee} \rightarrow \mathcal{O}_B$ . Consider the composition

$$\iota : \mathcal{V} \rightarrow (F^1V_{\mathcal{O}}^{\vee})^{\vee} \xrightarrow{\phi_{\lambda}^{\vee}} T^{\vee}B$$

under which the lattice  $V_{\mathbb{Z}} \subset \mathcal{V}$  embeds into  $T^{\vee}B$  as a Lagrangian submanifold. Similar to Theorem 2.2.2, we obtain a symplectic form from the canonical one on  $T^{\vee}B$  with Lagrangian condition on a generic fiber. Moreover, the total space  $\mathcal{J}(\mathbf{V})$

has a canonical Poisson structure associated to the given symplectic form. As the action of the affine torus on  $\mathcal{J}(\mathbf{V})$  is Hamiltonian, free and proper, the quotient space  $\mathcal{J}_{\text{cpt}}(\mathbf{V})$  is a Poisson manifold. Thus,  $\mathcal{J}_{\text{cpt}}(\mathbf{V})$  has a Poisson integrable system structure whose symplectic leaves are locally parametrized by  $\phi_\lambda^{-1}(\text{Gr}_2^W V_{\mathcal{O}}^\vee \cap F^1 V_{\mathcal{O}}^\vee)$  (see [KS14, Section 4.2] for more details). This proves the following proposition.

**Proposition 2.2.5.** *Let  $\mathbf{V} = (V_{\mathbb{Z}}, W_{\bullet} V_{\mathbb{Z}}, F^\bullet V_{\mathcal{O}}, Q)$  be an admissible variation of semi-polarized  $\mathbb{Z}$ -mixed Hodge structures of type  $\{(-1, -1), (-1, 0), (0, -1)\}$  over  $B$  and*

*$\lambda \in H^0(B, V_{\mathcal{O}}^\vee)$  be the Seiberg-Witten differential. Then, the relative Jacobian fibration*

$$p : \mathcal{J}(\mathbf{V}) := \mathbf{Tot}(V_{\mathcal{O}} / (F^0 V_{\mathcal{O}} + V_{\mathbb{Z}})) \rightarrow B \quad (2.2.5)$$

*forms a semi-polarized integrable system. In particular, the compact quotient  $\mathcal{J}_{\text{cpt}}(\mathbf{V}) \rightarrow B$  admits a Poisson integrable system structure.*

*Remark 16.* The reason we take a global section of the dual vector bundle in the definition of Seiberg-Witten differential is that, unlike the classical case, the semi-polarization  $Q$  does not induce the canonical identification between  $\mathbf{V}$  and  $\mathbf{V}^\vee$ . Moreover, this is also motivated by the geometric examples we will consider where  $V_{\mathbb{Z}}$  and  $V_{\mathbb{Z}}^\vee$  are torsion-free integral homology and cohomology of a non-singular quasi-projective variety, respectively.

*Remark 17.* In [KS14], Kontsevich and Soibelman introduce the notion of a central charge  $Z \in H^0(B, V_{\mathcal{O}}^\vee)$  which induces an local embedding of the base into  $V_{\mathcal{O}}^\vee$ . It

is equivalent to the data of an abstract Seiberg-Witten differential which suits our story better.

## 2.3 Moduli space of diagonally framed Higgs bundles

In this section, we will study the moduli space of (unordered) diagonally framed Higgs bundles and the associated Hitchin map as introduced in [BLP19]. In particular, we will give the spectral and Hodge theoretic description of the generic Hitchin fiber. Then we prove that it is a semi-polarized integrable system in two different ways: using deformation theory and using abstract Seiberg-Witten differentials. As mentioned in Section 1, parts of this section will follow the approach of [BLP19]. For basic properties of Hitchin systems and spectral covers, we refer to [DM96b].

### 2.3.1 The moduli space of (unordered) diagonally framed Higgs bundles

We fix  $\Sigma$  to be a smooth curve of genus  $g$ ,  $D$  a reduced divisor on  $\Sigma$  and  $\Sigma^\circ := \Sigma \setminus D$ .

**Definition 2.3.1.** A framed  $SL(n, \mathbb{C})$ -Higgs bundle on  $\Sigma$  is a triple  $(E, \theta, \delta)$ , where  $E$  is a vector bundle of rank  $n$  with trivial determinant,  $\delta : E_D \xrightarrow{\sim} \bigoplus_{i=1}^n \mathcal{O}_D$  is an isomorphism, i.e. a framing at  $D$ , and  $\theta \in \Gamma(\Sigma, \text{End}_0(E) \otimes K_\Sigma(D))$  is a traceless Higgs field.



A morphism between framed Higgs bundles  $(E, \theta, \delta)$  and  $(E', \theta', \delta')$  is a map  $f : E \rightarrow E'$  such that  $\delta \circ f|_D = \delta'$  and  $\theta' \circ f = (f \otimes Id_{K_\Sigma(D)}) \circ \theta$ .

*Remark 18.* A framed  $GL(n, \mathbb{C})$ -Higgs bundle and  $PGL(n, \mathbb{C})$ -Higgs bundle are defined in a similar way.

In order to discuss moduli spaces, we first define the stability conditions we will be using. We shall follow the definition of stability conditions in [BLP19]. Essentially, the stability condition for a framed Higgs bundle is just the stability condition for a  $K_\Sigma(D)$ -twisted Higgs bundle. More precisely, we say that a framed Higgs bundle  $(E, \theta, \delta)$  is stable (semistable respectively) if for every  $\theta$ -invariant proper subbundle  $F \subset E$ , that is,  $\theta(F) \subset F \otimes K(D)$ , we have  $\mu(F) < \mu(E)$  ( $\mu(F) \leq \mu(E)$  respectively). Here we write  $\mu$  for the slope  $\mu(E) = \deg(E)/\dim(E)$ .

The following lemma and the next corollary can be found in [BLP19, Lemma 2.3]. We record them here for future reference. Let  $(E, \theta)$  and  $(E', \theta')$  be  $K_\Sigma(D)$ -valued semistable Higgs bundles on  $\Sigma$  with  $\mu(E) = \mu(E')$ .

**Lemma 2.3.2.** *Let  $f : E \rightarrow E'$  be a  $\mathcal{O}_\Sigma$ -modules homomorphism such that*

$$(1) \theta' \circ f = (f \otimes Id_{K_\Sigma(D)}) \circ \theta,$$

$$(2) \text{ there is a point } x_0 \in \Sigma \text{ such that } f|_{x_0} = 0,$$

*then  $f$  vanishes identically.*

**Corollary 2.3.3.** *A semistable framed Higgs bundle admits no non-trivial automorphism.*

*Proof.* Indeed, suppose  $(E, \theta, \delta)$  admits an automorphism  $h$ , then the morphism  $h - Id_E$  vanishes on  $D$ . By the Lemma 2.3.2 above,  $h - Id_E$  vanishes identically or equivalently  $h = Id_E$ .  $\square$

We denote  $\mathfrak{g} := \mathfrak{sl}_n$  ( $\mathfrak{gl}_n$  respectively) and  $\mathfrak{g}_E := End_0(E)$  ( $End(E)$  respectively). For our discussion, we will only consider the case of  $\mathfrak{sl}_n$ . Let  $\mathfrak{t}$  be the vector subspace of diagonal traceless  $n \times n$  matrices and  $\mathfrak{q}$  be the orthogonal complement of  $\mathfrak{t}$  with respect to the Killing form, i.e. the vector subspace of  $n \times n$  matrices whose diagonal entries are all zero. We have  $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{q}$ . Given a framing  $\delta$  of  $E$ , we can define the  $\delta$ -restrictions to  $D$  as the compositions:

$$\begin{aligned} \mathfrak{g}_E &\twoheadrightarrow \mathfrak{g}_E \otimes \mathcal{O}_D \xrightarrow{ad_\delta} \mathfrak{g} \otimes \mathcal{O}_D \twoheadrightarrow \mathfrak{q} \otimes \mathcal{O}_D \\ \mathfrak{g}_E &\twoheadrightarrow \mathfrak{g}_E \otimes \mathcal{O}_D \xrightarrow{ad_\delta} \mathfrak{g} \otimes \mathcal{O}_D \twoheadrightarrow \mathfrak{t} \otimes \mathcal{O}_D \end{aligned}$$

where the maps  $\mathfrak{g} \otimes \mathcal{O}_D \twoheadrightarrow \mathfrak{q} \otimes \mathcal{O}_D$  and  $\mathfrak{g} \otimes \mathcal{O}_D \twoheadrightarrow \mathfrak{t} \otimes \mathcal{O}_D$  are given by the projections for the decomposition  $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{q}$ .

Given a framed bundle  $(E, \delta)$ , we define subsheaves  $\mathfrak{g}'_E, \mathfrak{g}''_E \subset \mathfrak{g}_E$  as the kernels

$$\begin{aligned} 0 &\rightarrow \mathfrak{g}'_E \rightarrow \mathfrak{g}_E \rightarrow \mathfrak{q}_D := i_*\mathfrak{q} \rightarrow 0 \\ 0 &\rightarrow \mathfrak{g}''_E \rightarrow \mathfrak{g}_E \rightarrow \mathfrak{t}_D := i_*\mathfrak{t} \rightarrow 0 \end{aligned}$$

where  $i : D \hookrightarrow \Sigma$  is the inclusion. In other words, a section of endomorphism in  $\mathfrak{g}'_E$  ( $\mathfrak{g}''_E$  respectively) restricted to  $p \in D$  is diagonal (anti-diagonal respectively) with respect to  $\delta$ .

**Definition 2.3.4.** We say that a framed Higgs bundle  $(E, \theta, \delta)$  is diagonally framed if  $\theta \in H^0(\Sigma, \mathfrak{g}'_E \otimes K_\Sigma(D)) \subset H^0(\Sigma, \mathfrak{g}_E \otimes K_\Sigma(D))$ .

By the results of [Sim94a][Sim94b] [BLP19, Section 2], it is shown that the moduli space of semistable framed  $SL(n, \mathbb{C})$ -Higgs bundles  $\mathcal{M}_F(n, D)$  exists as a fine moduli space that is a smooth irreducible quasi-projective variety. The moduli space we are interested in is the moduli space of semistable diagonally framed  $SL(n, \mathbb{C})$ -Higgs bundle, denoted by  $\overline{\mathcal{M}}^\Delta(n, D)$ . It is clear that  $\overline{\mathcal{M}}^\Delta(n, D)$  is a subvariety of  $\mathcal{M}_F(n, D)$ .

*Remark 19.* Unless mentioned otherwise, we will assume all diagonally framed Higgs bundles are semistable with structure group  $SL(n, \mathbb{C})$  throughout the paper.

For each  $p \in D$ , there is a natural  $S_n$ -action on  $\oplus_{i=1}^n \mathcal{O}_p$  by permuting the order of the components

$$\sigma : \oplus_{i=1}^n \mathcal{O}_p \xrightarrow{\sim} \oplus_{i=1}^n \mathcal{O}_p, \quad (s_1, \dots, s_n) \mapsto (s_{\sigma(1)}, \dots, s_{\sigma(n)}), \quad \text{where } \sigma \in S_n.$$

For each  $p \in D$ , this induces a  $S_n$ -action on the space of framings

$$\sigma \cdot \delta = \sigma \circ \delta : E|_p \rightarrow \oplus_{i=1}^n \mathcal{O}_p \xrightarrow{\sigma} \oplus_{i=1}^n \mathcal{O}_p.$$

Hence, the moduli spaces  $\overline{\mathcal{M}}^\Delta(n, D)$  and  $\mathcal{M}_F(n, D)$  admit a  $S_n^{|D|}$ -action: for  $\underline{\sigma} \in S_n^{|D|}$ ,

$$\underline{\sigma} : (E, \theta, \delta) \mapsto (E, \theta, \underline{\sigma} \cdot \delta), \quad \text{where } \underline{\sigma} \cdot \delta : E|_D \rightarrow \oplus_{i=1}^n \mathcal{O}_D \rightarrow \oplus_{i=1}^n \mathcal{O}_D.$$

Since the group is finite, we can consider the quotient  $\mathcal{M}_F(n, D)/(S_n^{|D|})$ . The effect of taking quotient is that, for a fixed Higgs bundle, framings that differ only in

reordering of components will be identified. More precisely, a morphism between unordered framed Higgs bundles  $(E, \theta, \delta)$  and  $(E', \theta', \delta')$  is a map  $f : E \rightarrow E'$  such that

$$\delta \circ f|_D = \underline{\sigma} \circ \delta' \text{ for some } \underline{\sigma} \in S_n^{|D|}, \quad \theta' \circ f = (f \otimes Id_{K_\Sigma(D)}) \circ \theta.$$

In other words,  $\mathcal{M}_F(n, D)/(S_n^{|D|})$  now parametrizes unordered framed Higgs bundles. However, this group action is not free. In order to get a free action by  $S_n^{|D|}$ , we will assume that the associated spectral curve is smooth and unramified over  $D$ , or equivalently, the residue of  $\theta$  at  $D$  has distinct eigenvalues. More precisely, we define  $B^{ur}$  to be the locus of smooth cameral curves (see Section 2.3.4) which are unramified over  $D$  and have simple ramifications. Of course, the associated spectral curve for  $b \in B^{ur}$  is automatically a smooth spectral curve that is unramified over  $D$ , and the necessity to work with smooth cameral curve with simple ramifications will be explained in Section 5. Moreover, we restrict to the subvariety  $\overline{\mathcal{M}}^\Delta(n, D)^{ur} := \overline{h}_\Delta^{-1}(B^{ur})$  where  $\overline{h}_\Delta$  denotes the composition  $\overline{\mathcal{M}}^\Delta(n, D) \hookrightarrow \mathcal{M}_F(n, D) \xrightarrow{f_1} \mathcal{M}(n, D) \xrightarrow{h} B$  and  $f_1$  denotes the forgetful map.

**Lemma 2.3.5.** *The  $S_n^{|D|}$ -action on  $\overline{\mathcal{M}}^\Delta(n, D)^{ur}$  is free.*

*Proof.* Consider  $(E, \theta, \delta) \in \overline{\mathcal{M}}^\Delta(n, D)^{ur}$  and suppose that there exists  $\underline{\sigma} \in S_n^{|D|}$  and an isomorphism  $\alpha : (E, \theta, \delta) \rightarrow (E, \theta, \underline{\sigma} \circ \delta)$ . The compatibility condition  $\delta \circ \alpha|_D = \underline{\sigma} \circ \delta$  implies that  $\delta \circ \alpha|_D \circ \delta^{-1} = \underline{\sigma}$ , while the compatibility condition  $\theta \circ \alpha = (\alpha \otimes Id_{K_\Sigma(D)}) \circ \theta$  restricted to  $D$  is equivalent to  $\theta_\delta \circ \underline{\sigma} = \underline{\sigma} \circ \theta_\delta$  where  $\theta_\delta := \delta^{-1}\theta|_D$ . The last relation  $\theta_\delta \circ \underline{\sigma} = \underline{\sigma} \circ \theta_\delta$  is clearly not possible as  $\theta_\delta$  is diagonal with distinct

eigenvalues at each  $p \in D$ . □

Since the  $S_n^{|D|}$ -action on  $\overline{\mathcal{M}}^\Delta(n, D)^{ur}$  is finite and free, we get a geometric quotient  $\mathcal{M}^\Delta(n, D)^{ur} := \overline{\mathcal{M}}^\Delta(n, D)^{ur}/(S_n^{|D|})$ . The variety  $\mathcal{M}^\Delta(n, D)^{ur}$  parametrizes unordered diagonally framed Higgs bundles.

Clearly, there is a morphism  $f_2 : \mathcal{M}^\Delta(n, D)^{ur} \rightarrow \mathcal{M}(n, D)^{ur} := h^{-1}(B^{ur})$  by forgetting the framings. For our purpose of proving Theorem 2.1.3, we will need to study the composition of the forgetful map  $f_2$  and the Hitchin map  $h$ , denoted by  $h_\Delta^{ur} : \mathcal{M}^\Delta(n, D)^{ur} \xrightarrow{f_2} \mathcal{M}(n, D)^{ur} \xrightarrow{h^{ur}} B^{ur}$ . We summarize the relation among the moduli spaces over  $B^{ur}$ :

$$\begin{array}{ccc}
 \overline{\mathcal{M}}^\Delta(n, D)^{ur} & \hookrightarrow & \mathcal{M}_F(n, D)^{ur} \\
 \downarrow q & & \downarrow f_1 \\
 \mathcal{M}^\Delta(n, D)^{ur} & \xrightarrow{f_2} & \mathcal{M}(n, D)^{ur} \\
 & \searrow h_\Delta^{ur} & \downarrow h^{ur} \\
 & & B^{ur}
 \end{array} \tag{2.3.1}$$

where  $\mathcal{M}_F(n, D)^{ur} := (h \circ f_1)^{-1}(B^{ur})$ .

### 2.3.2 Spectral correspondence

We explain the spectral correspondence for unordered diagonally framed Higgs bundles (see Proposition 2.3.8). After that, we describe the Hodge structures of a generic Hitchin fiber which will be used in the proof of the main theorem.

**Definition 2.3.6.** Let  $D$  be an effective reduced divisor on  $C$ . A  $D$ -framed line bundle on a curve  $C$  is a pair  $(L, \beta)$  where  $L$  is a line bundle and  $\beta : L|_D \xrightarrow{\sim} \mathcal{O}_D$  is

an isomorphism.

*Remark 20.* Unless mentioned otherwise, we will call  $(L, \beta)$  a framed line bundle whenever the divisor  $D$  is clear from the context.

**Proposition 2.3.7.** *Let  $C$  be a smooth curve and  $D$  a reduced divisor on  $C$ . Let  $C^\circ = C \setminus D$ ,  $j : C^\circ \rightarrow C$  and  $i : D \rightarrow C$  be the natural inclusions. The isomorphism classes of degree 0 framed line bundles on  $C$  are parametrized by the generalized Jacobian*

$$Jac(C^\circ) := \frac{H^0(C, \Omega_C(\log D))^\vee}{H_1(C^\circ, \mathbb{Z})}. \quad (2.3.2)$$

*Proof.* By duality, we can identify

$$Jac(C^\circ) = \frac{H^0(C, \Omega_C(\log D))^\vee}{H_1(C^\circ, \mathbb{Z})} \cong \frac{H^1(C, \mathcal{O}(-D))}{H^1(C, D, \mathbb{Z})}$$

Consider the exponential sequence

$$0 \rightarrow j_! \mathbb{Z} \rightarrow \mathcal{O}_C(-D) \xrightarrow{\exp(2\pi i(-))} \mathcal{O}_C^*(-D) \rightarrow 0$$

where  $\mathcal{O}_C^*(-D)$  is defined as the subsheaf of  $\mathcal{O}_C^*$  consisting of functions with value 1 on  $D$ . It induces a long exact sequence

$$\begin{aligned} \cdots \rightarrow H^1(C, j_! \mathbb{Z}) \cong H^1(C, D, \mathbb{Z}) \rightarrow H^1(C, \mathcal{O}_C(-D)) \rightarrow H^1(C, \mathcal{O}_C^*(-D)) \\ \xrightarrow{c_1} H^2(C, j_! \mathbb{Z}) \cong H^2(C, D, \mathbb{Z}) \rightarrow H^2(C, \mathcal{O}_C(-D)) \rightarrow H^2(C, \mathcal{O}_C^*(-D)) \rightarrow \cdots \end{aligned}$$

where the map  $c_1 : H^1(C, \mathcal{O}_C^*(-D)) \rightarrow H^2(C, j_! \mathbb{Z}) \cong H^2(C, D, \mathbb{Z}) \cong H^2(C, \mathbb{Z}) \cong \mathbb{Z}$  can be interpreted as the first Chern class map. The group  $H^1(C, \mathcal{O}_C^*(-D))$

naturally parametrizes all framed line bundles. Indeed, the sheaf  $\mathcal{O}_C^*(-D)$  sits in a short exact sequence

$$1 \rightarrow \mathcal{O}_C^*(-D) \rightarrow \mathcal{O}_C^* \rightarrow i_*\mathbb{C}^* \rightarrow 1$$

which induces a quasi-isomorphism  $\mathcal{O}_C^*(-D) \rightarrow F^\bullet := [\mathcal{O}_C^* \rightarrow i_*\mathbb{C}^*]$  and hence an isomorphism  $H^1(C, \mathcal{O}_C^*(-D)) \cong \mathbb{H}^1(C, F^\bullet)$ . By choosing a Čech covering  $(U_\alpha)$ , a 1-cocycle in  $Z^1(U_\alpha, F^\bullet)$  is a pair of  $f_{\alpha\beta} \in H^0(U_{\alpha\beta}, \mathcal{O}_C^*)$  and  $\eta_\alpha \in H^0(U_\alpha, i_*\mathbb{C}^*)$  such that  $\eta_\alpha/\eta_\beta = f_{\alpha\beta}|_D$ . The data  $f_{\alpha\beta}$  represents a line bundle. By assumption,  $f_{\alpha\beta}|_D = 1$  implies that  $\eta_\alpha|_D = \eta_\beta|_D \in \mathbb{C}^*$ . Since a framing of a line bundle at a point is equivalent to a choice of a non-zero complex number,  $(\eta_\alpha)$  defines a framing of the line bundle at  $D$ . In other words, the pair  $(f_{\alpha\beta}, \eta_\alpha)$  represents a framed line bundle, and a class in  $\mathbb{H}^1(C, F^\bullet)$  represents an isomorphism class of the framed line bundle.

In particular, we find that

$$Jac(C^\circ) \cong \frac{H^1(C, \mathcal{O}(-D))}{H^1(C, D, \mathbb{Z})} \cong \ker(c_1 : H^1(C, \mathcal{O}_C^*(-D)) \rightarrow \mathbb{Z})$$

which parametrizes degree 0 framed line bundles. □

We will apply the previous discussion to  $C = \bar{\Sigma}_b$ , a spectral curve of  $\Sigma$  corresponding to  $b \in B^{ur}$ .

*Remark 21.* Unless mentioned otherwise, we will omit the the subscript  $b$  in  $\bar{\Sigma}_b$  and  $\bar{\Sigma}_b^\circ$  in this section for convenience, as it is irrelevant to our discussion.

Since we are mainly interested in  $SL(n, \mathbb{C})$ -Higgs bundles, we will need to consider the Prym variety of the spectral cover  $\bar{p} : \bar{\Sigma} \rightarrow \Sigma$ . The norm map  $Nm : Jac(\bar{\Sigma}) \rightarrow Jac(\Sigma)$  induces a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathbb{C}^*)^{nd-1} & \longrightarrow & Jac(\bar{\Sigma}^\circ) & \longrightarrow & Jac(\bar{\Sigma}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow Nm^\circ & & \downarrow Nm & & \\ 0 & \longrightarrow & (\mathbb{C}^*)^{d-1} & \longrightarrow & Jac(\Sigma^\circ) & \longrightarrow & Jac(\Sigma) & \longrightarrow & 0 \end{array}$$

where  $d = |D|$  and  $Nm^\circ : Jac(\bar{\Sigma}^\circ) \rightarrow Jac(\Sigma^\circ)$  is defined by taking norms on line bundles and determinants on framings. Recall that  $Nm(L) = \det(\bar{p}_* L) \otimes \det(\bar{p}_* \mathcal{O}_{\bar{\Sigma}})^\vee$  and for a framed line bundle  $(L, \beta) \in Jac(\bar{\Sigma}^\circ)$ , the natural framing

$$\bar{p}_* L|_x \xrightarrow{\sim} \bigoplus_{y \in \bar{p}^{-1}(x)} L_y \xrightarrow[\beta]{\sim} \bigoplus_{y \in \bar{p}^{-1}(x)} \mathcal{O}_y$$

induces a framing on  $\det(\bar{p}_* L)|_x$  over each  $x \in D$ . Also, there is a natural framing on  $\det(\bar{p}_* \mathcal{O}_{\bar{\Sigma}})^\vee|_x$  induced from the identity  $Id : \mathcal{O}_{\bar{\Sigma}}|_{\bar{p}^{-1}(x)} \rightarrow \mathcal{O}_{\bar{\Sigma}}|_{\bar{p}^{-1}(x)}$ . Both framings determine a framing on  $Nm(L)$  and hence the map  $Nm^\circ$ .

By taking the kernel of this morphism, we get a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathbb{C}^*)^{(n-1)d} & \longrightarrow & Prym(\bar{\Sigma}^\circ/\Sigma^\circ) & \longrightarrow & Prym(\bar{\Sigma}/\Sigma) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (\mathbb{C}^*)^{nd-1} & \longrightarrow & Jac(\bar{\Sigma}^\circ) & \longrightarrow & Jac(\bar{\Sigma}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow Nm^\circ & & \downarrow Nm & & \\ 0 & \longrightarrow & (\mathbb{C}^*)^{d-1} & \longrightarrow & Jac(\Sigma^\circ) & \longrightarrow & Jac(\Sigma) & \longrightarrow & 0 \end{array} \quad (2.3.3)$$

where  $Prym(\bar{\Sigma}^\circ/\Sigma^\circ) := \ker(Nm^\circ)$ .

**Proposition 2.3.8.** *(Spectral correspondence [BLP19]).*

*For a fixed  $b \in B^{ur}$ , there is a one-to-one correspondence between degree zero framed*



line bundles on  $\bar{\Sigma}_b$  and unordered diagonally framed Higgs bundles on  $\Sigma$ . Moreover, the following results hold:

(1) The fiber  $h_{\Delta, GL(n)}^{-1}(b)$  is isomorphic to  $Jac(\bar{\Sigma}_b^\circ)$ ;

(2) The fiber  $h_{\Delta, SL(n)}^{-1}(b)$  is isomorphic to  $Prym(\bar{\Sigma}_b^\circ/\Sigma^\circ)$ .

*Proof.* For simplicity, we assume  $D = \{x\}$ ,  $\bar{D} = \bar{p}^{-1}(x)$  in this proof. Let  $L$  be a line bundle on  $\bar{\Sigma}_b$  and  $(E, \theta)$  a Higgs bundle on  $\Sigma$ . Recall that there is a bijection between line bundles on  $\bar{\Sigma}_b$  and Higgs bundles on  $\Sigma$

$$\begin{array}{ccc}
 & \bar{p}_* & \\
 & \curvearrowright & \\
 \text{Line bundle } L \text{ on } \bar{\Sigma}_b & & \text{Higgs bundle } (E, \theta) \text{ on } \Sigma \\
 & \curvearrowleft & \\
 & \text{coker}(\bar{p}^*\theta - \lambda Id) & 
 \end{array} \quad (2.3.4)$$

where  $\lambda$  denotes the tautological section of  $K_\Sigma(D)$ . It remains to verify the bijection on framings.

Pushing forward a  $\bar{D}$ -framed line bundle  $(L, \beta)$  gives an unordered framed Higgs bundle  $(\bar{p}_*L, \bar{p}_*\lambda, \delta)$  where

$$\delta : E|_x \xrightarrow{\sim} \bigoplus_{y \in \bar{p}^{-1}(x)} L_y \xrightarrow{\sim}_{\beta} \bigoplus_{y \in \bar{p}^{-1}(x)} \mathcal{O}_y$$

is well-defined as an unordered framing. With respect to the unordered framing, the Higgs field  $\bar{p}_*\lambda$  is diagonal as  $\theta|_x := \bar{p}_*\lambda$  defines multiplication by  $\lambda_i$  on each eigenline  $L_i$ .

Conversely, given an unordered diagonally framed Higgs bundle  $(E, \theta, \delta)$ , since we assume that  $\theta|_x$  has distinct eigenvalues, for each  $\lambda_i \in \bar{p}^{-1}(D)$ , the natural

composition

$$\ker(\bar{p}^*\theta - \lambda_i Id) \rightarrow E|_x \rightarrow \text{coker}(\bar{p}^*\theta - \lambda_i Id)$$

is an isomorphism. The assumption that  $\theta|_x$  is diagonal with respect to  $\delta$  implies that there is a component  $\mathcal{O}_x \xrightarrow{\alpha_i} \bigoplus_i^n \mathcal{O}_x$  such that

$$\begin{array}{ccc} \ker(\bar{p}^*\theta - \lambda_i Id) & \longrightarrow & E|_x \\ \cong \uparrow & & \cong \uparrow \\ \mathcal{O}_x & \xrightarrow{\alpha_i} & \bigoplus_i^n \mathcal{O}_x \end{array}$$

In particular, we get a framing  $\mathcal{O}_x \xrightarrow{\sim} \text{coker}(\bar{p}^*\theta - \lambda_i Id)$  for each  $\lambda_i$ .

Finally, claims (1), (2) follow from Proposition 2.3.7.  $\square$

Hodge structures Recall that since  $\bar{\Sigma}^\circ$  is non-compact,  $H^1(\bar{\Sigma}^\circ, \mathbb{Z})$  carries the  $\mathbb{Z}$ -mixed Hodge structure whose Hodge filtration is given by

$$F^0 = H^1(\bar{\Sigma}^\circ, \mathbb{C}) \supset F^1 = H^0(\bar{\Sigma}, \Omega_{\bar{\Sigma}}^1(\log D)) \supset F^2 = 0. \quad (2.3.5)$$

This induces the mixed Hodge structure on  $(H^1(\bar{\Sigma}^\circ, \mathbb{Z}))^\vee$  which is isomorphic to  $H_1(\bar{\Sigma}^\circ, \mathbb{Z})/(\text{torsion}) \cong H_1(\bar{\Sigma}^\circ, \mathbb{Z})$  by the universal coefficient theorem. Note that  $\text{Ext}(H_0(\bar{\Sigma}^\circ, \mathbb{Z}), \mathbb{Z}) = \text{Ext}(\mathbb{Z}, \mathbb{Z}) = 0$ , so there is no torsion in this case. The Hodge filtration of this dual mixed Hodge structure is given by

$$F^{-1} = H^1(\bar{\Sigma}^\circ, \mathbb{C})^\vee \supset F^0 = \left( \frac{H^1(\bar{\Sigma}^\circ, \mathbb{C})}{H^0(\bar{\Sigma}, \Omega_{\bar{\Sigma}}^1(\log D))} \right)^\vee \supset F^1 = 0$$

Note that the weight filtration on  $H_1(\bar{\Sigma}^\circ, \mathbb{Z})$  is

$$W_{-3} = 0 \subset W_{-2} = \mathbb{Z}^{nd-1} \subset W_{-1} = H_1(\bar{\Sigma}^\circ, \mathbb{Z}).$$

Thus we can define as in [Car79] the Jacobian of this Hodge structure as

$$J(H_1(\bar{\Sigma}^\circ, \mathbb{Z})) := \frac{H_1(\bar{\Sigma}^\circ, \mathbb{C})}{F^0 + H_1(\bar{\Sigma}^\circ, \mathbb{Z})} \quad (2.3.6)$$

**Lemma 2.3.9.** *There is an isomorphism between*

$$J(H_1(\bar{\Sigma}^\circ, \mathbb{Z})) \cong \text{Jac}(\bar{\Sigma}^\circ)$$

*Proof.* Indeed,

$$J(H_1(\bar{\Sigma}^\circ, \mathbb{Z})) = \frac{H_1(\bar{\Sigma}^\circ, \mathbb{C})}{F^0 + H_1(\bar{\Sigma}^\circ, \mathbb{Z})} = \frac{F^{-1}}{F^0 + H_1(\bar{\Sigma}^\circ, \mathbb{Z})} \cong \frac{H^0(\bar{\Sigma}, \Omega_{\bar{\Sigma}}^1(\log(D)))}{H_1(\bar{\Sigma}^\circ, \mathbb{Z})}.$$

□

Taking the first integral homology of every term in the diagram (2.3.3), we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathbb{Z})^{(n-1)d} & \longrightarrow & H_{\Delta, SL(n)} & \longrightarrow & H_{SL(n)} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (\mathbb{Z})^{nd-1} & \longrightarrow & H_1(\bar{\Sigma}^\circ, \mathbb{Z}) & \longrightarrow & H_1(\bar{\Sigma}, \mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow Nm^\circ & & \downarrow Nm & & \\ 0 & \longrightarrow & (\mathbb{Z})^{d-1} & \longrightarrow & H_1(\Sigma^\circ, \mathbb{Z}) & \longrightarrow & H_1(\Sigma, \mathbb{Z}) & \longrightarrow & 0 \end{array}$$

where we define

$$H_{\Delta, SL(n)} := H_1(\text{Prym}(\bar{\Sigma}^\circ/\Sigma^\circ), \mathbb{Z}) \cong \ker(Nm^\circ : H_1(\bar{\Sigma}^\circ, \mathbb{Z}) \rightarrow H_1(\Sigma^\circ, \mathbb{Z})), \quad (2.3.7)$$

$$H_{SL(n)} := H_1(\text{Prym}(\bar{\Sigma}/\Sigma), \mathbb{Z}) \cong \ker(Nm : H_1(\bar{\Sigma}, \mathbb{Z}) \rightarrow H_1(\Sigma, \mathbb{Z})). \quad (2.3.8)$$

Since the norm map is a morphism of mixed Hodge structures and taking the Jacobian is functorial, we immediately get the following result.

**Corollary 2.3.10.** *The Prym lattice  $H_{\Delta,SL(n)}$  is torsion free and admits the  $\mathbb{Z}$ -mixed Hodge structure of type  $\{(-1, -1), (-1, 0), (0, -1)\}$  induced by the map  $H_1(Nm^\circ) : H_1(\bar{\Sigma}^\circ, \mathbb{Z}) \rightarrow H_1(\Sigma^\circ, \mathbb{Z})$ . In particular, the Jacobian  $J(H_{\Delta,SL(n)})$  is isomorphic to  $\text{Prym}(\bar{\Sigma}^\circ/\Sigma^\circ)$ .*

*Remark 22.* Note that the mixed Hodge structure of the above type on  $H_{\Delta,SL(n)}$  is equivalent to the data of semi-abelian variety  $J(H_{\Delta,SL(n)})$ . A review is included in Appendix (2.6).

*Remark 23.* The Prym lattice  $H_{\Delta,SL(n)}$  admits a sheaf-theoretic formulation which will be needed in later sections. Consider the short exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow \bar{p}_*\underline{\mathbb{Z}} \xrightarrow{Tr} \underline{\mathbb{Z}} \rightarrow 0$$

The trace map  $\bar{p}_*\underline{\mathbb{Z}} \xrightarrow{Tr} \underline{\mathbb{Z}}$  is defined by

$$\bar{p}_*\underline{\mathbb{Z}}(U) = \underline{\mathbb{Z}}(\bar{p}^{-1}(U)) \rightarrow \underline{\mathbb{Z}}(U), \quad (s_1, \dots, s_n) \mapsto \sum_{i=1}^n s_i$$

if  $U$  is away from the ramification divisor, where  $s_i$  is a section on each component of  $\bar{p}^{-1}(U)$ .

This short exact sequence induces a long exact sequence:

$$\begin{aligned} 0 \rightarrow H_c^0(\Sigma, \mathcal{K}) \rightarrow H_c^0(\Sigma, \bar{p}_*\underline{\mathbb{Z}}_{\bar{\Sigma}}) &\cong H_c^0(\bar{\Sigma}, \mathbb{Z}) \rightarrow H_c^0(\Sigma, \mathbb{Z}) \\ \rightarrow H_c^1(\Sigma, \mathcal{K}) \rightarrow H_c^1(\Sigma, \bar{p}_*\underline{\mathbb{Z}}) &\cong H_c^1(\bar{\Sigma}, \mathbb{Z}) \rightarrow H_c^1(\Sigma, \mathbb{Z}) \end{aligned}$$

Since the cokernel of the map  $H_c^0(\bar{\Sigma}, \mathbb{Z}) \rightarrow H_c^0(\Sigma, \mathbb{Z})$  is torsion and  $H_c^1(\bar{\Sigma}, \mathbb{Z})$  is torsion-free, it follows that the maximal torsion free quotient  $H_c^1(\Sigma, \mathcal{K})_{\text{tf}} :=$

$H_c^1(\Sigma, \mathcal{K})/H_c^1(\Sigma, \mathcal{K})_{\text{tors}}$  can be identified as follows

$$H_c^1(\Sigma, \mathcal{K})_{\text{tf}} \cong \ker(H_c^1(\bar{\Sigma}, \mathbb{Z}) \rightarrow H_c^1(\Sigma, \mathbb{Z})) \cong \ker(H_1(\bar{\Sigma}, \mathbb{Z}) \rightarrow H_1(\Sigma, \mathbb{Z}))$$

by Poincaré duality. Note that we could have used cohomology instead of compactly supported cohomology since the curve  $\Sigma$  is compact, but the above argument also works for the noncompact curve  $\Sigma^\circ$ . In particular, the same argument implies that

$$H_c^1(\Sigma^\circ, \mathcal{K}|_{\Sigma^\circ})_{\text{tf}} \cong \ker(H_1(\bar{\Sigma}^\circ, \mathbb{Z}) \rightarrow H_1(\Sigma^\circ, \mathbb{Z})) \cong H_{\Delta, SL(n)}. \quad (2.3.9)$$

Note that  $H_c^1(\Sigma^\circ, \mathcal{K}|_{\Sigma^\circ})_{\text{tf}}$  can also be written as  $H^1(\Sigma, D, \mathcal{K})_{\text{tf}}$ .

### 2.3.3 Deformation theory

In this section, we show that the moduli space of diagonally framed Higgs bundle  $\overline{\mathcal{M}}^\Delta(n, D)$  is symplectic. For the following discussion in this section, we fix a diagonally framed Higgs bundle  $(E, \theta, \delta)$ . Recall that we assume  $b \in B^{ur}$  which means that the associated cameral curve is smooth, unramified over  $D$ , and has simple ramification. In particular, the residue of  $\theta$  at  $D$  is diagonal with distinct eigenvalues with respect to the framing  $\delta$ .

Denote by  $\Sigma[\epsilon]$  the fiber product  $\Sigma \times \text{Spec}(\mathbb{C}[\epsilon])$ .

**Definition 2.3.11.** An infinitesimal deformation of diagonally framed Higgs bundle is a triple  $(E_\epsilon, \theta_\epsilon, \delta_\epsilon)$  such that

- $E_\epsilon$  is a locally free sheaf on  $\Sigma[\epsilon]$ ,

- $\theta_\epsilon \in H^0(\Sigma[\epsilon], \mathfrak{g}'_{E_\epsilon, D[\epsilon]} \otimes p_\Sigma^* K_\Sigma(D))$ ,
- $\delta_\epsilon : E|_{D[\epsilon]} \rightarrow \mathcal{O}_{D[\epsilon]}^{\oplus n}$  is an isomorphism,
- $(E_\epsilon, \theta_\epsilon, \delta_\epsilon)|_{D \times 0} \cong (E, \theta, \delta)$ ,

where as before  $\mathfrak{g}'_{E_\epsilon, D[\epsilon]}$  is defined as the kernel of the map  $\mathfrak{g}_{E_\epsilon} \rightarrow \mathfrak{q} \otimes \mathcal{O}_{D[\epsilon]}$  induced by  $\delta_\epsilon$  and  $p_\Sigma : \Sigma[\epsilon] \rightarrow \Sigma$  denotes the natural projection.

**Proposition 2.3.12.** *The space of infinitesimal deformations of a diagonally framed Higgs bundle  $(E, \theta, \delta)$  is canonically isomorphic to  $\mathbb{H}^1(C^\bullet)$  where*

$$C^\bullet : C^0 = \mathfrak{g}_E(-D) \xrightarrow{[\cdot, \theta]} C^1 = \mathfrak{g}'_E \otimes K_\Sigma(D) \quad (2.3.10)$$

*Proof.* Recall that [Mar94] the space of infinitesimal deformation of a framed Higgs bundles  $(E, \theta, \delta)$  is canonically isomorphic to  $\mathbb{H}^1(C_F^\bullet)$  where

$$C_F^\bullet : C_F^0 = \mathfrak{g}_E(-D) \xrightarrow{[\cdot, \theta]} C_F^1 = \mathfrak{g}_E \otimes K_\Sigma(D). \quad (2.3.11)$$

Choose a Čech cover  $U := (U_\alpha)$  of  $\Sigma$  which induces cover  $U[\epsilon] := (U_\alpha[\epsilon])$  of  $\Sigma[\epsilon]$ . Imposing further the condition that the Higgs bundles are diagonally framed implies that  $\theta \in \mathfrak{g}'_E \otimes K_\Sigma(D) \subset \mathfrak{g}_E \otimes K_\Sigma(D)$ . Suppose that a 1-cocycle  $(\dot{f}_{\alpha\beta}, \dot{\varphi}_\alpha)$  in  $Z^1(U[\epsilon], C_F^\bullet)$  represents an infinitesimal deformation of  $(E, \theta, \delta)$  as framed Higgs bundles where  $\dot{f}_{\alpha\beta} \in H^0(U_{\alpha\beta}[\epsilon], \mathfrak{g}_E(-D))$  and  $\dot{\varphi}_\alpha \in H^0(U_\alpha[\epsilon], \mathfrak{g}_E \otimes p_\Sigma^* K_\Sigma(D))$ . Then  $(\dot{f}_{\alpha\beta}, \dot{\varphi}_\alpha)$  is an infinitesimal deformation of  $(E, \theta, \delta)$  as diagonally framed Higgs bundles if and only if  $\dot{\varphi}_\alpha \in H^0(U_\alpha[\epsilon], \mathfrak{g}'_E \otimes p_\Sigma^* K_\Sigma(D))$ . Hence, it follows that  $\mathbb{H}^1(C^\bullet)$  parametrizes the infinitesimal deformations of diagonally framed Higgs bundles.

□

Recall that the Serre duality says that  $\mathbb{H}^1(C^\bullet) \xrightarrow{\sim} (\mathbb{H}^1(\check{C}^\bullet))^\vee$  where

$$\check{C}^\bullet : (\mathfrak{g}'_E)^\vee \otimes \mathcal{O}_\Sigma(-D) \xrightarrow{[-, \theta]^\dagger} \mathfrak{g}_E^\vee \otimes K_\Sigma(D) \quad (2.3.12)$$

is the Serre dual to  $C^\bullet$ . Combining the Serre duality isomorphism with the isomorphism in the next proposition, we get a non-degenerate skew-symmetric pairing on  $\mathbb{H}^1(C^\bullet)$ .

**Proposition 2.3.13.** *There is a canonical isomorphism*

$$\mathbb{H}^1(\check{C}^\bullet) \cong \mathbb{H}^1(C^\bullet). \quad (2.3.13)$$

*Proof.* We consider an auxiliary complex<sup>2</sup>

$$C_1^\bullet : \mathfrak{g}''_E \rightarrow \mathfrak{g}_E \otimes K_\Sigma(D)$$

and show that this is isomorphic to both  $C^\bullet$  and  $\check{C}^\bullet$ .

First, consider the morphism of complexes  $t : C^\bullet \rightarrow C_1^\bullet$ :

$$\begin{array}{ccccc} C^\bullet & \mathfrak{g}_E \otimes \mathcal{O}_\Sigma(-D) & \longrightarrow & \mathfrak{g}'_E \otimes K_\Sigma(D) & \\ \downarrow t & \downarrow t_0 & & \downarrow t_1 & \\ C_1^\bullet & \mathfrak{g}''_E & \longrightarrow & \mathfrak{g}_E \otimes K_\Sigma(D) & \end{array}$$

Both  $t_0$  and  $t_1$  are injective. The diagram clearly commutes away from  $D$ , hence commutes everywhere. In particular, around  $D$ , choose an open subset  $U$  that trivializes all the bundles, we see that the maps become the natural maps

$$\begin{array}{ccc} \mathfrak{t}(-D) \oplus \mathfrak{q}(-D) & \longrightarrow & (\mathfrak{t} \oplus \mathfrak{q}(-D)) \otimes K_\Sigma(D)|_U \\ \downarrow t_0|_U & & \downarrow t_1|_U \\ \mathfrak{t}(-D) \oplus \mathfrak{q} & \longrightarrow & (\mathfrak{t} \oplus \mathfrak{q}) \otimes K_\Sigma(D)|_U \end{array}$$

---

<sup>2</sup>The complex  $C_1^\bullet$  here coincides with the complex " $\mathcal{C}_\bullet^\Delta$ " that is defined in [BLP19, Section 5].

where we abuse notations by denoting  $\mathfrak{t}$  and  $\mathfrak{q}$  the trivial bundles with fibers  $\mathfrak{t}$  and  $\mathfrak{q}$ , respectively. The cokernel of  $t$  is

$$\text{coker}(t) : \mathfrak{q}_D \xrightarrow{[\cdot, \theta]|_D} \mathfrak{q}_D \otimes K_\Sigma(D)$$

**Lemma 2.3.14.**

$$\mathbb{H}^i(\text{coker}(t)) = 0, \text{ for all } i.$$

*Proof.* Since the complex is supported at  $D$ , it reduces to a complex of  $\mathbb{C}$ -vector spaces. Assume  $D$  consists of a single point for simplicity. The complex reduces to

$$\mathfrak{q} \xrightarrow{[\cdot, \theta]|_D} \mathfrak{q}.$$

Recall our assumption that the associated spectral curve is unramified over  $D$ . The restriction  $\theta|_D$  of the Higgs field to  $D$  is a diagonal matrix with distinct eigenvalues with respect to  $\delta$ . In particular,  $\theta|_D$  is regular and semisimple, so its centralizer  $Z_{\mathfrak{g}}(\theta|_D) = \{x \in \mathfrak{g} | [x, \theta|_D] = 0\}$  is a Cartan subalgebra and coincides with  $\mathfrak{t}$ . Since  $\ker([\cdot, \theta]|_D : \mathfrak{g} \rightarrow \mathfrak{g}) = Z_{\mathfrak{g}}(\theta|_D) = \mathfrak{t}$  which intersects  $\mathfrak{q}$  trivially, it follows that the restricted map  $([\cdot, \theta]|_D)|_{\mathfrak{q}} : \mathfrak{q} \rightarrow \mathfrak{q}$  is an isomorphism. Hence, all the cohomologies of the complex  $\text{coker}(t)$  must be zero.

□

The long exact sequence induced by  $0 \rightarrow C^\bullet \rightarrow C_1^\bullet \rightarrow \text{coker}(t) \rightarrow 0$  is:

$$\begin{aligned} 0 \rightarrow \mathbb{H}^0(C^\bullet) \rightarrow \mathbb{H}^0(C_1^\bullet) \rightarrow \mathbb{H}^0(\text{coker}(t)) = 0 \\ \rightarrow \mathbb{H}^1(C^\bullet) \rightarrow \mathbb{H}^1(C_1^\bullet) \rightarrow \mathbb{H}^1(\text{coker}(t)) = 0 \rightarrow \dots \end{aligned}$$



and hence  $\mathbb{H}^0(C^\bullet) \cong \mathbb{H}^0(C_1^\bullet)$  and  $\mathbb{H}^1(C^\bullet) \cong \mathbb{H}^1(C_1^\bullet)$ .

Finally, we claim that there is an isomorphism of complexes  $C_1^\bullet \cong \check{C}^\bullet$

$$\begin{array}{ccc} C_1^\bullet & \mathfrak{g}_E'' \longrightarrow & \mathfrak{g}_E \otimes K_\Sigma(D) \\ \downarrow \cong & \downarrow r_0 & \downarrow r_1 \\ \check{C}^\bullet & (\mathfrak{g}'_E)^\vee \otimes \mathcal{O}_\Sigma(-D) \longrightarrow & (\mathfrak{g}_E)^\vee \otimes K_\Sigma(D) \end{array} \quad (2.3.14)$$

The map  $r_0$  is defined as follows. Consider the composition of morphisms

$$\mathfrak{g}_E'' \hookrightarrow \mathfrak{g}_E \xrightarrow{\sim} \mathfrak{g}_E^\vee \hookrightarrow (\mathfrak{g}'_E)^\vee \rightarrow (\mathfrak{g}'_E)^\vee \otimes \mathcal{O}_D. \quad (2.3.15)$$

where the isomorphism  $\mathfrak{g}_E \rightarrow \mathfrak{g}_E^\vee$  is given by the trace pairing. If we know that this composition is zero, then we will get a map

$$r_0 : \mathfrak{g}_E'' \rightarrow \ker((\mathfrak{g}'_E)^\vee \rightarrow (\mathfrak{g}'_E)^\vee \otimes \mathcal{O}_D) = (\mathfrak{g}'_E)^\vee \otimes \mathcal{O}_\Sigma(-D).$$

Away from  $D$ , the map (2.3.15) is clearly zero. Around  $D$ , we can find an open subset  $U$  such that each sheaf in the composition is trivial, then

$$\begin{array}{ccccccccc} \mathfrak{g}_E''|_U & \longrightarrow & \mathfrak{g}_E|_U & \longrightarrow & \mathfrak{g}_E^\vee|_U & \longrightarrow & (\mathfrak{g}'_E)^\vee|_U & \longrightarrow & ((\mathfrak{g}'_E)^\vee \otimes \mathcal{O}_D)|_U \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \mathfrak{t}(-D) \oplus \mathfrak{q} & \longrightarrow & \mathfrak{t} \oplus \mathfrak{q} & \longrightarrow & \mathfrak{t}^\vee \oplus \mathfrak{q}^\vee & \longrightarrow & \mathfrak{t}^\vee \oplus \mathfrak{q}^\vee(D) & \longrightarrow & (\mathfrak{t}^\vee \otimes \mathcal{O}_D) \oplus (\mathfrak{q}^\vee \otimes \mathcal{O}_D(D)) \end{array}$$

Each component of the bottom row clearly composes to zero, hence the whole composition is zero. Locally over  $U$ , the map  $r_0 : \mathfrak{g}_E'' \rightarrow (\mathfrak{g}'_E)^\vee \otimes \mathcal{O}_\Sigma(-D)$  is induced by the trace pairing:  $\mathfrak{t} \xrightarrow{\sim} \mathfrak{t}^\vee$  and  $\mathfrak{q} \xrightarrow{\sim} \mathfrak{q}^\vee$ ,

$$r_0|_U : \mathfrak{g}_E''|_U \cong \mathfrak{t}(-D) \oplus \mathfrak{q} \xrightarrow{\sim} \mathfrak{t}^\vee(-D) \oplus \mathfrak{q}^\vee \cong (\mathfrak{g}'_E)^\vee \otimes \mathcal{O}_\Sigma(-D)|_U$$

Since  $r_0$  is clearly an isomorphism away from  $D$ , it follows that  $r_0$  is an isomorphism.

The commutativity can be argued in the same way. Again, the diagram commutes away from  $D$ . Around  $D$ , the bundles trivialize and we get the diagram

$$\begin{array}{ccc} \mathfrak{t}(-D) \oplus \mathfrak{q} & \longrightarrow & \mathfrak{t} \oplus \mathfrak{q} \otimes K_{\Sigma}(D)|_U \\ \downarrow & & \downarrow \\ \mathfrak{t}^{\vee}(-D) \oplus \mathfrak{q}^{\vee} & \longrightarrow & \mathfrak{t}^{\vee} \oplus \mathfrak{q}^{\vee} \otimes K_{\Sigma}(D)|_U \end{array}$$

which commutes on the nose.

All of this together gives

$$\mathbb{H}^1(C^{\bullet}) \cong \mathbb{H}^1(C_1^{\bullet}) \cong \mathbb{H}^1(\check{C}^{\bullet}). \quad (2.3.16)$$

as claimed.  $\square$

Let  $\omega_{\Delta} : \mathbb{H}^1(C^{\bullet}) \times \mathbb{H}^1(C^{\bullet}) \rightarrow \mathbb{C}$  be the non-degenerate skew-symmetric pairing induced by Serre duality and the isomorphism in Proposition 2.3.13.

**Proposition 2.3.15.** *The nondegenerate 2-form  $\omega_{\Delta}$  is closed.*

*Proof.* Consider the following inclusion of complexes  $C^{\bullet} \xrightarrow{u} C_F^{\bullet}$ :

$$\begin{array}{ccc} C^{\bullet} & \mathfrak{g}_E \otimes \mathcal{O}_{\Sigma}(-D) \longrightarrow \mathfrak{g}'_E \otimes K_{\Sigma}(D) \\ \downarrow u & \downarrow u_0 & \downarrow u_1 \\ C_F^{\bullet} & \mathfrak{g}_E \otimes \mathcal{O}_{\Sigma}(-D) \longrightarrow \mathfrak{g}_E \otimes K_{\Sigma}(D) \end{array}$$

where as before  $C_F^{\bullet}$  is the complex whose first hypercohomology controls the deformations of the framed Higgs bundle  $(E, \theta, \delta)$ . By the same argument as in Proposition 2.3.13, since  $u_0$  is isomorphic and  $u_1$  is injective whose cokernel has zero-dimensional support and concentrated in degree one, we have an injection

$$i : \mathbb{H}^1(C^{\bullet}) \hookrightarrow \mathbb{H}^1(C_F^{\bullet}).$$

Note that Serre duality induces a non-degenerate bilinear pairing on  $\mathbb{H}^1(C_F^\bullet)$  which corresponds to the well-known symplectic form  $\omega_F$  on  $\mathcal{M}_F(n, D)$ , see [BLP19]. We claim that the pairing  $\omega_\Delta$  is obtained by restricting  $\omega_F$  to  $\mathbb{H}^1(C^\bullet) \subset \mathbb{H}^1(C_F^\bullet)$ . In other words, the corresponding 2-form on  $\overline{\mathcal{M}}^\Delta(n, D)$  is obtained by pulling back the symplectic form  $\omega_F$  on  $\mathcal{M}_F(n, D)$ . It then follows that  $\omega_F$  is closed as well.

Our claim is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccccccc}
\mathbb{H}^1(C_F^\bullet) & \longrightarrow & \mathbb{H}^1(\check{C}_F^\bullet)^\vee & \longrightarrow & \mathbb{H}^1(C_F^\bullet)^\vee & & \\
\uparrow i & & \uparrow & & \downarrow i^\vee & & \\
\mathbb{H}^1(C^\bullet) & \longrightarrow & \mathbb{H}^1(\check{C}^\bullet)^\vee & \longrightarrow & \mathbb{H}^1(C_1^\bullet)^\vee & \longrightarrow & \mathbb{H}^1(C^\bullet)^\vee
\end{array}$$

The left square diagram commutes by the functoriality of Serre duality. Then it remains to check the commutativity of the right square diagram. This follows from the commutativity of the diagram of complexes:

$$\begin{array}{ccccc}
\check{C}_F^\bullet & \longleftarrow & & C_F^\bullet & \\
\downarrow & & & \uparrow & \\
\check{C}^\bullet & \longleftarrow & C_1^\bullet & \longleftarrow & C^\bullet
\end{array}$$

Away from  $D$ , the diagram clearly commutes. Around  $D$ , we again trivialize the bundles and the diagram looks like

$$\begin{array}{ccccc}
\boxed{\mathfrak{t}^\vee(-D) \oplus \mathfrak{q}^\vee(-D) \rightarrow (\mathfrak{t}^\vee \oplus \mathfrak{q}^\vee)L} & \longleftarrow & & \boxed{\mathfrak{t}(-D) \oplus \mathfrak{q}(-D) \rightarrow (\mathfrak{t} \oplus \mathfrak{q})L} & \\
\downarrow & & & \uparrow & \\
\boxed{\mathfrak{t}^\vee(-D) \oplus \mathfrak{q}^\vee \rightarrow (\mathfrak{t}^\vee \oplus \mathfrak{q}^\vee)L} & \longleftarrow & \boxed{\mathfrak{t}(-D) \oplus \mathfrak{q} \rightarrow (\mathfrak{t} \oplus \mathfrak{q})L} & \longleftarrow & \boxed{\mathfrak{t}(-D) \oplus \mathfrak{q}(-D) \rightarrow (\mathfrak{t} \oplus \mathfrak{q}(-D))L}
\end{array}$$

where we denote by  $L$  the operation " $\otimes K_\Sigma(D)$ ".

□

**Proposition 2.3.16.**

(1)  $\mathbb{H}^0(C^\bullet) = \mathbb{H}^2(C^\bullet) = 0$ . In particular, the deformations of a diagonally framed Higgs bundle  $(E, \theta, \delta)$  are unobstructed.

(2)  $\dim(\mathbb{H}^1(C^\bullet)) = (n^2 - 1)(2g - 2 + d) + (n - 1)d$ .

*Proof.* (1) Since morphisms between diagonally framed Higgs bundles are in particular morphisms between framed Higgs bundles, automorphisms of diagonally framed Higgs bundles are the same as automorphisms as framed Higgs bundles. So Corollary 2.3.3 implies that the diagonally framed Higgs bundles are rigid. Hence,  $\mathbb{H}^0(C^\bullet) = 0$ .

On the other hand, again by Serre duality,

$$\mathbb{H}^2(C^\bullet) \cong (\mathbb{H}^0(\check{C}^\bullet))^\vee \cong (\mathbb{H}^0(C_1^\bullet))^\vee$$

where the second isomorphism comes from the isomorphism of the complex (2.3.14).

Finally, recall that from the long exact sequence above, we have that  $\mathbb{H}^0(C^\bullet) \cong \mathbb{H}^0(C_1^\bullet)$  which vanishes as we just proved, hence  $\mathbb{H}^2(C^\bullet) = 0$ .

(2) By the definition of  $\mathfrak{g}'_E$ , we have a short exact sequence

$$0 \rightarrow \mathfrak{g}_E \otimes \mathcal{O}_\Sigma(-D) \rightarrow \mathfrak{g}'_E \rightarrow i_* \mathfrak{t} \rightarrow 0$$

and thus

$$\begin{aligned} \chi(\mathfrak{g}'_E \otimes K_\Sigma(D)) &= \chi(\mathfrak{g}'_E) + (n^2 - 1) \deg(K_\Sigma(D)) \\ &= \chi(\mathfrak{t} \otimes \mathcal{O}_D) + \chi(\mathfrak{g}_E(-D)) + (n^2 - 1) \deg(K_\Sigma(D)) \\ &= (n - 1)d + \chi(\mathfrak{g}_E) + (n^2 - 1) \deg(\mathcal{O}_\Sigma(-D)) + (n^2 - 1) \deg(K_\Sigma(D)) \\ &= \chi(\mathfrak{g}_E) + (n - 1)d + (n^2 - 1)(2g - 2). \end{aligned}$$

By (1),  $\chi(C^\bullet) = \mathbb{H}^1(C^\bullet)$ , so

$$\begin{aligned}
\mathbb{H}^1(C^\bullet) &= \chi(\mathfrak{g}'_E \otimes K_\Sigma(D)) - \chi(\mathfrak{g}_E(-D)) \\
&= \chi(\mathfrak{g}_E) + (n-1)d + (n^2-1)(2g-2) - \chi(\mathfrak{g}_E) + (n^2-1)d \\
&= (n^2-1)(2g-2+d) + (n-1)d.
\end{aligned}$$

□

*Remark 24.* In the case of  $\mathfrak{g} = \mathfrak{gl}_n$ , a similar computation shows that

$$\mathbb{H}^1(C^\bullet) = n^2(2g-2+d) + nd.$$

*Remark 25.* A direct computation by applying the Riemann-Roch theorem shows that

$$\begin{aligned}
\dim(B) &= \sum_{i=2}^n h^0(\Sigma, (K(D))^{\otimes i}) = (2g-2+d) \left( \frac{n(n+1)}{2} - 1 \right) + (n-1)(1-g) \\
&= \frac{1}{2} \left( (n^2-1)(2g-2+d) + (n-1)d \right) \\
&= \frac{1}{2} \dim(\mathbb{H}^1(C^\bullet)).
\end{aligned}$$

**Proposition 2.3.17.** *The open subset  $\overline{\mathcal{M}}^\Delta(n, D)^{ur}$  of the moduli space  $\overline{\mathcal{M}}^\Delta(n, D)$  is a smooth quasi-projective variety of dimension  $(n^2-1)(2g-2+d) + (n-1)d$ . The tangent space  $T_{[(E, \theta, \delta)]} \overline{\mathcal{M}}^\Delta(n, D)^{ur}$  is canonically isomorphic to  $\mathbb{H}^1(C^\bullet)$ . Moreover,  $\overline{\mathcal{M}}^\Delta(n, D)^{ur}$  admits a symplectic form  $\omega_\Delta$  which is the restriction of the symplectic form  $\omega_F$  on  $\mathcal{M}_F(n, D)$ .*

*Proof.* All the claims follow immediately from Proposition 2.3.12, 2.3.15 and 2.3.16.

The argument to show that  $\omega_\Delta$  is a restriction of  $\omega_F$  is contained in the proof of Proposition 2.3.15.  $\square$

**Proposition 2.3.18.** *The fiber of the map  $\bar{h}_\Delta : \overline{\mathcal{M}}^\Delta(n, D)^{ur} \rightarrow B^{ur}$  is Lagrangian with respect to  $\omega_\Delta$ .*

*Proof.* Denote by  $(h_1, \dots, h_l) := h \circ f_1 : \mathcal{M}_F(n, D) \rightarrow \mathcal{M}(n, D) \rightarrow \mathbb{C}^l = B$  the composition of the forgetful map and the Hitchin map. According to [BLP19, Theorem 5.1], the functions  $h_i$  Poisson-commute. Since the symplectic form  $\omega_\Delta$  on  $\overline{\mathcal{M}}^\Delta(n, D)^{ur}$  is the restriction of the symplectic form  $\omega_F$  on  $\mathcal{M}_F(n, D)$ , the functions  $h_i$  Poisson-commute as well when restricted to  $\overline{\mathcal{M}}^\Delta(n, D)^{ur}$ .

Since the dimension of the fiber  $h_\Delta^{-1}(b)$  for  $b \in B^{ur}$  is exactly  $\frac{1}{2}\dim(\overline{\mathcal{M}}^\Delta(n, D)^{ur})$  by Remark (25), it suffices to show that  $\omega_\Delta$  restricted to  $\bar{h}_\Delta^{-1}(b)$  vanishes to prove our claim. This follows from Poisson-commutativity of  $(h_i)|_{\overline{\mathcal{M}}^\Delta(n, D)^{ur}}$ .  $\square$

**Proposition 2.3.19.** *The tangent space  $T_{[(E, \theta, \delta)]}\mathcal{M}^\Delta(n, D)^{ur}$  is canonically isomorphic to  $\mathbb{H}^1(C^\bullet)$ . Moreover, the symplectic form  $\omega_\Delta$  on  $\overline{\mathcal{M}}^\Delta(n, D)^{ur}$  is invariant under the  $S_n^{|D|}$ -action. In particular,  $\omega_\Delta$  descends to a symplectic form  $\omega'_\Delta$  on  $\mathcal{M}^\Delta(n, D)^{ur}$ .*

*Proof.* In the proof of Proposition 2.3.12, given an infinitesimal deformation  $(E_\epsilon, \theta_\epsilon, \delta_\epsilon)$ , the assignment of a 1-cocycle  $(\dot{f}_{\alpha\beta}, \dot{\varphi}_\alpha)$  in  $\mathbb{H}^1(C^\bullet)$  is independent of the reordering

of components. That means we have the following commutative diagram

$$\begin{array}{ccc}
T_{[(E,\theta,\delta)]}\overline{\mathcal{M}}^\Delta(n,D)^{ur} & \xrightarrow{\sim} & \mathbb{H}^1(C^\bullet) \\
\downarrow d\sigma & & \downarrow = \\
T_{[(E,\theta,\underline{\sigma},\delta)]}\overline{\mathcal{M}}^\Delta(n,D)^{ur} & \xrightarrow{\sim} & \mathbb{H}^1(C^\bullet)
\end{array}$$

for  $\underline{\sigma} \in S_n^{|D|}$ . The differential of the quotient map

$$dq : T_{[(E,\theta,\delta)]}\overline{\mathcal{M}}^\Delta(n,D)^{ur} \rightarrow T_{[(E,\theta,S_n^{|D|},\delta)]}\mathcal{M}^\Delta(n,D)^{ur}$$

is an isomorphism. Hence, the canonical identification  $T_{[(E,\theta,\delta)]}\overline{\mathcal{M}}^\Delta(n,D)^{ur} \cong \mathbb{H}^1(C^\bullet)$  descends to the tangent space  $T_{[(E,\theta,S_n^{|D|},\delta)]}\mathcal{M}^\Delta(n,D)^{ur}$  via  $dq$  and yields a canonical isomorphism  $T_{[(E,\theta,S_n^{|D|},\delta)]}\mathcal{M}^\Delta(n,D)^{ur} \cong \mathbb{H}^1(C^\bullet)$ . Since the group action of  $S_n^{|D|}$  is trivial on  $\mathbb{H}^1(C^\bullet)$ , the symplectic form  $\omega_\Delta$  on  $\overline{\mathcal{M}}^\Delta(n,D)^{ur}$  is invariant under  $S_n^{|D|}$ .  $\square$

**Corollary 2.3.20.** *The map  $h_\Delta^{ur} : \mathcal{M}^\Delta(n,D)^{ur} \rightarrow B^{ur}$  forms a semi-polarized integrable system.*

*Proof.* By the spectral correspondence proved in Proposition 2.3.8, the fibers are semi-abelian varieties. Since  $\omega'_\Delta$  descends from the symplectic form  $\omega_\Delta$ , it follows immediately from Proposition 2.3.18 that the fiber of the map  $h_\Delta^{ur} : \mathcal{M}^\Delta(n,D)^{ur} \rightarrow B^{ur}$  is Lagrangian with respect to  $\omega'_\Delta$ .  $\square$

*Remark 26.* For a fixed  $b \in B^{ur}$ , the fiber  $(h_\Delta^{ur})^{-1}(b)$  is a semi-abelian variety  $\text{Prym}(\overline{\Sigma}_b^\circ, \Sigma^\circ)$  which admits a  $(\mathbb{C}^*)^{(n-1)d}$ -action. This group action can be seen by viewing  $\text{Prym}(\overline{\Sigma}_b^\circ, \Sigma^\circ)$  as parametrizing framed line bundles on  $\overline{\Sigma}_b$  which correspond

to unordered diagonally framed  $SL(n, \mathbb{C})$ -Higgs bundles under spectral correspondence. Then  $(\mathbb{C}^*)^{(n-1)d}$  acts simply transitively on the space of framings over  $D$  for each fixed line bundle, and the quotient map is equivalent to the natural map  $Prym(\bar{\Sigma}_b^\circ, \Sigma^\circ) \rightarrow Prym(\bar{\Sigma}_b, \Sigma)$  of forgetting the framings. Applying this fiberwise quotient by  $(\mathbb{C}^*)^{(n-1)d}$  to the fibration  $\mathcal{M}^\Delta(n, D)^{ur} \rightarrow B^{ur}$ , we see that the quotient map is precisely the forgetful map  $f_1 : \mathcal{M}^\Delta(n, D)^{ur} \rightarrow \mathcal{M}(n, D)^{ur}$ . Thus, this provides a geometric interpretation of the fact that the Poisson integrable system  $\mathcal{M}(n, D)^{ur} \rightarrow B^{ur}$  is realized as the fiberwise compact quotient of the semi-polarized integrable system  $\mathcal{M}^\Delta(n, D) \rightarrow B^{ur}$  as discussed in Section 2.2.2.

### 2.3.4 Cameral description

Although the spectral curve description is more intuitive and straightforward, it only works for classical groups. To describe the general fiber of Hitchin system for any reductive group  $G$  as well as prove DDP-type results, it is more natural to use the cameral curve description and *generalized Prym variety*. In this section, we focus on the extension of classical results in our case (A-type). We refer to [DG02][DP12] for more basics and details about the cameral description.

In this section, we use general notation from algebraic group theory with an eye towards a generalization of the previous arguments to any reductive group  $G$  (see Remark 27).

As the Hitchin base  $B$  can be considered as the space of sections of  $K_\Sigma(D) \otimes \mathfrak{t}/W$ ,



we have the following commutative diagram

$$\begin{array}{ccc}
\tilde{\Sigma} & \longrightarrow & \tilde{U} := \mathbf{Tot}(K_{\Sigma}(D) \otimes \mathfrak{t}) \\
\downarrow \tilde{p} & & \downarrow \phi \\
\Sigma \times B & \xrightarrow{ev} & U := \mathbf{Tot}(K_{\Sigma}(D) \otimes \mathfrak{t}/W)
\end{array} \tag{2.3.17}$$

where  $\tilde{\Sigma}$  is the *universal cameral curve* of  $\Sigma$ . By projecting to  $B$ , we have a family of cameral curves  $\tilde{\Sigma} \rightarrow B$  whose fiber is a  $W$ -Galois cover of the base curve  $\Sigma$ . An interesting observation is that in the meromorphic case, one can consider the universal cameral pair  $(\tilde{\Sigma}, \tilde{D} := \tilde{p}^{-1}(D \times B))$ , which allows us to extend the notion of generalized Prym variety [Don93]. Let's recall the definition of the generalized Prym variety. For a generic  $b \in B$ , we define a sheaf of abelian groups  $\mathcal{T}_b$  by

$$\mathcal{T}_b(U) := \{t \in \tilde{p}_{b*}(\Lambda_G \otimes \mathcal{O}_{\tilde{\Sigma}}^*)^W(U) \mid \alpha(t)|_{M^\alpha} = +1 \quad \forall \alpha \in R(G)\}$$

where  $R(G)$  is a root system and  $\Lambda_G$  is the cocharacter lattice and  $M^\alpha$  is the ramification locus of  $\tilde{p}_b : \tilde{\Sigma}_b \rightarrow \Sigma$  fixed by the reflection  $S_2 \in W$  corresponding to  $\alpha$ . We define the generalized Prym variety of  $\tilde{\Sigma}_b$  over  $\Sigma$  as the sheaf cohomology  $H^1(\Sigma, \mathcal{T}_b)$ .

**Theorem 2.3.21** ([DG02][HHP10]). *For  $b \in B^\circ$ , the fiber  $h^{-1}(b)$  in the meromorphic Hitchin system is isomorphic to the generalized Prym variety  $H^1(\Sigma, \mathcal{T}_b)$ :*

$$h^{-1}(b) \cong H^1(\Sigma, \mathcal{T}_b)$$

where  $B^\circ$  is the locus of smooth cameral curves with simple ramifications.

Let  $i_D : D \hookrightarrow \Sigma \hookleftarrow \Sigma \setminus D : j_D$  be inclusions. Associated to the cameral pair  $(\tilde{\Sigma}_b, \tilde{D}_b)$ , one can extend the generalized Prym variety to  $H^1(\Sigma, j_{D!} j_D^* \mathcal{T}_b)$  which is isomorphic to  $h_{\Delta}^{-1}(b)$ .

**Proposition 2.3.22.** *For  $b \in B^{ur}$ , the unordered diagonally framed Hitchin fiber  $(h_\Delta)^{-1}(b)$  is isomorphic to  $H^1(\Sigma, j_{D!}j_D^*\mathcal{T}_b)$ . In particular, it is a semi-abelian variety which corresponds to the  $\mathbb{Z}$ -mixed Hodge structure*

$$(H^1(\Sigma, D, (\tilde{p}_{b*}\Lambda_{SL(n)})^W)_{tf}, H^1(\tilde{\Sigma}_b, \tilde{D}_b, \mathfrak{t})^W)$$

whose weight and Hodge filtration are induced from Hodge structure of  $H^1(\tilde{\Sigma}_b, \mathfrak{t})^W$  and  $H^0(\tilde{D}_b, \mathfrak{t})^W$ .

*Proof.* For completeness, we use the spectral description of unordered diagonally framed Higgs bundles. The fiber  $(h_\Delta)^{-1}(b)$  is isomorphic to the Jacobian of the relative  $\mathbb{Z}$ -mixed Hodge structure on  $H^1(\Sigma, D, \mathcal{K}_b)$  where  $\mathcal{K}_b := \ker(\text{Tr} : \bar{p}_{b*}\mathbb{Z} \rightarrow \mathbb{Z})$  (see Remark 23). To relate with the cameral description, we consider an isomorphism of sheaves,

$$(\tilde{p}_{b*}\Lambda_{SL(n)})^W \cong \mathcal{K}_b \tag{2.3.18}$$

proved in Lemma 2.5.4. It induces the isomorphism of  $\mathbb{Z}$ -mixed Hodge structures on the relative sheaf cohomology:

$$H^1(\Sigma, D, (\tilde{p}_{b*}\Lambda_{SL(n)})^W) \cong H^1(\Sigma, D, \mathcal{K}_b).$$

They agree on the torsion free part, hence we obtain the result by complexifying the lattice. □

*Remark 27.* In the forthcoming paper [LL], we develop the theory of diagonally framed Higgs bundle for arbitrary reductive group  $G$  and its abelianization by following [DG02]. In summary, note that an additional data of diagonal framing

amounts to specifying  $W$ -equivariant section of  $T$ -bundle at  $D$ . This can be formulated as  $H^0(D_b, \mathcal{T}) = H^0(D, (\tilde{p}_{b*} \Lambda_{SL(n)})^W \otimes \mathbb{C}^*)$  modulo the action of the center  $Z(G)$ . Moreover, the distinguished triangle in the constructible derived category of  $\Sigma$ ,  $D_c^b(\Sigma)$

$$j_{b!} j_b^* \rightarrow id \rightarrow i_{b*} i_b^*$$

induces the long exact sequence as follows

$$H^0(\Sigma, j_{b!} j_b^* \mathcal{T}_b) \rightarrow H^0(\Sigma, \mathcal{T}_b) \xrightarrow{i_D^*} H^0(D, \mathcal{T}_b) \rightarrow H^1(\Sigma, j_{b!} j_b^* \mathcal{T}_b) \rightarrow H^1(\Sigma, \mathcal{T}_b) \rightarrow 0. \quad (2.3.19)$$

Here,  $H^0(\Sigma, \mathcal{T}_b)$  is the space of  $W$ -equivariant maps,  $\text{Hom}_W(\tilde{\Sigma}_b, T)$ , which takes values 1 on  $M_{\Sigma_b}^\alpha$  for every root  $\alpha$ . Note that

$$Z(G) = \{t \in T^W \mid \alpha(t) = 1 \text{ for all } \alpha \in R(G)\}.$$

Therefore, the cokernel of  $i_D^* : H^0(\Sigma, \mathcal{T}_b) \rightarrow H^0(D, \mathcal{T}_b)$  can be identified with  $T^{|D|}/Z(G)$ , a level subgroup. Clearly this is a copy of  $\mathbb{C}^*$ 's, so we have the semi-abelian variety  $H^1(\Sigma, j_{b!} j_b^* \mathcal{T}_b)$  as an extension of  $H^1(\Sigma, \mathcal{T}_b)$  by  $T^{|D|}/Z(G)$ . In order to get the complete description of the general fiber, we should verify the precise torsor structure. For type A, this can be done easily with the help of spectral description.

### 2.3.5 Abstract Seiberg-Witten differential

Using the cameral description, one can define an abstract Seiberg-Witten differential. Note that in the classical case, the Seiberg-Witten differential is a holomorphic

one-form which is obtained by the tautological section of the pullback of  $K_\Sigma$  under  $\mathbf{Tot}(K_\Sigma) \rightarrow \Sigma$ . Similarly, in the meromorphic case, the tautological section of the pullback of  $K_\Sigma(D)$  under  $\mathbf{Tot}(K_\Sigma(D)) \rightarrow \Sigma$  gives the logarithmic 1-form  $\theta$ . For each  $b \in B$ , we define the Seiberg-Witten differential to be the restriction

$$\lambda_{\Delta,b} := \theta|_{\tilde{\Sigma}_b} \in H^0(\tilde{\Sigma}_b, \mathfrak{t} \otimes \Omega_{\tilde{\Sigma}_b}(\log \tilde{D}_b))^W = F^1 H^1(\tilde{\Sigma}_b \setminus \tilde{D}_b, \mathfrak{t})^W$$

where  $(H^1(\Sigma \setminus D, (\tilde{p}_{b,*} \Lambda_{SL(n)})^W), H^1(\tilde{\Sigma}_b \setminus \tilde{D}_b, \mathfrak{t})^W)$  is the  $\mathbb{Z}$ -mixed Hodge structure associated to the cameral pair  $(\tilde{\Sigma}_b, \tilde{p}_b^{-1}(D))$ . This is the dual to the one we described earlier and is of type  $\{(0, 1), (1, 0), (1, 1)\}$ . For simplicity, let's denote it by  $V_b^\vee = H^1(\tilde{\Sigma}_b \setminus \tilde{D}_b, \mathfrak{t})^W$ .

Note that having a variation of  $\mathbb{Z}$ -mixed Hodge structures over  $B$  corresponds to having the classifying map to mixed period domain;  $\Phi : B \rightarrow \mathcal{D}/\Gamma$ . It admits a holomorphic lift [SSU87]  $\tilde{\Phi} : B \rightarrow \mathcal{D}$  which factors through relative Kodaira-Spencer map  $\kappa : T_{B,b} \rightarrow H^1(\tilde{\Sigma}_b, T_{\tilde{\Sigma}_b}(-\log \tilde{D}_b))$

$$\begin{array}{ccc} T_{B,b} & \xrightarrow{d\tilde{\Phi}} & T_{\mathcal{D}, \tilde{\Phi}(b)} \\ & \searrow \kappa & \nearrow m^\vee \\ & H^1(\tilde{\Sigma}_b, T_{\tilde{\Sigma}_b}(-\log \tilde{D}_b)) & \end{array} \quad (2.3.20)$$

where  $m^\vee : H^0(\tilde{\Sigma}_b, \mathfrak{t} \otimes \Omega_{\tilde{\Sigma}_b}(\log \tilde{D}_b))^W \otimes H^1(\tilde{\Sigma}_b, T_{\tilde{\Sigma}_b}(-\log \tilde{D}_b)) \rightarrow H^1(\tilde{\Sigma}_b, \mathcal{O}_{\tilde{D}_b})^W$  is the logarithmic contraction.

**Proposition 2.3.23.** *For each  $b \in B^{ur}$ , applying the Gauss-Manin connection to*

$\lambda_\Delta$ , one can obtain an isomorphism

$$\begin{aligned} \nabla^{GM} : T_b B &\xrightarrow{\cong} F^1 V_b^\vee \\ \mu &\mapsto \nabla_\mu^{GM}(\lambda_{\Delta,b}) \end{aligned}$$

*Proof.* The idea is to follow the local computation as in the original proof of the classical case [HHP10]. We can apply the same arguments because we restrict to cameral covers with no ramification over the divisors. First, given  $\mu \in T_b B$ , one can compute  $\nabla_\mu^{GM}$  by using the above diagram (2.3.20). Let's consider

$$\begin{aligned} C_\mu &:= pr \circ \nabla_\mu^{GM} : F^1 \rightarrow V_b^\vee \rightarrow V_b^\vee / F^1 \\ C_\mu(\alpha) &= \alpha \cup \kappa(\mu). \end{aligned}$$

One can see that  $\nabla_\mu^{GM}(\lambda_\Delta) \in F^1$  for all  $\mu \in T_b B$  by noticing that  $C_\mu(\lambda_{\Delta,b}) = 0$ . This also follows from Griffiths' transversality of variation of mixed Hodge structures [PS08, Section 14.4]. On the other hand, using the isomorphism  $T_b B \cong F^1 V_b^\vee \cong H^0(\tilde{\Sigma}_b, \mathfrak{t} \otimes K_{\tilde{\Sigma}_b}(\tilde{D}_b))^W$ , we can assign a logarithmic one form  $\alpha_\mu$  to every  $\mu \in T_b B$ . From the definition of the Seiberg-Witten differential form, it now follows that

$$\nabla_\mu^{GM}(\lambda_{\Delta,b}) = \alpha_\mu$$

for all  $\mu \in T_b B$ . □

## 2.4 Calabi-Yau integrable systems

### 2.4.1 Construction

In this section, we shall generalize Smith's elementary modification idea [Smi15] to construct a (semi-polarized) Calabi-Yau integrable system.

First, we describe the construction of a family of Calabi-Yau threefolds. Let  $V := \mathbf{Tot}(K_\Sigma(D) \oplus (K_\Sigma(D))^{n-1} \oplus K_\Sigma(D))$  and consider the short exact sequence

$$0 \rightarrow \mathcal{O}_\Sigma(-D) \xrightarrow{\alpha} \mathcal{O}_\Sigma \rightarrow i_{D*}\mathcal{O}_D \rightarrow 0.$$

Suppose  $u$  is a local frame of  $\mathcal{O}_\Sigma(-D)$ . In terms of a local coordinate  $z$  around a point of  $D$  where  $z = 0$ ,  $\alpha(u)$  is represented by  $f \cdot u$  where  $f$  is a locally defined function that vanishes at  $z = 0$ . We define an elementary modification  $W$  of  $V$  along the first component:

$$W := \mathbf{Tot}(K_\Sigma(D-D) \oplus (K_\Sigma(D))^{n-1} \oplus K_\Sigma(D)) \rightarrow \mathbf{Tot}(K_\Sigma(D) \oplus (K_\Sigma(D))^{n-1} \oplus K_\Sigma(D))$$

and denote the projection map by  $\pi_W : W \rightarrow \Sigma$ .

For  $b = (b_2(z), \dots, b_n(z)) \in B = \bigoplus_{i=2}^n H^0(\Sigma, K_\Sigma(D))^{\otimes i}$ , we define the threefold  $X_b$  as the zero locus of a section in  $\Gamma(W, \pi_W^* K_\Sigma(D)^{\otimes n})$ :

$$X_b := \{\alpha(x)y - s^n - \pi_W^* b_2(z)s^{n-2} - \dots - \pi_W^* b_n(z) = 0\} \subset W \quad (2.4.1)$$

with the projection  $\pi_b : X_b \rightarrow \Sigma$ . Here we denote by  $x, y$  and  $s$  the tautological sections of  $K_\Sigma, (K_\Sigma(D))^{n-1}$  and  $K_\Sigma(D)$ , respectively. Note that each term in the

equation (2.4.1) is a section of  $\pi_W^* K_\Sigma(D)^{\otimes n}$ . More explicitly, we have

$$\begin{aligned} x &\in \Gamma(W, \pi_W^* K_\Sigma), & \alpha(x) &\in \Gamma(W, \pi_W^* K_\Sigma(D)), & y &\in \Gamma(W, \pi_W^* (K_\Sigma(D))^{n-1}) \\ s &\in \Gamma(W, \pi_W^* K_\Sigma(D)), & \pi^* b_i &\in \Gamma(W, \pi_W^* (K_\Sigma(D))^i) \end{aligned}$$

This construction gives rise to a family of quasi-projective threefolds

$$pr_2 \circ \pi : \mathcal{X} \rightarrow B.$$

Next, we show that the threefold  $X_b$  is indeed a non-singular Calabi-Yau threefold.

**Proposition 2.4.1.** *The threefold  $X_b$  has trivial canonical bundle.*

*Proof.* By the adjunction formula,

$$K_{X_b} = K_W \otimes \pi_W^* (K_\Sigma(D))^{\otimes n}|_{X_b}.$$

where  $\pi_W : W \rightarrow \Sigma$ . Note that

$$K_W = \pi_W^* \det(W^\vee) \otimes \pi_W^* K_\Sigma \cong \pi_W^* (K_\Sigma^{-n-1}(-nD)) \otimes \pi_W^* K_\Sigma \cong \pi_W^* (K_\Sigma^{-n}(-nD)).$$

So it follows that

$$K_{X_b} = \pi_W^* (K_\Sigma^{-n}(-nD)) \otimes \pi_W^* (K_\Sigma(D))^{\otimes n}|_{X_b} \cong \mathcal{O}_{X_b}.$$

□

**Proposition 2.4.2.** *For each  $b \in B^{ur}$ , the threefold  $X_b$  is non-singular.*

*Proof.* This is a local statement, so we can restrict to neighbourhoods in  $\Sigma$ . Around a point of  $D$  with local coordinate  $z$ , the local model of  $X_b$  is

$$\{f(z)xy - s^n - \tilde{b}_2(z)s^{n-2} - \dots - \tilde{b}_n(z) = 0\} \subset \mathbb{C}_{(x,y,s)}^3 \times \mathbb{C}_z,$$

where  $\tilde{b}_i$  are now functions of  $z$ , and  $f(z)$  is function with zero only at  $z = 0$ . We check smoothness by examining the Jacobian criterion. The equation

$$\frac{\partial}{\partial s} (s^n - \tilde{b}_2(z)s^{n-2} - \dots - \tilde{b}_n(z)) = 0$$

implies that, for each  $z$ , the equation  $s^n - \tilde{b}_2(z)s^{n-2} - \dots - \tilde{b}_n(z) = 0$  must have repeated solutions, this happens only when  $z$  is at a critical value. The remaining equations in the Jacobian criterion are

$$f(z)y = 0, \quad f(z)x = 0, \quad f'(z)xy + \frac{\partial}{\partial z} (s^n - \tilde{b}_2(z)s^{n-2} - \dots - \tilde{b}_n(z)) = 0$$

When  $x = y = 0$ , the equation  $\frac{\partial}{\partial z} (s^n - \tilde{b}_2(z)s^{n-2} - \dots - \tilde{b}_n(z)) = 0$  has no solution since we assume that the spectral curve associated to  $b$  is smooth. Hence, it must be the case  $f(z) = 0$  or equivalently  $z = 0$ . However, since we assume  $b \in B^{ur}$ , this cannot happen and  $X_b$  is non-singular around  $D$ .

Away from  $D$ , a similar argument shows that the threefold is non-singular over the local neighbourhood. Hence,  $X_b$  is non-singular everywhere.

□

Again, by examining the defining equation (2.4.1), we can list the types of fibers of the map  $\pi_b : X_b \rightarrow \Sigma$ :



- For  $p \in D$  with coordinate  $z = 0$ , the fiber is defined by the equation  $s^n - \tilde{b}_2(z)s^{n-2} - \dots - \tilde{b}_n(z) = 0$ , i.e. disjoint union of  $n$  copies of  $\mathbb{C}^2$ .
- For a critical value  $p$  of  $\pi_b$ , the fiber is defined by  $xy - \prod_{i=1}^m (s - s_i)^{k_i}$  where  $\sum_{i=1}^m k_i = n$  ( $m < n$ ). Hence, the fiber is a singular surface with  $A_{k_i-1}$ -singularity at  $s_i$ .
- For  $p$  away from  $D$  and the discriminant locus of  $\pi_b$ , the fiber is defined by  $(xy - s^n) - \tilde{b}_2(z)s^{n-2} - \dots - \tilde{b}_n(z) = 0$  and smooth, so it is isomorphic to a smooth fiber of the universal unfolding of  $A_{n-1}$ -singularity  $\mathbb{C}^2/\mathbb{Z}_n$ .

Next, we study the mixed Hodge structure of  $X_b$ . Let's denote the complement of  $\pi_b^{-1}(D)$  by  $X_b^\circ$ . The long exact sequence of compactly supported cohomologies associated to the pair  $(X_b, \pi_b^{-1}(D))$  is

$$\cdots \rightarrow H_c^2(\pi_b^{-1}(D), \mathbb{Z}) \rightarrow H_c^3(X_b^\circ, \mathbb{Z}) \rightarrow H_c^3(X_b, \mathbb{Z}) \rightarrow H_c^3(\pi_b^{-1}(D), \mathbb{Z}) \rightarrow \cdots$$

As  $H_c^2(\pi_b^{-1}(D), \mathbb{Z}) = H_c^3(\pi_b^{-1}(D), \mathbb{Z}) = 0$ , we have an isomorphism of  $\mathbb{Z}$ -mixed Hodge structures

$$H_c^3(X_b, \mathbb{Z}) \cong H_c^3(X_b^\circ, \mathbb{Z}) \tag{2.4.2}$$

Moreover, the Leray spectral sequence for compactly supported cohomology associated to  $\pi_b^\circ := \pi_b|_{X_b^\circ} : X_b^\circ \rightarrow \Sigma^\circ$  implies

$$H_c^3(X_b^\circ, \mathbb{Z}) \cong H_c^1(\Sigma^\circ, R^2\pi_b^\circ\mathbb{Z}) \tag{2.4.3}$$

because the (compactly supported) cohomology of a fiber is non-trivial only for degree 0 and 2 [DDP07, Lemma 3.1]. As the Leray spectral sequence is compatible

with mixed Hodge structures ([Ara05], [De 09, Corollary 2.10]), it is enough to compute the Hodge type of  $H_c^1(\Sigma^\circ, R^2\pi_{b!}^\circ\mathbb{Z})$ . For this, we need to deal with critical values of  $\pi_b^\circ$  and the monodromy around  $D$ . First, note that the critical values do not determine Hodge type. This is because a local system  $\mathcal{F}$  having finite monodromies  $M$  (only around the critical values) can be trivialized by pulling back to an order  $|M|$  covering  $\pi_M : \tilde{\Sigma}_M \rightarrow \Sigma$ . Then the sheaf cohomology  $H^1(\Sigma, \mathcal{F})$  is the same as  $H^1(\tilde{\Sigma}_M, \pi_M^*\mathcal{F})^M$  whose Hodge type is determined by  $H^1(\tilde{\Sigma}_M, \pi_M^*\mathcal{F})$ . Applying this to our case, we can ignore the critical values and it is enough to consider only the monodromy of  $R^2\pi_{b!}^\circ\mathbb{Z}$  around  $D$  to compute the Hodge type. Since  $X_b$  is constructed via elementary modification from another threefold which has smooth fibers everywhere around  $D$ , we see that the monodromy of  $R^2\pi_{b!}\mathbb{Z}$  around  $D$  is trivial. As  $H_c^1(\Sigma^\circ, R^2\pi_{b!}^\circ\mathbb{Z}) \cong H^1(\Sigma, D, R^2\pi_{b!}\mathbb{Z})$ , it admits the  $\mathbb{Z}$ -mixed Hodge structure of type  $\{(-2, -2), (-2, -1), (-1, -2)\}$  due to the relative version of Zucker's theorem [Zuc79]. Therefore, we have the following result.

**Proposition 2.4.3.** *For  $b \in B^{ur}$ , the third homology group  $H_3(X_b, \mathbb{Z})$  admits a  $\mathbb{Z}$ -mixed Hodge structure of type  $\{(-2, -2), (-2, -1), (-1, -2)\}$ . Moreover, the third cohomology group  $H^3(X_b, \mathbb{Z})$  admits a  $\mathbb{Z}$ -mixed Hodge structure of type  $\{(1, 2), (2, 1), (2, 2)\}$ .*

The homology version of the second intermediate Jacobian of  $X_b$  is defined to be Jacobian associated to the  $\mathbb{Z}$ -mixed Hodge structure of  $H_3(X_b, \mathbb{Z})(1)$

$$J_2(X_b) := J(H_3(X_b, \mathbb{Z})(1)) = \frac{H_3(X_b, \mathbb{C})}{F^{-1}H_3(X_b, \mathbb{C}) + H_3(X_b, \mathbb{Z})} \quad (2.4.4)$$

*Remark 28.* The homology group  $H_3(X_b, \mathbb{Z})(1)$  turns out to have torsion (see Theorem 2.5.2). To get the  $\mathbb{Z}$ -mixed Hodge structure on the lattice of the semi-abelian variety  $J_2(X_b)$ , we should consider the  $\mathbb{Z}$ -mixed Hodge structure on the torsion-free part  $H_3(X_b, \mathbb{Z})_{\text{tf}}(1)$ .

**Corollary 2.4.4.** *For  $b \in B^{ur}$ , the homology version of the second intermediate Jacobian  $J_2(X_b)$  is a semi-abelian variety.*

*Remark 29.* (Adjoint Type) Unlike the classical case, the cohomological intermediate Jacobian on  $H^3(X_b, \mathbb{Z})$  is not a semi-abelian variety. This is one of the new features, so we need to consider different data to describe the case of  $PGL(n, \mathbb{C})$ , the adjoint group of type A. It turns out that the right object is a mixture of compactly supported cohomology and ordinary cohomology associated to  $\pi_b : X_b \rightarrow \Sigma$ :

$$H_c^1(\Sigma, R^2\pi_{b*}\mathbb{Z}) \cong H_c^1(\Sigma^\circ, R^2\pi_{b*}^\circ\mathbb{Z}).$$

## 2.4.2 Calabi-Yau integrable systems

Having constructed the family of Calabi-Yau threefolds  $\mathcal{X}^{ur} \rightarrow B^{ur}$ , we can consider the relative intermediate Jacobian fibration  $\pi^{ur} : \mathcal{J}(\mathcal{X}^{ur}/B^{ur}) \rightarrow B^{ur}$  whose fiber is  $J_2(X_b) = H_3(X_b, \mathbb{C}) / (F^{-1}H_3(X_b, \mathbb{C}) + H_3(X_b, \mathbb{Z}))$ . One way to equip it with an integrable system structure is to find an abstract Seiberg-Witten differential (see Section 2). In the case of an intermediate Jacobian fibration, this can be achieved by finding a global nowhere-vanishing holomorphic volume form in each

fiber. The resulting semi-polarized integrable system will again be called the Calabi-Yau integrable system.

Consider the subfamily of Calabi-Yau threefolds

$$(\mathcal{X}^\circ)^{ur} := \mathcal{X}^{ur} \setminus \pi^{-1}(D \times B^{ur}) \subset \mathcal{X}^{ur} \rightarrow B^{ur}.$$

whose fiber is  $X_b^\circ := X_b \setminus \pi_b^{-1}(D)$ . From the relation (2.4.2), it is enough to find global holomorphic volume forms for the family  $(\mathcal{X}^\circ)^{ur} \rightarrow B^{ur}$ . The idea is that the family  $(\mathcal{X}^\circ)^{ur} \rightarrow B^{ur}$  can be constructed alternatively by gluing Slodowy slices as in [DDP07][Bec20], which is the key ingredient used for the existence of global volume forms.

**Claim 2.4.5.** *The family of quasi-projective Calabi-Yau threefolds  $\pi^{ur} : (\mathcal{X}^\circ)^{ur} \rightarrow B^{ur}$  can be obtained by gluing Slodowy slices.*

Recall that in the classical case [Slo80], the Slodowy slice  $S \subset \mathfrak{g}$  provides a semi-universal  $\mathbb{C}^*$ -deformation  $\sigma : S \rightarrow \mathfrak{t}/W$  of simple singularities via the adjoint map  $\sigma : \mathfrak{g} \rightarrow \mathfrak{t}/W$ . However, if we denote by  $d_j$  the standard ( $\mathbb{C}^*$ -action) weights of the generators of the coordinate ring  $\mathbb{C}[\chi_1, \dots, \chi_j]$  of  $\mathfrak{t}/W$ , then the weights on  $\mathbb{C}[\chi_1, \dots, \chi_j]$  must be chosen as  $2d_j$  for  $\sigma$  to be  $\mathbb{C}^*$ -equivariant (see [BDW20, Remark 2.5.3], [Slo80]).

Now we choose a theta characteristic  $L$  on  $\Sigma$ , i.e.  $L^2 \cong K_\Sigma$ . Since  $L^2|_{\Sigma^\circ} \cong K_\Sigma|_{\Sigma^\circ} \cong K_\Sigma(D)|_{\Sigma^\circ}$ , we have an isomorphism of associated bundles over  $\Sigma^\circ$

$$L|_{\Sigma^\circ} \times_{\mathbb{C}^*} \mathfrak{t}/W \cong K_\Sigma(D)|_{\Sigma^\circ} \times_{\mathbb{C}^*} \mathfrak{t}/W$$

where the weights of the  $\mathbb{C}^*$ -action on both sides are different: the left hand side has weights  $2d_j$  and the right hand side has weights  $d_j$ . As the map  $\sigma : S \rightarrow \mathfrak{t}/W$  is  $\mathbb{C}^*$ -equivariant, we can glue it along  $\mathbf{Tot}(L)$  to obtain

$$\sigma : \mathbf{S} := \mathbf{Tot}(L \times_{\mathbb{C}^*} S) \rightarrow \mathbf{Tot}(L \times_{\mathbb{C}^*} \mathfrak{t}/W)$$

and its restriction

$$\sigma|_{\Sigma^\circ} : \mathbf{S}|_{\Sigma^\circ} := \mathbf{Tot}(L \times_{\mathbb{C}^*} S)|_{\Sigma^\circ} \rightarrow \mathbf{Tot}(L|_{\Sigma^\circ} \times_{\mathbb{C}^*} \mathfrak{t}/W) \cong \mathbf{Tot}(K_\Sigma(D))|_{\Sigma^\circ} \times_{\mathbb{C}^*} \mathfrak{t}/W = \mathbf{U}|_{\Sigma^\circ}.$$

Pulling back under the evaluation map from  $\Sigma \times B$ , one gets a family of quasi-projective threefolds  $(\mathcal{Y}^\circ)^{ur}$  as follows:

$$\begin{array}{ccc} (\mathcal{Y}^\circ)^{ur} & \longrightarrow & \mathbf{S}|_{\Sigma^\circ} \\ \downarrow \pi' & & \downarrow \sigma|_{\Sigma^\circ} \\ \Sigma^\circ \times B^{ur} & \xrightarrow{ev} & \mathbf{U}|_{\Sigma^\circ} \end{array} \quad (2.4.5)$$

**Lemma 2.4.6.** *We have an isomorphism of the families  $(\mathcal{Y}^\circ)^{ur} \cong (\mathcal{X}^\circ)^{ur}$  over  $B^{ur}$  and, in particular,  $Y_b^\circ \cong X_b^\circ$  where  $Y_b^\circ$  is a member of the family  $\mathcal{Y}^\circ$ .*

*Proof.* For type  $A$ , we have a semi-universal  $\mathbb{C}^*$ -deformation of  $A_{n-1}$  singularities (see [KM92, Theorem 1]) as follows:

$$\begin{aligned} \sigma' : H := \{xy - s^n - b_2 s^{n-2} - \dots - b_n = 0\} \subset \mathbb{C}^3 \times \mathbb{C}^{n-1} &\rightarrow \mathbb{C}^{n-1} \cong \mathfrak{t}/W \\ (x, y, s, b_2, \dots, b_n) &\mapsto (b_2, \dots, b_n) \end{aligned} \quad (2.4.6)$$

The map  $\sigma'$  is  $\mathbb{C}^*$ -equivariant if we endow the following  $\mathbb{C}^*$ -actions on  $\mathbb{C}^3$  and  $\mathbb{C}^{n-1}$ :

$$(x, y, s) \mapsto (\lambda^2 x, \lambda^{2(n-1)} y, \lambda^2 s), \quad (b_2, \dots, b_n) \mapsto (\lambda^4 b_2, \dots, \lambda^{2n} b_n). \quad (2.4.7)$$

Since the semi-universal  $\mathbb{C}^*$ -deformation of a simple singularity is unique up to isomorphism, the two deformations  $\sigma$  and  $\sigma'$  are isomorphic. In other words, the Slodowy slice  $S$  contained in  $\mathfrak{g}$  is isomorphic to the hypersurface  $H$  in  $\mathbb{C}^3 \times \mathbb{C}^{n-1}$  as semi-universal  $\mathbb{C}^*$ -deformation. Note that it is important to choose the  $\mathbb{C}^*$ -action on  $\mathbb{C}^{n-1} \cong \mathfrak{t}/W$  and  $\mathbb{C}^3$  as above for  $S$  and  $H$  to be isomorphic as  $\mathbb{C}^*$ -deformation (see [BDW20, Remark 2.5.3]).

Next, let's turn to the global situation. We again have the isomorphism of associated bundles

$$L|_{\Sigma^\circ} \times_{\mathbb{C}^*} \mathbb{C}^* \cong K_\Sigma(D)|_{\Sigma^\circ} \times_{\mathbb{C}^*} \mathbb{C}^*$$

with the weights of the  $\mathbb{C}^*$ -action on the left hand side being twice the weights on the right hand side. Hence, the associated bundle  $L|_{\Sigma^\circ} \times_{\mathbb{C}^*} \mathbb{C}^3$  is

$$L^2|_{\Sigma^\circ} \oplus L^{2(n-1)}|_{\Sigma^\circ} \oplus L^2|_{\Sigma^\circ} \cong (K_\Sigma(D) \oplus K_\Sigma(D)^{\otimes n-1} \oplus K_\Sigma(D))|_{\Sigma^\circ} \cong V|_{\Sigma^\circ}$$

Also, since the elementary modification is an isomorphism i.e.  $V|_{\Sigma^\circ} \cong W|_{\Sigma^\circ}$  away from  $D$ , the previous construction (2.4.1) of the family  $\pi^\circ : (\mathcal{X}^\circ)^{ur} \rightarrow B^{ur}$  as a family of hypersurfaces in the total space of  $W|_{\Sigma^\circ}$  is equivalent to the construction as the pullback of the gluing of  $H$  and  $\sigma'$  over  $K_\Sigma(D)|_{\Sigma^\circ}$ :

$$\begin{array}{ccc} (\mathcal{X}^\circ)^{ur} & \longrightarrow & \mathbf{H}|_{\Sigma^\circ} \subset \mathbf{Tot}(K_\Sigma(D)|_{\Sigma^\circ} \times_{\mathbb{C}^*} \mathbb{C}^3) \times \mathbf{Tot}(K_\Sigma(D)|_{\Sigma^\circ} \times_{\mathbb{C}^*} \mathfrak{t}/W) \\ \downarrow \pi^\circ & & \downarrow (\sigma')|_{\Sigma^\circ} \\ \Sigma^\circ \times B^{ur} & \xrightarrow{ev} & \mathbf{U}|_{\Sigma^\circ} = \mathbf{Tot}(K_\Sigma(D)|_{\Sigma^\circ} \times_{\mathbb{C}^*} \mathfrak{t}/W) \end{array} \quad (2.4.8)$$

where we define  $\sigma' : \mathbf{H} = \mathbf{Tot}(K_\Sigma(D) \times_{\mathbb{C}^*} H) \rightarrow \mathbf{U}$  and all the  $\mathbb{C}^*$ -actions in the diagram are understood as having half the weights in (2.4.7). By the argument

that  $S$  and  $H$  are isomorphic as  $\mathbb{C}^*$ -deformation, we have that  $\sigma|_{\Sigma^\circ} : \mathbf{S}|_{\Sigma^\circ} = \mathbf{Tot}(L|_{\Sigma^\circ} \times_{\mathbb{C}^*} S) \rightarrow \mathbf{U}|_{\Sigma^\circ}$  and  $\sigma'|_{\Sigma^\circ} : \mathbf{H}|_{\Sigma^\circ} \rightarrow \mathbf{U}|_{\Sigma^\circ}$  are also isomorphic. By pulling back this isomorphism along the evaluation map to  $\Sigma^\circ \times B^{ur}$ , we get the isomorphism  $(\mathcal{Y}^\circ)^{ur} \cong (\mathcal{X}^\circ)^{ur}$ .

□

**Proposition 2.4.7.** *The relative intermediate Jacobian fibration  $\pi^{ur} : \mathcal{J}(\mathcal{X}^{ur}/B^{ur}) \rightarrow B^{ur}$  is a semi-polarized integrable system.*

*Proof.* By the relation (4.2), it is enough to show that there exists a Seiberg-Witten differential associated to the subfamily  $(\mathcal{X}^\circ)^{ur} \rightarrow B^{ur}$ . In other words, we need to construct a holomorphic volume form  $\lambda_{CY^\circ}$  on  $(\mathcal{X}^\circ)^{ur}$  which yields the nowhere vanishing holomorphic volume form  $\lambda_{CY,b}^\circ \in H^0(X_b^\circ, K_{X_b^\circ})$  for each  $b \in B^{ur}$  and satisfies the condition (2.2.4).

First, the holomorphic volume form  $\lambda_{CY^\circ}$  is obtained from the holomorphic 3-form  $\lambda$  on  $\mathbf{S}$ . Note that the Kostant-Kirillov form on  $\mathfrak{g}$  induces the nowhere vanishing section in  $\nu \in H^0(S, K_\sigma)$ . One can glue the sections over  $L$  by tensoring with local frames in the pullback of  $K_\Sigma$ , which turns out to be the holomorphic 3-form  $\lambda$  on  $\mathbf{S}$  [DDP07][Bec20]. By restricting  $\lambda$  to  $\Sigma^\circ$ , it becomes a global holomorphic 3-form whose pullback to  $(\mathcal{X}^\circ)^{ur}$  is the desired volume form  $\lambda_{CY^\circ}$ .

Next, the proof that  $\lambda_{CY^\circ}$  becomes the Seiberg-Witten differential relies on our main result (Theorem 2.5.1). In particular, we identify the volume form  $\lambda_{CY^\circ}$  with the Seiberg-Witten differential for the Hitchin system so that the form  $\lambda_{CY^\circ}$  au-

tomatically satisfies the condition (2.2.4). Therefore, it follows from Proposition (2.2.5) that  $\mathcal{J}(\mathcal{X}^{ur}/B^{ur}) \rightarrow B^{ur}$  is a semi-polarized integrable system.  $\square$

## 2.5 Meromorphic DDP correspondence

### 2.5.1 Isomorphism of semi-polarized integrable systems

The goal of this section is to prove an isomorphism between the two semi-polarized integrable systems that have been studied so far: the moduli space of unordered diagonally framed Higgs bundles  $\mathcal{M}^\Delta(n, D)^{ur} \rightarrow B^{ur}$  and the relative intermediate Jacobian fibration  $\mathcal{J}(\mathcal{X}^{ur}/B^{ur}) \rightarrow B^{ur}$  of the family of Calabi-Yau threefolds  $\mathcal{X}^{ur} \rightarrow B^{ur}$ . The main result is stated as follows.

**Theorem 2.5.1.** *There is an isomorphism of semi-polarized integrable systems:*

$$\begin{array}{ccc}
 \mathcal{J}(\mathcal{X}^{ur}/B^{ur}) & \xrightarrow{\cong} & \mathcal{M}^\Delta(n, D)^{ur} \\
 \searrow \pi^{ur} & & \swarrow h_\Delta^{ur} \\
 & & B^{ur}
 \end{array} \tag{2.5.1}$$

Recall that we have shown in Proposition 2.3.8 and Corollary 2.3.10 that  $(h_\Delta^{ur})^{-1}(b) \cong \text{Prym}(\overline{\Sigma}_b^\circ, \Sigma^\circ) \cong J(H_{\Delta, SL(n), b})$  where  $H_{\Delta, SL(n), b} := H_1(\text{Prym}(\overline{\Sigma}_b^\circ, \Sigma^\circ), \mathbb{Z}) = H_1(\Sigma^\circ, \mathcal{K}_b|_{\Sigma^\circ})_{\text{tf}}$  and  $\mathcal{K}_b := \ker(\text{Tr} : \overline{p}_{b*} \mathbb{Z} \rightarrow \mathbb{Z})$ . By definition, the fiber  $(\pi^{ur})^{-1}(b) = J_2(X_b) = J(H_3(X_b, \mathbb{Z})(1))$ . The specialization of Theorem 2.5.1 to  $b \in B^{ur}$  is equivalent to an isomorphism between the semi-abelian varieties  $J_2(X_b)$  and  $\text{Prym}(\overline{\Sigma}_b^\circ, \Sigma^\circ)$ , or equivalently, between the  $\mathbb{Z}$ -mixed Hodge structures  $H_3(X, \mathbb{Z})_{\text{tf}}(1)$  and  $H_{\Delta, SL(n), b}$  of type  $\{(-1, -1), (-1, 0), (0, -1)\}$ . We begin by proving the following result.



**Theorem 2.5.2.** *For  $b \in B^{ur}$ , there is an isomorphism of  $\mathbb{Z}$ -mixed Hodge structures:*

$$(H_3(X_b, \mathbb{Z})_{tf}(1), W_{\bullet}^{CY}, F_{CY}^{\bullet}) \cong (H_{\Delta, SL(n), b}, W_{\bullet}^{\Delta, b}, F_{\Delta, b}^{\bullet}). \quad (2.5.2)$$

*Proof.* We first fix some notations. Denote by  $\Sigma^1 := \Sigma^{\circ} \setminus Br(\tilde{p}_b^{\circ})$ ,  $\tilde{\Sigma}_b^1 := \tilde{\Sigma}_b^{\circ} \setminus Ram(\tilde{p}_b^{\circ})$  the complement of the ramification and branch divisors in  $\Sigma_b^{\circ}, \tilde{\Sigma}_b^{\circ}$  respectively. Since the branch divisor of the spectral cover  $\bar{p}_b : \bar{\Sigma}_b \rightarrow \Sigma$  is contained in the branch divisor of the cameral cover  $\tilde{p}_b^{\circ} : \tilde{\Sigma}_b^{\circ} \rightarrow \Sigma^{\circ}$ , we write  $\bar{\Sigma}_b^1 := \bar{\Sigma}_b^{\circ} \setminus (\bar{p}_b^{\circ})^{-1} Br(\tilde{p}_b^{\circ})$ . The restricted maps of the spectral cover  $\bar{p}_b^1 : \bar{\Sigma}_b^1 \rightarrow \Sigma^1$  and the cameral cover  $\tilde{p}_b^1 : \tilde{\Sigma}_b^1 \rightarrow \Sigma^1$  are then unramified. Similarly, we write  $X_b^1 \subset X_b^{\circ}$  the complement of  $(\pi_b^{\circ})^{-1}(D)$  in  $X_b^{\circ}$  and the restricted map as  $\pi_b^1 : X_b^1 \rightarrow \Sigma^1$ .

**Step 1.** As argued in (2.4.2) and (2.4.3) of the previous section, we have the isomorphisms of  $\mathbb{Z}$ -mixed Hodge structures of type  $\{(-1, -1), (-1, 0), (0, -1)\}$

$$H_3(X_b, \mathbb{Z})(1) \cong H_c^3(X_b, \mathbb{Z})(1) \cong H_c^3(X_b^{\circ}, \mathbb{Z})(1) \cong H_c^1(\Sigma^{\circ}, R^2\pi_{b!}^{\circ}\mathbb{Z})(1). \quad (2.5.3)$$

**Step 2.**

**Lemma 2.5.3.** *Over  $\Sigma^{\circ}$ , we have an isomorphism of sheaves,*

$$R^2\pi_{b!}^{\circ}\mathbb{Z} \cong (\tilde{p}_{b*}^{\circ}\Lambda_{SL(n)})^W. \quad (2.5.4)$$

*Proof.* In the classical work of [Slo80], Slodowy provided a detailed study of the topology of the maps in the following diagram via its simultaneous resolution:

$$\begin{array}{ccc} \tilde{S} & \longrightarrow & S \\ \downarrow \tilde{\sigma} & & \downarrow \sigma \\ \mathfrak{t} & \xrightarrow{\phi} & \mathfrak{t}/W \end{array} \quad (2.5.5)$$

It can be shown that there is an isomorphism of constructible sheaves

$$R^2\sigma_*^1\mathbb{Z} \cong (\phi_*^1\Lambda_{SL(n)})^W \quad (2.5.6)$$

over an open subset  $\mathfrak{t}^1/W \subset \mathfrak{t}/W$  defined as the image of another open subset  $\mathfrak{t}^1 \subset \mathfrak{t}$  under  $\phi$ . Here we denote  $\phi^1 := \phi|_{\mathfrak{t}^1}$  and  $\sigma^1 : \sigma^{-1}(\mathfrak{t}^1/W) \rightarrow \mathfrak{t}^1/W$ . For details, see [Bec20, Lemma 5.1.3].

Next, we glue the maps  $\sigma$  and  $\phi$  along  $K_\Sigma(D)|_{\Sigma^\circ}$  as in (2.3.17) and (2.4.8)

$$\begin{array}{ccc} & & \mathbf{S}|_{\Sigma^\circ} = \mathbf{Tot}(L|_{\Sigma^\circ} \times_{\mathbb{C}^*} S) \\ & & \downarrow \sigma|_{\Sigma^\circ} \\ \widetilde{\mathbf{U}}|_{\Sigma^\circ} := \mathbf{Tot}(K_\Sigma(D)|_{\Sigma^\circ} \times_{\mathbb{C}^*} \mathfrak{t}) & \xrightarrow{\phi|_{\Sigma^\circ}} & \mathbf{U}|_{\Sigma^\circ} = \mathbf{Tot}(K_\Sigma(D)|_{\Sigma^\circ} \times_{\mathbb{C}^*} \mathfrak{t}/W) \cong \mathbf{Tot}(L|_{\Sigma^\circ} \times_{\mathbb{C}^*} \mathfrak{t}/W) \end{array} \quad (2.5.7)$$

Let us define  $\mathbf{U}^1 := \mathbf{Tot}(K_\Sigma(D) \times_{\mathbb{C}^*} \mathfrak{t}^1/W) \subset \mathbf{U}$ . Since the varieties here are glued using the same cocycle of  $L|_{\Sigma^\circ}$  (again, in taking the associated bundles here,  $L|_{\Sigma^\circ}$  as a  $\mathbb{C}^*$ -bundle acts with twice the weights of the action by  $K_\Sigma(D)|_{\Sigma^\circ}$ ), the isomorphism of constructible sheaves (2.5.6) over  $\mathfrak{t}^1/W$  also glues together to another isomorphism of constructible sheaves over  $\mathbf{U}^1|_{\Sigma^\circ}$ :

$$R^2(\sigma)_!\mathbb{Z} \cong (\phi_*\Lambda_{SL(n)})^W. \quad (2.5.8)$$

As argued in Claim (2.4.5),  $\sigma|_{\Sigma^\circ} : \mathbf{S}|_{\Sigma^\circ} \rightarrow \mathbf{U}|_{\Sigma^\circ}$  is equivalent to  $(\sigma')|_{\Sigma^\circ} : \mathbf{H}|_{\Sigma^\circ} \rightarrow \mathbf{U}|_{\Sigma^\circ}$ , so we obtain

$$R^2(\sigma')_!\mathbb{Z} \cong (\phi_*\Lambda_{SL(n)})^W \quad (2.5.9)$$

over  $\mathbf{U}^1|_{\Sigma^\circ}$ . In both (2.5.8) and (2.5.9), we drop the notation of the restrictions of  $\sigma$ ,  $\sigma'$  and  $\phi$  to  $\mathbf{U}^1|_{\Sigma^\circ}$  for convenience.

Recall from Claim (2.4.5) that  $\pi_b^\circ : X_b^\circ \rightarrow \Sigma^\circ$  can be obtained by pulling back from  $\sigma'|_{\Sigma^\circ} : \mathbf{H}|_{\Sigma^\circ} \rightarrow \mathbf{U}|_{\Sigma^\circ}$  along the composition of the inclusion and the evaluation map  $\Sigma^\circ \times \{b\} \hookrightarrow \Sigma^\circ \times B \rightarrow \mathbf{U}|_{\Sigma^\circ}$ . For  $b \in B^{ur}$ , the section  $b : \Sigma \rightarrow \mathbf{U}$  factorizes through  $\mathbf{U}^1$  and then restricts to  $b|_{\Sigma^\circ} : \Sigma^\circ \rightarrow \mathbf{U}^1|_{\Sigma^\circ}$ , so the isomorphism (2.5.9) specializes to  $R^2\pi_b^\circ\mathbb{Z} \cong (\tilde{p}_{b*}^\circ \Lambda_{SL(n)})^W$  by pulling back along  $b|_{\Sigma^\circ}$ .

□

### Step 3.

**Lemma 2.5.4.** *Over  $\Sigma^\circ$ , we have an isomorphism of sheaves,*

$$(\tilde{p}_{b*}^\circ \Lambda_{SL(n)})^W \cong \mathcal{K}_b|_{\Sigma^\circ}. \quad (2.5.10)$$

*Proof.* To simplify the notation, we will write  $\mathcal{K}_b^\circ := \mathcal{K}_b|_{\Sigma^\circ}$  in this proof. Recall that there is an isomorphism (see [Don93, (6.5)]) between the two sheaves away from the branch locus:

$$\bar{p}_{b*}^1 \mathbb{Z} \cong (\tilde{p}_{b*}^1 R)^W \quad (2.5.11)$$

where  $R := \mathbb{Z}[W/W_0]$  denote the free abelian group generated by the set of right (or left) cosets  $W/W_0$ . Then we see that

$$\mathcal{K}_b|_{\Sigma^1} = \ker(\bar{p}_{b*}^1 \mathbb{Z} \rightarrow \mathbb{Z}) \cong \ker((\tilde{p}_{b*}^1 R)^W \rightarrow \mathbb{Z}) \cong (\tilde{p}_{b*}^1 \Lambda_{SL(n)})^W,$$

the last isomorphism holds because  $\ker(R \rightarrow \mathbb{Z}) = \Lambda_{SL(n)}$ .

Denote by  $j : \Sigma^1 \rightarrow \Sigma^\circ$  the inclusion map. We first write  $\mathcal{K}_b^\circ$  as  $j_* j^* \mathcal{K}_b^\circ$ . Indeed, as  $\bar{p}_{b*}^\circ \mathbb{Z} = j_* \bar{p}_{b*}^1 \mathbb{Z}$  and  $\mathbb{Z} \cong j_* \mathbb{Z}$ , applying the functor  $j_*$  to the short exact sequence

$0 \rightarrow j^* \mathcal{K}_b^\circ \rightarrow \bar{p}_{b*}^1 \mathbb{Z} \xrightarrow{Tr|_{\Sigma^1}} \mathbb{Z} \rightarrow 0$ , we get

$$0 \rightarrow j_* j^* \mathcal{K}_b^\circ \rightarrow j_* \bar{p}_{b*}^1 \mathbb{Z} = \bar{p}_{b*}^1 \mathbb{Z} \xrightarrow{Tr} j_* \mathbb{Z} = \mathbb{Z} \rightarrow R^1 j_* j^* \mathcal{K}_b^\circ \rightarrow \dots$$

In particular, it follows that  $j_* j^* \mathcal{K}_b^\circ \cong \ker(Tr) = \mathcal{K}_b^\circ$ .

Hence, we get

$$(\tilde{p}_{b*}^\circ \Lambda_{SL(n)})^W \cong j_* (\tilde{p}_{b*}^1 \Lambda_{SL(n)})^W \cong j_* j^* \mathcal{K}_b^\circ \cong \mathcal{K}_b^\circ$$

which means that the isomorphism (2.5.11) above extends from  $\Sigma^1$  to  $\Sigma^\circ$ .  $\square$

**Step 4.** Finally, since the isomorphic local systems  $R^2 \pi_{b!}^\circ \mathbb{Z} \cong (\tilde{p}_{b*}^\circ \Lambda_{SL(n)})^W \cong \mathcal{K}_b|_{\Sigma^\circ}$  have trivial monodromy at  $D$ , one can argue as in [DDP07, Lemma 3.1] and the argument for Proposition 2.4.3 that it induces the  $\mathbb{Z}$ -mixed Hodge structure of type  $\{(-1, -1), (-1, 0), (0, -1)\}$  on

$$H_c^1(\Sigma^\circ, R^2 \pi_{b!}^\circ \mathbb{Z})(1) \cong H_c^1(\Sigma^\circ, (\tilde{p}_{b*}^\circ \Lambda_{SL(n)})^W) \cong H_c^1(\Sigma^\circ, \mathcal{K}_b|_{\Sigma^\circ}).$$

Hence, taking the torsion free part, we achieve the isomorphism of  $\mathbb{Z}$ -mixed Hodge structures

$$H_3(X_b, \mathbb{Z})_{\text{tf}}(1) \cong H_c^1(\Sigma^\circ, \mathcal{K}_b|_{\Sigma^\circ})_{\text{tf}} \cong H_{\Delta, SL(n), b}.$$

$\square$

By the equivalence between semi-abelian varieties and torsion free  $\mathbb{Z}$ -mixed Hodge structures of type  $\{(-1, -1), (-1, 0), (0, -1)\}$ , we immediately get the following result:

**Corollary 2.5.5.** *We have an isomorphism of semi-abelian varieties*

$$J_2(X_b) \cong h_{\Delta}^{-1}(b) \cong \mathbf{Prym}(\overline{\Sigma}_b^{\circ}/\Sigma^{\circ}). \quad (2.5.12)$$

Now we return to the main theorem.

*Proof of Theorem 2.5.1.* Clearly, the argument in Theorem 2.5.2 works globally for the family of CY threefolds  $pr_2 \circ \pi^{ur} : \mathcal{X}^{ur} \rightarrow \Sigma \times B^{ur} \rightarrow B^{ur}$  and the family of punctured spectral curves  $pr_2 \circ \bar{p}^{ur} : \overline{\Sigma}^{\circ} \rightarrow \Sigma^{\circ} \times B^{ur} \rightarrow B^{ur}$ , so it yields an isomorphism of admissible variations of  $\mathbb{Z}$ -mixed Hodge structures:

$$R^3(pr_2 \circ \pi^{ur})_! \mathbb{Z}(1) \cong R^1(pr_2)_!(\mathcal{K}) \quad (2.5.13)$$

where  $\mathcal{K} := \ker(\mathbf{Tr} : \bar{p}_*^{ur} \mathbb{Z} \rightarrow \mathbb{Z})$ . By taking the relative Jacobian fibrations of both sides, we immediately get an isomorphism of varieties:

$$\begin{array}{ccc} \mathcal{J}(\mathcal{X}^{ur}/B^{ur}) & \xrightarrow{\cong} & \mathbf{Prym}(\overline{\Sigma}^{\circ}, \Sigma^{\circ}) \cong \mathcal{M}^{\Delta}(n, D)^{ur} \\ & \searrow \pi^{ur} & \swarrow h_{\Delta}^{ur} \\ & B^{ur} & \end{array} \quad (2.5.14)$$

where  $\mathbf{Prym}(\overline{\Sigma}^{\circ}, \Sigma^{\circ})$  is the relative Prym fibration of the family of punctured spectral curves  $\overline{\Sigma}^{\circ} \rightarrow B^{ur}$ . By the spectral correspondence proved in Proposition 2.3.8, we have  $\mathbf{Prym}(\overline{\Sigma}^{\circ}, \Sigma^{\circ}) \cong \mathcal{M}^{\Delta}(n, D)^{ur}$ .

It remains to verify that the morphism  $\mathcal{J}(\mathcal{X}^{ur}/B^{ur}) \rightarrow \mathcal{M}^{\Delta}(n, D)^{ur}$  intertwines the abstract Seiberg-Witten differentials constructed on each side. This can be easily obtained by modifying the classical results in [DDP07] [Bec20] to our punctured case. Note that both the abstract Seiberg-Witten differentials come from the tautological section on  $\widetilde{U}$ . In order to compare them, we again look at the simultaneous

resolution of  $S \rightarrow \mathfrak{t}/W$ :

$$\begin{array}{ccc} \tilde{S} & \longrightarrow & S \\ \downarrow \tilde{\sigma} & & \downarrow \sigma \\ \mathfrak{t} & \xrightarrow{\phi} & \mathfrak{t}/W \end{array} \quad (2.5.15)$$

and recall that  $\tilde{\sigma}$  is  $C^\infty$ -trivial.

Taking a step further in (2.5.7), we can glue all the maps in the simultaneous resolution diagram to a commutative diagram

$$\begin{array}{ccc} \tilde{\mathcal{S}}|_{\Sigma^\circ} & \xrightarrow{\Psi} & \mathcal{S}|_{\Sigma^\circ} \\ \downarrow \tilde{\sigma}|_{\Sigma^\circ} & & \downarrow \sigma|_{\Sigma^\circ} \\ \tilde{\mathcal{U}}|_{\Sigma^\circ} & \xrightarrow{\phi|_{\Sigma^\circ}} & \mathcal{U}|_{\Sigma^\circ} \end{array} \quad (2.5.16)$$

where  $\tilde{\mathcal{S}}|_{\Sigma^\circ} := \mathbf{Tot}(L|_{\Sigma^\circ} \times_{\mathbb{C}^*} \tilde{S})$ .

The map  $\Psi$  induces an inclusion of cohomologies

$$\Psi^* : \mathcal{H}^3((\mathcal{X}^\circ)^{ur}/B^{ur}, \mathbb{C}) \rightarrow \mathcal{H}^3((\tilde{\mathcal{X}}^\circ)^{ur}/B^{ur}, \mathbb{C}) \quad (2.5.17)$$

so that we can lift  $\lambda_{CY}^\circ$  to  $\tilde{\mathcal{X}}^\circ$ . As both are induced from the tautological section on  $\tilde{\mathcal{U}}$ , under the following isomorphism

$$\mathcal{H}^3((\tilde{\mathcal{X}}^\circ)^{ur}/B^{ur}, \mathbb{C}) \cong \mathcal{H}^1(\tilde{\Sigma}^\circ, \mathfrak{t})$$

the two abstract Seiberg-Witten differentials  $\lambda_{CY}^\circ$  and  $\lambda_\Delta$  coincide [Bec20, Theorem 5.2.1].

□

*Remark 30.* (Adjoint type) The above argument is easily applied to the adjoint case,  $PGL(n, \mathbb{C})$ , so that there is an isomorphism between (unordered) diagonally framed  $PGL(n, \mathbb{C})$ -Hitchin system and Calabi-Yau integrable system. On the Hitchin side,

we consider dual Prym sheaf  $\mathcal{K}^\vee$ . The key is to construct the relevant family of semi-abelian varieties on the Calabi-Yau side as mentioned in Remark 29.

## 2.6 Appendix: Summary of Deligne's theory of 1-motives

In [Del74], Deligne gave a motivic description of variations of (polarized)  $\mathbb{Z}$ -mixed Hodge structures of type  $\{(-1, -1), (-1, 0), (0, -1), (0, 0)\}$ . We recall the arguments in [Del74] and study the special case which is of main interest in this paper.

**Definition 2.6.1.** An 1-motive  $M$  over  $\mathbb{C}$  consists of

- (1)  $X$  free abelian group of finite rank, a complex abelian variety  $A$ , and a complex affine torus  $\mathbb{T}$ .
- (2) A complex semi-abelian variety  $G$  which is an extension of  $A$  by  $\mathbb{T}$ .
- (3) A homomorphism  $u : X \rightarrow G$ .

We will denote a 1-motive by  $(X, A, T, G, u)$  or  $M = [X \xrightarrow{u} G]$ .

**Proposition 2.6.2.** *The category of (polarizable) mixed Hodge structures of type  $\{(-1, -1), (-1, 0), (0, -1), (0, 0)\}$  is equivalent to the category of 1-motives.*

*Proof.* Given a 1-motive  $M$ , Deligne constructed a mixed Hodge structure  $(T(M)_{\mathbb{Z}}, W, F)$  of type  $\{(-1, -1), (-1, 0), (0, -1), (0, 0)\}$  as follows. Define a lattice  $T(M)_{\mathbb{Z}}$  as the

fiber product

$$\begin{array}{ccc}
 T(M)_{\mathbb{Z}} & \xrightarrow{\beta} & X \\
 \downarrow \alpha & & \downarrow u \\
 Lie(G) & \xrightarrow{\exp} & G
 \end{array} \tag{2.6.1}$$

The weight filtration on  $T(M)_{\mathbb{Z}}$  is given by setting  $W_{-1}T(M)_{\mathbb{Z}} := H_1(G, \mathbb{Z}) = \ker(\beta)$  and  $W_{-2}T(M)_{\mathbb{Z}} = H_1(T, \mathbb{Z})$ . Also, by linearly extending  $\alpha : T(M)_{\mathbb{Z}} \rightarrow Lie(G)$  to  $\mathbb{C}$ , we define  $F^0(T(M)_{\mathbb{Z}} \otimes \mathbb{C}) := \ker(\alpha_{\mathbb{C}})$ . By construction  $\mathrm{Gr}_{-1}^W(T(M)_{\mathbb{Z}}) = H_1(A, \mathbb{Z})$  with the usual Hodge filtration and is therefore polarizable.

Conversely, if  $H := (H_{\mathbb{Z}}, W, F)$  is a mixed Hodge structure of the given type with  $\mathrm{Gr}_{-1}^W(H_{\mathbb{Z}})$  polarizable, then one can construct a 1-motive by taking

$$(1) \quad A := \mathrm{Gr}_{-1}^W(H_{\mathbb{C}}) / (F^0 \mathrm{Gr}_{-1}^W(H_{\mathbb{C}}) + \mathrm{Gr}_{-1}^W(H_{\mathbb{Z}}))$$

$$(2) \quad \Gamma := \mathrm{Gr}_{-2}^W(H_{\mathbb{C}}) / \mathrm{Gr}_{-2}^W(H_{\mathbb{Z}})$$

$$(3) \quad G := H_{\mathbb{C}} / (F^0 H_{\mathbb{C}} + H_{\mathbb{Z}})$$

$$(4) \quad X := \mathrm{Gr}_0^W(H_{\mathbb{Z}})$$

□

In particular, if  $X$  is trivial, the 1-motive  $M$  is equivalent to a semi-abelian variety  $G$ . By Proposition 2.6.2, we have an equivalence between the abelian category of semi-abelian varieties and the abelian category of (polarizable)  $\mathbb{Z}$ -mixed Hodge structures of type  $\{(-1, -1), (-1, 0), (0, -1)\}$ .

*Example 2.6.3.* A typical example coming from geometry is the mixed Hodge structure on the first homology group of a punctured curve. Let  $C$  be a Riemann



surface and  $D \subset C$  be a reduced divisor. The first homology group  $H_{\mathbb{Z}} = H_1(C \setminus D, \mathbb{Z})$  carries a  $\mathbb{Z}$ -mixed Hodge structure of type  $\{(-1, -1), (-1, 0), (0, -1)\}$  where  $\mathrm{Gr}_{-1}^W(H_{\mathbb{C}}) = H_1(C, \mathbb{Z}) \otimes \mathbb{C}$ . Moreover, it admits a degenerate intersection pairing  $Q : H_{\mathbb{Z}} \times H_{\mathbb{Z}} \rightarrow \mathbb{Z}$  whose kernel is  $W_{-2}H_{\mathbb{C}} \cap H_{\mathbb{Z}}$ . Note that it induces a polarization on  $\mathrm{Gr}_{-1}^W(H_{\mathbb{C}})$  and so gives rise to the type of object in proposition 2.6.2. In other words, we get a semi-abelian variety  $G$  by taking the Jacobian of  $(H_{\mathbb{Z}}, W_{\bullet}, F^{\bullet})$  as follows

$$G := J(H) = H_{\mathbb{C}} / (F^0 H_{\mathbb{C}} + H_{\mathbb{Z}})$$

$$A := J_{\mathrm{cpt}}(H) = \mathrm{Gr}_{-1}^W H_{\mathbb{C}} / (\mathrm{Gr}_{-1}^W F^0 H_{\mathbb{C}} + H_{\mathbb{Z}})$$

$$\mathbb{T} := W_{-2}H_{\mathbb{C}} / W_{-2}H_{\mathbb{Z}}$$

We call such integral mixed Hodge structure a *semi-polarized  $\mathbb{Z}$ -mixed Hodge structure*. Moreover, consider the dual mixed Hodge structure  $H^{\vee}$  which is of type  $\{(0, 1), (1, 0), (1, 1)\}$ . Geometrically it corresponds to the first cohomology  $H^1(C \setminus D)$  of the punctured Riemann surface  $C \setminus D$ . The associated Jacobian  $J(H^{\vee}) = H_{\mathbb{C}}^{\vee} / (F^1 H_{\mathbb{C}}^{\vee} + H_{\mathbb{Z}})$  is no longer a semi-abelian variety, but just a complex torus.

# Bibliography

- [AHK14] L. B. Anderson, J. J. Heckman, and S. Katz. “T-branes and geometry”.  
In: *Journal of High Energy Physics* 2014.5 (2014), p. 80.
- [AKO06] D. Auroux, L. Katzarkov, and D. Orlov. “Mirror symmetry for del Pezzo  
surfaces: vanishing cycles and coherent sheaves.” In: *Invent. Math.* 166(3)  
(2006), pp. 537–582.
- [And+17] L. B. Anderson, J. J. Heckman, S. Katz, and L. P. Schaposnik. “T-  
Branes at the Limits of Geometry”. In: *Journal of High Energy Physics*  
2017.10 (2017), p. 58.
- [Ara05] D. Arapura. “The Leray spectral sequence is motivic”. In: *Invent. Math.*  
160.3 (2005), pp. 567–589.
- [AS] M. Abouzaid and P. Seidel. “Lefschetz fibration techniques in wrapped  
Floer cohomology”. In: *In preparation* ().
- [Asp+09] P. Aspinwall, T. Bridgeland, A. Craw, M. Douglas, M. Gross, A. Ka-  
pustin, G. Moore, G. Segal, B. Szendrői, and P. Wilson. “Dirichlet

- branes and mirror symmetry”. In: *volume 4 of Clay Mathematics Monographs*. American Mathematical Society, Providence, RI, 2009.
- [Aur07] D. Auroux. “Mirror symmetry and T-duality in the complement of an anticanonical divisor”. In: *J. Gökova Geom. Topol.* 1 (2007), pp. 51–91.
- [Aur13] D. Auroux. “A beginner’s introduction to Fukaya categories”. In: *arxiv.org/abs/1301.70* (2013).
- [Bal+13] M. Ballard, C. Diemer, D. Favero, L. Katzarkov, and G. Kerr. “The Mori Program and Non-Fano Toric Homological Mirror Symmetry”. In: *Transactions of the American Mathematical Society* 367.12 (2013).
- [BBD82] A. A. Beilinson, J. Bernstein, and P. Deligne. “Faisceaux pervers”. In: *Paris: Société Mathématique de France*. 100 (1982).
- [BDW20] F. Beck, R. Donagi, and K. Wendland. “Folding of Hitchin systems and crepant resolutions”. In: *International Mathematics Research Notices (to appear)*, *arXiv preprint arXiv:2004.04245* (2020).
- [Bec19] F. Beck. “Calabi–Yau orbifolds over Hitchin bases”. In: *Journal of Geometry and Physics* 136 (2019), pp. 14–30.
- [Bec20] F. Beck. “Hitchin and Calabi-Yau integrable systems via variations of Hodge structures”. In: *Q. J. Math.* 71.4 (2020), pp. 1345–1375. ISSN: 0033-5606.

- [BLP19] I. Biswas, M. Logares, and A. Peón-Nieto. “Symplectic geometry of a moduli space of framed Higgs bundles”. In: *International Mathematics Research Notices* (Feb. 2019).
- [BLP20] I. Biswas, M. Logares, and A. Peón-Nieto. “Moduli Spaces of Framed  $G$ -Higgs Bundles and Symplectic Geometry”. In: *Comm.Math.Physics* 376 (2020), pp. 1875–1908.
- [Car79] J. Carlson. “Extensions of mixed Hodge structures”. In: *Journées de géométrie algébrique d’Angers* (1979), pp. 107–128.
- [CHM12] M. A. A. de Cataldo, T. Hausel, and L. Migliorini. “Topology of Hitchin systems and Hodge theory of character varieties: the case  $A_1$ ”. In: *Annals of Mathematics* (2012), pp. 1329–1407.
- [CM10] M. A. A. de Cataldo and L. Migliorini. “The perverse filtration and the Lefschetz hyperplane theorem”. In: *Ann.Math.* 171:3 (2010), pp. 2089–2113.
- [CMS19] M. A. A. de Cataldo, D. Maulik, and J. Shen. “Hitchin fibrations, abelian surfaces, and the  $P=W$  conjecture”. In: *arXiv preprint arXiv:1909.11885* (2019).
- [DDP07] D. E. Diaconescu, R. Donagi, and T. Pantev. “Intermediate Jacobians and  $ADE$  Hitchin systems”. In: *Math. Res. Lett.* 14.5 (2007), pp. 745–756.

- [De 09] M. A. A. De Cataldo. “The standard filtration on cohomology with compact supports with an appendix on the base change map and the Lefschetz hyperplane theorem”. In: *Contemporary Mathematics* 496 (2009), p. 199.
- [Del74] P. Deligne. “Théorie de Hodge: III”. In: *Publications Mathématiques de l’IHÉS*. Vol. 44. 1974, pp. 5–77.
- [DG02] R. Donagi and D. Gaitsgory. “The gerbe of Higgs bundles”. In: *Transform. Groups* 7(2) (2002), pp. 109–153.
- [DI87] P. Deligne and L. Illusie. “Relèvements modulo  $p^2$  et décomposition du complexe de de Rham”. In: *Invent. Math.* 89(2) (1987), pp. 247–270.
- [Dia+06] D.-E. Diaconescu, R. Dijkgraaf, R. Donagi, C. Hofman, and T. Pantev. “Geometric transitions and integrable systems”. In: *Nuclear Physics B* 752.3 (2006), pp. 329–390.
- [DM96a] R. Donagi and E. Markman. “Cubics, integrable systems, and Calabi-Yau threefolds”. In: *Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993)*. Vol. 9. Israel Math. Conf. Proc. Bar-Ilan Univ., Ramat Gan, 1996, pp. 199–221.
- [DM96b] R. Donagi and E. Markman. “Spectral covers, algebraically completely integrable, Hamiltonian systems, and moduli of bundles”. In: *Integrable systems and quantum groups*. Springer, 1996, pp. 1–119.

- [Don93] R. Donagi. “Decomposition of spectral covers”. In: *Astérisque* 218 (1993), pp. 145–175.
- [Don97] R. Donagi. “Seiberg-Witten integrable systems”. In: *Proc. Sympos. Pure Math.* 62 (1997), pp. 3–43.
- [DP12] R. Donagi and T. Pantev. “Langlands duality for Hitchin systems”. In: *Invent. Math.* 189(3) (2012), pp. 653–735.
- [ESY17] H. Esnault, C. Sabbah, and J.-D. Yu. “ $E_1$ -degeneration of the irregular Hodge filtration”. In: *J. reine. angew. Math.* 729 (2017), pp. 171–227.
- [Fuk+09] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono. “Lagrangian Intersection Floer Theory Anomaly and Obstruction, Part II”. In: *Studies in Advanced Mathematics*. AMS/IP, 2009.
- [Gan13] S. Ganatra. “Symplectic cohomology and duality for the wrapped Fukaya category”. In: *ArXiv preprint arXiv:1304.7312* (2013).
- [GJS02] V. Guillemin, L. Jeffrey, and R. Sjamaar. “Seiberg-Witten integrable systems”. In: *Trans. Groups* 7 (2002), pp. 155–185.
- [GM80] M. Goresky and R. MacPherson. “Intersection homology theory”. In: *Topology* 19.2 (1980), pp. 135–162.
- [GM83] M. Goresky and R. MacPherson. “Intersection homology theory. II”. In: *Inventiones Mathematicae* 72.1 (1983), pp. 77–129.

- [Gro+18] M. Gross, P. Hacking, S. Keel, and M. Kontsevich. “Canonical bases for cluster algebras”. In: *J. Amer. Math. Soc.* 31 (2018), pp. 497–608.
- [HHP10] C. Hertling, L. Hoevenaars, and H. Posthuma. “Frobenius manifolds, projective special geometry and Hitchin systems”. In: *J. Reine Angew. Math.* 649 (2010), pp. 117–165.
- [Hit87a] N. Hitchin. “Stable bundles and integrable systems”. In: *Duke mathematical journal* 54.1 (1987), pp. 91–114.
- [Hit87b] N. Hitchin. “The self-duality equations on a Riemann surface”. In: *Proceedings of the London Mathematical Society* 3.1 (1987), pp. 59–126.
- [HKP18] A. Harder, L. Katzarkov, and V. Przyjalkowski. “P=W phenomena”. In: *arXiv preprint arXiv:1905.08706* (2018).
- [HKR62] G. Hochschild, B. Kostant, and A. Rosenberg. “Differential forms on regular affine algebras”. In: *Transactions AMS* 102.3 (1962), pp. 383–408.
- [Hor+03] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow. “Mirror symmetry”. In: *volume 1 of Clay Mathematics Monographs*. American Mathematical Society, Providence, RI, 2003.

- [HT03] T. Hausel and M. Thaddeus. “Mirror symmetry, Langlands duality, and the Hitchin system”. In: *Inventiones Mathematicae* 153.1 (July 2003), pp. 197–229. ISSN: 1432-1297.
- [Kel99] B. Keller. “On the cyclic homology of exact categories”. In: *J. Pure Appl. Algebra* 136.1 (1999), pp. 1–56.
- [KKP08] L. Katzarkov, M. Kontsevich, and T. Pantev. “Hodge theoretic aspects of mirror symmetry”. In: *Proc. Sympos. Pure Math.* Vol. 78. Amer. Math. Soc., Providence, RI, 2008, pp. 87–174.
- [KKP17] L. Katzarkov, M. Kontsevich, and T. Pantev. “Bogomolov–Tian–Todorov theorems for Landau–Ginzburg models”. In: *J. Differential Geom.* 105 (2017), pp. 55–117.
- [KM92] S. Katz and D. R. Morrison. “Gorenstein threefold singularities with small resolutions via invariant theory for Weyl groups”. In: *J. Algebraic Geom.* 1.3 (1992), pp. 449–530.
- [Kon95] M. Kontsevich. “Homological algebra of mirror symmetry”. In: *In Proceedings of the International Congress of Mathematicians (Zürich, 1994)* Vol. 1, 2 (1995), pp. 120–139.
- [KPH19] L. Katzarkov, V. Przyjalkowski, and A. Harder. “Filtrations on Cohomology and Mirror Symmetry”. In: *Unpublished* (2019).



- [KS06] M. Kashiwara and P. Schapira. “Categories and Sheaves”. In: *Grundlehren der Mathematischen Wissenschaften*. Springer, 2006, p. 495.
- [KS14] M. Kontsevich and Y. Soibelman. “Wall-crossing structures in Donaldson–Thomas invariants, integrable systems and mirror symmetry”. In: *Homological mirror symmetry and tropical geometry*. Springer, 2014, pp. 197–308.
- [KW07] A. Kapustin and E. Witten. “Electric-magnetic duality and the geometric Langlands program”. In: *Communications in Number Theory and Physics* 1.1 (2007), pp. 1–236. ISSN: 1931-4531.
- [LL] J.-c. Lee and S. Lee. “Geometry of relatively framed  $G$ -Higgs bundles”. In preparation.
- [LL20] J.-c. Lee and S. Lee. “Semi-polarized meromorphic Hitchin and Calabi–Yau integrable systems”. [arxiv.org/abs/2010.15812](https://arxiv.org/abs/2010.15812). 2020.
- [Mar94] E. Markman. “Spectral curves and integrable systems”. In: *Compositio mathematica* 93.3 (1994), pp. 255–290.
- [Ngô06] B. C. Ngô. “Fibration de Hitchin et endoscopie”. In: *Inventiones mathematicae* 164.2 (2006), pp. 399–453.
- [Ngô10] B. C. Ngô. “Le lemme fondamental pour les algèbres de Lie”. In: *Publications Mathématiques de l’IHÉS* 111.1 (2010), pp. 1–169.

- [PS08] C. A. Peters and J. H. Steenbrink. *Mixed Hodge structures*. Vol. 52. Springer Science & Business Media, 2008.
- [PS15] V. Przyjalkowski and C. Shramov. “On Hodge Numbers of Complete Intersections and Landau–Ginzburg Models”. In: *International Mathematics Research Notices* 2015.21 (2015), pp. 11302–11332.
- [Sei08] P. Seidel. “ $A_\infty$ -subalgebras and natural transformations”. In: *Homology Homotopy Appl.* 2 (2008), pp. 83–114.
- [Sei12] P. Seidel. “Fukaya  $A_\infty$ -structures associated to Lefschetz fibrations”. In: *I. J. Symplectic Geom.* 10(3) (2012), pp. 325–388.
- [Sha18] Y. Shamoto. “Hodge–Tate Conditions for Landau–Ginzburg Models”. In: *European Mathematical Society* 54.3 (2018), pp. 469–515.
- [Sim94a] C. T. Simpson. “Moduli of representations of the fundamental group of a smooth projective variety I”. In: *Publications Mathématiques de l’Institut des Hautes Études Scientifiques* 79.1 (1994), pp. 47–129.
- [Sim94b] C. T. Simpson. “Moduli of representations of the fundamental group of a smooth projective variety. II”. In: *Publications Mathématiques de l’Institut des Hautes Études Scientifiques* 80 (1994), pp. 5–79.
- [Slo80] P. Slodowy. *Four lectures on simple groups and singularities*. Mathematical Institute, Rijksuniversiteit, 1980.

- [Smi15] I. Smith. “Quiver algebras as Fukaya categories”. In: *Geometry & Topology* 19.5 (2015), pp. 2557–2617.
- [SSU87] M.-H. Saito, M.-H. Shimizu, and S. Usui. “Variation of Mixed Hodge Structure and the Torelli Problem”. In: *Advanced Studies in Pure Mathematics* 10 (1987), pp. 649–693.
- [SYZ96] A. Strominger, S.-T. Yau, and E. Zaslow. “Mirror symmetry is T - duality”. In: *Nucl. Phys. B* 479 (1996), pp. 243–259.
- [Zuc79] S. Zucker. “Hodge theory with degenerating coefficients.  $L_2$  cohomology in the Poincaré metric.” In: *Ann. of Math. (2)* 109(3) (1979), pp. 415–476.