MIXED CARLITZ MOTIVES AND COLORED MULTIZETA VALUES IN CHARACTERISTIC P

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ABSTRACT

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This thesis studies characteristic p multizeta values, which are function field analogs of the Euler-Riemann multizeta values. The objective of this thesis is two-fold. We first explicitly construct the category of mixed Carlitz motives, which is a counterpart to the category of mixed Tate motives in characteristic zero. After that, we identify specific mixed Carlitz motives, and use them to derive algebraic independence properties of colored multizeta values. The former includes all known t-motives related to multizeta values, while the latter complements results in the literature on algebraic relations between multizeta values.

Contents

1 Introduction **1**

1.1 Overview of classical case 1

1.2 Outline 3

2 Background **6**

2.1 List of notations 6

2.2 Definitions of t-motives 8

2.3 Uniformizable t-motives 15

2.4 The motivic Galois group 19

2.5 Aside: Some aspects of Hodge-Pink theory 20

3 Mixed Carlitz Motives **24**

3.1 The category of mixed Carlitz motives 24

3.2 Explicit period computations for mixed Carlitz motives 26

4 Application to Colored Multizeta Values **30**

4.1 Multizeta values and multipolylogarithms 30

4.2 Some combinatorial properties 37

4.3 Linear relations on polylogarithms 41

4.4 Algebraic independence of multipolylogarithms 47

4.5 Implications on colored multizeta values 53

4.6 Remarks on linear relations 64

5 Future Directions 71

5.1 Adelic multizeta values 71

5.2 Some other classes of multizeta values 73

Chapter 1

Introduction

1.1 Overview of classical case

The *Euler-Riemann multizeta value* is defined for a list of positive integers $s_1, \ldots, s_r$, with $s_1 \geq 2$, as

$$\zeta^{ER}(s_1, \ldots, s_r) = \sum_{n_1 > \cdots > n_r \geq 1} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}}.$$

More generally, for a positive integer N, let μ_N be the N^{th} roots of unity. Then the *Euler-Riemann colored multizeta value* for $\varepsilon_1, \ldots, \varepsilon_r \in \mu_N$ and $s_1, \ldots, s_r \in \mathbb{Z}_{\geq 1}$, with $(\varepsilon_1, s_1) \neq (1, 1)$, is defined in [26] as

$$\zeta^{ER}_{s_1,\ldots,s_r}(\varepsilon_1, \ldots, \varepsilon_r) = \sum_{n_1 > \cdots > n_r \geq 1} \frac{\varepsilon_1^{n_1} \cdots \varepsilon_r^{n_r}}{n_1^{s_1} \cdots n_r^{s_r}}.$$

These special values has connections to physics and enumeration problems, and a main goal is to understand algebraic relations between the Euler-Riemann multizeta values. Many relations between them are known, and one can either construct these relations combinatorially, or understand them better by reinterpreting them as periods of mixed Tate motives (see [33] for a collection of examples). However, almost nothing is known about algebraic independence. For example, we still do not know if $\zeta^{ER}(3)$ is transcendental!

Despite this, conjectures on algebraic relations between Euler-Riemann multizeta values has been formulated. For example, it is believed that the values $\pi, \zeta^{ER}(3), \zeta^{ER}(5), \ldots$ at odd positive integers greater than 1 are algebraically independent over $\mathbb{Q}$, and so is the set of multiple-zeta values $\zeta^{ER}(s_1, \ldots, s_r)$ with $s_i \in \{2, 3\}$ and $s_1 \cdots s_r$ forming a *Lyndon word* (see [33, Conjecture 12]). As far as we know, no such conjectures has been formulated for colored multizeta values.

Analogs of these multizeta values in the function field case has been defined, and are called *Thakur's multizeta values* and *colored multizeta values* respectively in this thesis. In stark contrast to the number field case, not many explicit linear relations between them are known, but one can construct arbitrarily large subsets of multizeta values that are algebraically independent. These questions in the case of Thakur's zeta values are solved completely in [9], and work has been done for multizeta values in higher depth. One consequence of this thesis is a generalization of these results to colored multizeta values.

1.2 Outline

In this thesis we will study the structure of certain t-motives and apply them to algebraic independence of multizeta values in characteristic p. The idea is to carry out a function field analog of a similar program in the case of number fields; see [16, 17] for an exposition.

Section 2 contains some background in t-motives that is essential to our discussion. Of upmost importance is the concept of uniformizability (Section 2.3), which is needed to interpret multizeta values as periods of certain t-motives. The main tool to analyze multizeta values after this interpretation is the motivic Galois group (Section 2.4). A general framework for the concepts surrounding t-motives is Hodge-Pink theory, and we will highlight some aspects of it that is related to our discussions later (Section 2.5).

Section 3 defines mixed Carlitz motives (Definition 3.1.1). This is a subclass of those t-motives that are successive extensions of tensor products of the Carlitz module. The category of mixed Carlitz motives is motivated from Hodge-Pink theory and the examples in [3, 5, 9, 19, 22], and all the special values studied in this thesis (and in the literature) can be realized as periods of certain mixed Carlitz motives. This category satisfies basic properties such as closure under tensor products and direct sums (Proposition 3.1.2), and we explicitly compute the period matrices for them (Theorem 3.2.2).

Section 4 can be split into two parts. The first half defines our main object of study: multizeta values and multipolylogarithms (Definition 4.1.2). These multizeta values are special values of Goss's analytic continuation of the Euler-Riemann multizeta function for function fields in [18], and generalizes Thakur's definition in [27]. We will discuss some geometrical and combinatorial properties of them, such as their interpretations as periods of mixed Carlitz motives, their shuffle relations, and the decomposition of multizeta values into multipolylogarithms (Sections 4.1 and 4.2). The next half studies linear independence properties of multipolylogarithms by way of the motivic Galois group (Sections 4.3 and 4.4), and applies this to infer algebraic independence of large classes of *colored multizeta values*, defined by

$$\zeta_{s_1,\ldots,s_r}(\varepsilon_1,\ldots,\varepsilon_r) = \sum_{\substack{\deg(a_1)>\cdots>\deg(a_r)\geq 0 \\ a_i\in A_+}} \frac{\varepsilon_1^{\deg(a_1)}\cdots\varepsilon_r^{\deg(a_r)}}{a_1^{s_1}\cdots a_r^{s_r}},$$

where s_i are positive integers, $\varepsilon_i \in \mathbb{F}_q^\times$ and A_+ denotes the set of monic polynomials in $\mathbb{F}_q[\theta]$. Thakur's multizeta values are the colored multizeta values with $\varepsilon_i = 1$ for all i. Furthermore, in case $r = 1$ Carlitz had indirectly computed that $\zeta_{(q-1)n}(1)$ is a $\mathbb{F}_q(\theta)$-rational multiple of $\tilde{\pi}^{(q-1)n}$, where $\tilde{\pi}$ is the "fundamental period" of the so-called Carlitz module. Our main results concerning colored multizeta values are stated in Section 4.5 (Theorems 4.5.9, 4.5.12, 4.5.13, and a recipe before Theorem 4.5.12), and will follow from the steps carried out in proving the following.

Theorem (c.f. Corollary 4.5.10)**.** *Let* $\mathcal{Z}_n = \{\tilde{\pi}, \zeta_s(\varepsilon) : 1 \leq s \leq n$ *and* $\varepsilon \in \mathbb{F}_q^\times\}$.

Then

$$\operatorname{trdeg}_{\overline{k}} \overline{k}(\mathcal{Z}_n) = 1 - \left\lfloor \frac{n}{q-1} \right\rfloor + \left\lfloor \frac{n}{p(q-1)} \right\rfloor + (q-1)\left(n - \left\lfloor \frac{n}{p} \right\rfloor\right).$$

The results we obtained are an extension of work done on Thakur's multizeta values in [9, 21, 34]. In particular, it subsumes results in these references as special cases, and we were motivated to obtain such a result due to recent work of [19] proving transcendence of colored multizeta values. After this, we conclude Section 4 with some remarks on writing down linear relations using known ones in [24, 30, 31] (Section 4.6).

Section 5 is the final section, and discusses some future directions of research stemming from our work.

Chapter 2

Background

2.1 List of notations

Below is a list of frequently used notation we will use throughout the thesis.

Notation	Meaning
q	a fixed power of a prime p
F	arbitrary field extension of $\mathbb{F}_q$ that is perfect
$F(t)[\sigma, \sigma^{-1}]$	ring defining t-motives (c.f. Definition 2.2.2)
A	$\mathbb{F}_q[\theta]$, with θ transcendental over $\mathbb{F}_q$
A_+	monic polynomials in A
k	$\mathbb{F}_q(\theta)$
k_∞	$\mathbb{F}_q((\frac{1}{\theta}))$, i.e. completion of k with respect to the valuation ∞
$\mathbb{C}_\infty$	completion at ∞ of a fixed algebraic closure of k_∞

$\overline{k}$	a fixed algebraic closure of k in $\mathbb{C}_\infty$
$\lvert \cdot \rvert$	a fixed absolute value on $\mathbb{C}_\infty$ with $\lvert\theta\rvert = q$
$.^{(i)}$	the i^{th} Frobenius twist (c.f. Definition 2.2.1)
$C^{\otimes n}$	Carlitz motive (c.f. Example 2.2.6)
$\Omega(t)$	Carlitz function (c.f. Definition 2.3.6)
$\tilde{\pi}$	Carlitz period; function field analog of π (c.f. Definition 2.3.7)
$\mathbb{T}$	Tate algebra of $\mathbb{C}_\infty[[t]]$
$\mathbb{L}$	fraction field of $\mathbb{T}$
l_i	$\prod_{j=1}^{i}(t-\theta^{q^j})$ if $i>0$, and 1 if $i=0$ (c.f. Definition 4.1.2)
$\mathcal{L}_i$	l_i evaluated at $t=\theta$ (c.f. Definition 4.1.2)
Φ	matrix for the σ-action of a t-motive (c.f. Definition 2.2.4)
Ψ	period matrix of a t-motive (c.f. Definition 2.3.4)
Γ_M	motivic Galois group of a t-motive M (c.f. Definition 2.3.9)
$\zeta_{\vec{s}}(\vec{x})$	multizeta values; MZ (c.f. Definition 4.1.2)
$\vec{\varepsilon}$	tuple of elements ε_i in $\mathbb{F}_q^\times$
γ_i	fixed $(q-1)^{st}$ root in $\mathbb{C}_\infty$ of an element $\varepsilon_i \in \mathbb{F}_q^\times$
$\mathrm{Li}_{\vec{s}}(\vec{\varepsilon},\vec{z})$	multipolylogarithms; MP (c.f. Definition 4.1.2)
r	depth of an arbitrary MZ or MP (c.f. Definition 4.1.2)
w	weight of an arbitrary MZ or MP (c.f. Definition 4.1.2)
Γ_s	Carlitz gamma (c.f. Definition 4.1.5)
$H_n(t)$	Anderson-Thakur polynomials (c.f Theorem 4.1.6)

$\zeta_{\vec{s}}(\vec{\varepsilon})$	colored multizeta values; CMZ (c.f. Definition 4.5.1)
$\mathrm{Li}_{\vec{s}}(\vec{\varepsilon}, \vec{u})$	colored multipolylogarithms; CMP (c.f. Definition 4.5.1)

2.2 Definitions of t-motives

The main goal of this section and the next is to sketch the main ideas of Papanikolas in [22], inserting in comments from other related papers as necessary. Let F be a field extension of $k = \mathbb{F}_q(\theta)$ that is perfect. The fields F we are most interested in is the function field complex numbers $\mathbb{C}_\infty$, and the algebraic closure $\overline{k}$ of k in $\mathbb{C}_\infty$.

Definition 2.2.1. For every $f = \sum_j f_j t^j \in F[[t]]$, define the i^{th} *Frobenius twist* to be

$$f^{(i)} := \sum_j f_j^{q^i} t^j.$$

Definition 2.2.2. The ring $F[t, \sigma]$ is the non-commutative ring defined by the relations

$$t\sigma = \sigma t, \qquad ft = tf, \qquad \sigma f = f^{(-1)}\sigma; \qquad f \in F[t].$$

The ring $F(t)[\sigma, \sigma^{-1}]$ is the ring consisting of finite sums $\sum_i f_i \sigma^i$ and satisfying the same relations above.

We define three kinds of t-motives over F in Definitions 2.2.3–2.2.5.

Definition 2.2.3. A *pre t-motive* is a left $F(t)[\sigma, \sigma^{-1}]$-module that is of finite dimension over $F(t)$. Morphisms between pre t-motives are left $F(t)[\sigma, \sigma^{-1}]$-module homomorphisms.

Definition 2.2.4. An *effective t-motive* is a left $F[t,\sigma]$-module M that is finite free over $F[t]$ and satisfying the following condition for one (and hence all) $F[t]$-basis $\mathbf{m}$ of M: if Φ is the matrix corresponding to the σ-action, so that $\sigma \cdot \mathbf{m} = \Phi\mathbf{m}$, then $\det \Phi = c(t-\theta)^s$ for some $c \in F^\times$ and nonnegative integer s. Morphisms between effective t-motives are left $F[t,\sigma]$-module homomorphisms.

Definition 2.2.5. An *Anderson t-motive* is an effective t-motive M that is also finite free over $F[\sigma]$, and satisfying

$$(t-\theta)^n M \subset \sigma M \text{ for } n >> 0.$$

Morphisms between Anderson t-motives are left $F[t,\sigma]$-module homomorphisms.

Remark. Our definitions of t-motives are sometimes called dual t-motives in the literature.

The tensor product of two pre t-motives M, M' over $F[t]$ is denoted $M \otimes M'$ with diagonal σ-action: $\sigma(m \otimes m') = \sigma(m) \otimes \sigma(m')$.

Example 2.2.6. Here are three key motives we will deal with.

- The trivial motive $\mathbb{1}$ is the free rank-one $F[t]$-module $F[t]$ with trivial σ-action $\sigma(f) = f^{(-1)}$. This is effective but not Anderson.
- The Carlitz motive C is the Anderson t-motive $F[t]$ with σ-action $\sigma(f) = (t-\theta)f^{(-1)}$.
- The motive $C^{\otimes n}$ is the n-fold tensor product of C over $F[t]$, which we also call a Carlitz motive. Here $\sigma(f) = (t-\theta)^n f^{(-1)}$.

Let $\mathcal{T}^o$ be the (exact) category of effective t-motives with left $F[t,\sigma]$-module homomorphisms as morphisms. Because of the noncommutativity of $F[t,\sigma]$, the Hom sets in $\mathcal{T}^o$ are modules over $\mathbb{F}_q[t]$ and not $F[t]$. In fact, for any effective t-motives $M, N \in \mathcal{T}^o$, the set $\mathrm{Hom}_{\mathcal{T}^o}(N, M)$ is a finite free $\mathbb{F}_q[t]$-module. This can be seen by a straightforward argument showing that the map

$$\mathrm{Hom}_{\mathcal{T}^o}(N, M) \otimes_{\mathbb{F}_q} F \longrightarrow \mathrm{Hom}_{F[t]}(N, M)$$

is injective, where the right hand side is the free $F[t]$-module consisting of all $F[t]$-linear maps from N to M.

Furthermore, the category $\mathcal{T}^o$ is a full subcategory of the category of all $F[t,\sigma]$-modules, but is not an abelian category: the multiplication by t map $t : \mathbb{1} \longrightarrow \mathbb{1}$ has trivial kernel and cokernel, but the morphism is not invertible. We will fix this later by enlarging the Hom set and defining a new category $\mathcal{T}$.

Now let N and M be effective t-motives, and denote their σ-actions by σ_N and σ_M. After fixing $F[t]$-bases $\mathbf{n}$ and $\mathbf{m}$, call their respective matrices $\Phi_{N,\mathbf{n}}$ and $\Phi_{M\mathbf{m}}$, so that

$$\sigma_N \cdot \mathbf{n} = \Phi_{N,\mathbf{n}}\mathbf{n}.$$

(Similarly $\sigma_M \cdot \mathbf{m} = \Phi_{M,\mathbf{m}}\mathbf{m}$). Note that N and M are determined up to isomorphism by $\Phi_{N,\mathbf{n}}$ and $\Phi_{M,\mathbf{m}}$.

Consider the group $\mathrm{Ext}^1_{\mathcal{T}^o}(N, M)$ of extensions of M by N, with group structure given by Baer sum. It is easy to check the following.

- An extension $[E] \in \operatorname{Ext}^1_{\mathcal{T}^o}(N, M)$ is specified by a matrix

$$\begin{bmatrix} \Phi_{M,\mathbf{m}} & 0 \\ \mathbf{e} & \Phi_{N,\mathbf{n}} \end{bmatrix}$$

for some e.

- If $[E], [E'] \in \operatorname{Ext}^1_{\mathcal{T}^o}(N, M)$ are classes corresponding to matrices

$$\begin{bmatrix} \Phi_{M,\mathbf{m}} & 0 \\ \mathbf{e} & \Phi_{N,\mathbf{n}} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \Phi_{M,\mathbf{m}} & 0 \\ \mathbf{e}' & \Phi_{N,\mathbf{n}} \end{bmatrix},$$

then $[E] + [E']$ is the class corresponding to the matrix

$$\begin{bmatrix} \Phi_{M,\mathbf{m}} & 0 \\ \mathbf{e} + \mathbf{e}' & \Phi_{N,\mathbf{n}} \end{bmatrix}.$$

- Let $0 \longrightarrow M \longrightarrow E \longrightarrow N \longrightarrow 0$ be an exact sequence corresponding to $[E] \in \operatorname{Ext}^1_{\mathcal{T}^o}(N, M)$. For any $a \in \mathbb{F}_q[t]$, consider the pushout diagram

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d, "a\cdot"'] & E \arrow[d] \\ M \arrow[r, hook] & M \sqcup_a E \end{tikzcd}$$

of t-motives. If $[M]$ corresponds to the matrix

$$\begin{bmatrix} \Phi_{M,\mathbf{m}} & 0 \\ \mathbf{e} & \Phi_{N,\mathbf{n}} \end{bmatrix},$$

then $[M \sqcup_a E]$ corresponds to the matrix

$$\begin{bmatrix} \Phi_{M,\mathbf{m}} & 0 \\ a\mathbf{e} & \Phi_{N,\mathbf{n}} \end{bmatrix}.$$

Hence $\mathrm{Ext}^1_{\mathcal{T}^o}(N, M)$ has an $\mathbb{F}_q[t]$-module structure defined by Baer sums and Cartesian pushouts.

Proposition 2.2.7. *Let M, N as above.*

(a) Fix a choice of $F[t]$-bases $\mathbf{n}, \mathbf{m}$ of N, M, and identify any homomorphism $u : N \longrightarrow M$ with the matrix $U \in \mathrm{Mat}_{n\times m}(F[t])$ satisfying $u(\mathbf{n}) = U\mathbf{m}$. Then there is a $\mathbb{F}_q[t]$-module isomorphism

$$\mathfrak{h}_{\mathbf{n},\mathbf{m}} \colon \mathrm{Hom}_{\mathcal{T}^o}(N, M) \xrightarrow{\sim} \left\{U \in \mathrm{Mat}_{n\times m}(F[t]) : \Phi_{N,\mathbf{n}} U = U^{(-1)}\Phi_{M,\mathbf{m}}\right\},$$

and $\mathrm{Hom}_{\mathcal{T}^o}(N \otimes C, M \otimes C) = \mathrm{Hom}_{\mathcal{T}^o}(N, M)$.

(b) There is an $\mathbb{F}_q[t]$-module isomorphism

$$\mathfrak{e}_{\mathbf{n},\mathbf{m}} \colon \mathrm{Ext}^1_{\mathcal{T}^o}(N, M) \xrightarrow{\sim} \frac{M^{\oplus n}}{(\sigma_M I_n - \Phi_{N,\mathbf{n}})M^{\oplus n}}$$

where $n = \mathrm{rank}_{F[t]} N$ and $m = \mathrm{rank}_{F[t]} M$. (The product $\Phi_{N,\mathbf{n}} M^{\oplus n}$ above is defined by identifying M with $\mathrm{Mat}_{1\times m}(F[t])$ via the chosen $F[t]$-basis $\mathbf{m}$ of M.)

Remark. Explicitly, writing $\mathbf{m} = (b_1, \ldots, b_m)$, we use the identification $\iota_{\mathbf{m}} \colon M \longrightarrow \mathrm{Mat}_{1\times m}(F[t])$ given by $\iota_{\mathbf{m}}(c_1b_1 + \cdots + c_mb_m) = (c_1, \ldots, c_m)$. This is extended to an identification $\iota_{\mathbf{m}} \colon M^{\oplus n} \longrightarrow \mathrm{Mat}_{n\times m}(F[t])$, and we make sense of multiplying $\Phi_{N,\mathbf{n}}$ by an element of $M^{\oplus n}$ by viewing this element as an element of $\mathrm{Mat}_{n\times m}(F[t])$ under $\iota_{\mathbf{m}}$.

Proof of Proposition 2.2.7. (a) This is a straightforward computation.

(b) Assume E is equivalent to $M \oplus N$ in $\mathrm{Ext}^1_{\mathcal{T}^o}(N, M)$. Then there is a commutative diagram as below.

$$\begin{tikzcd}
1 \arrow[r] & M \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "B"] & N \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & M \arrow[r] & M \oplus N \arrow[r] & N \arrow[r] & 1
\end{tikzcd}$$

In matrix notation, we require

$$B = \begin{bmatrix} I_m & 0 \\ V & I_n \end{bmatrix}, \qquad V \in \mathrm{Mat}_{n\times m}(F[t]),$$

and for B to be a morphism, by part (a) we require

$$\begin{bmatrix} \Phi_M & 0 \\ \mathbf{e} & \Phi_N \end{bmatrix} \begin{bmatrix} I_m & 0 \\ V & I_n \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ V^{(-1)} & I_n \end{bmatrix} \begin{bmatrix} \Phi_M & 0 \\ 0 & \Phi_N \end{bmatrix}.$$

This is equivalent to $V\Phi_M + \mathbf{e} = \Phi_N V^{(-1)}$. Thus E is equivalent to $M \oplus N$ if and only if $\mathbf{e}$ satisfies the relation

$$\mathbf{e} = -\Phi_N V + V^{(-1)}\Phi_M$$

for some $V \in \mathrm{Mat}_{n\times m}(F[t])$. We are done by observing that, for the basis $\mathbf{m}$ of M giving rise to Φ_M, there is an $\mathbb{F}_q[t]$-module homomorphism

$$\mathrm{Mat}_{n\times m}(F[t]) \longrightarrow \frac{M^{\oplus n}}{(\sigma_M I_n - \Phi_N)M^{\oplus n}}$$

$$\begin{bmatrix} — & \mathbf{x}_1 & — \\ & \vdots & \\ — & \mathbf{x}_n & — \end{bmatrix} \longmapsto (\mathbf{x}_1 \cdot \mathbf{m}, \ldots, \mathbf{x}_n \cdot \mathbf{m})$$

having kernel the $\mathbb{F}_q[t]$-module consisting of all matrices satisfying the same relation as $\mathbf{e}$ above. $\square$

Remark. Let us note the immediate corollary

$$\mathrm{Ext}^1_{\mathcal{T}^o}(\mathbb{1}, M) \cong M/(\sigma_M - 1)M,$$

which is mentioned in [12], and credited to an unpublished manuscript of Anderson. In [23], Papanikolas and Ramachandran has also given an interpretation of $\mathrm{Ext}^1_{\mathcal{T}^o}(N, M)$ using the language of biderivations.

Definition 2.2.8. Define the category $\mathcal{T}$ of effective t-motives with objects the same ones as $\mathcal{T}^o$ and Hom sets

$$\mathrm{Hom}_{\mathcal{T}^o}(-,-) \otimes_{\mathbb{F}_q[t]} \mathbb{F}_q(t).$$

Proposition 2.2.9 ([22, Proposition 3.4.5])**.** *For $M, N \in \mathcal{T}$, there is an isomorphism of $\mathbb{F}_q(t)$-spaces*

$$\mathrm{Hom}_{\mathcal{T}^o}(M, N) \otimes_{\mathbb{F}_q[t]} \mathbb{F}_q(t) \longrightarrow \mathrm{Hom}_{\textit{pre t-motives}}(M(t), N(t)),$$

where $M(t) = F(t) \otimes_{F[t]} M$ with σ-action $\sigma(\alpha \otimes m) = \alpha^{(-1)} \otimes \sigma(m)$ for all $m \in M$.

□

We can now define the internal Hom in the category of pre t-motives $\mathcal{P}$, defined as $\mathrm{Hom}_{F(t)}(M, N)$ with σ-action

$$\sigma(f) = \sigma_N \circ f \circ \sigma_M^{-1}.$$

(Note that this internal Hom cannot be constructed in $\mathcal{T}$, since an "inverse" to the Carlitz module does not exist there.) With this, we define the dual of an effective t-motive M by $M^\vee = \mathrm{Hom}_{F(t)}(M, \mathbb{1})$. The dual also lies in $\mathcal{P}$, and satisfies $M^{\vee\vee} = M$.

Example 2.2.10. The dual $C^\vee$ of the Carlitz motives C is an object in $\mathcal{P}$, and is isomorphic to $F(t)$ with σ-action

$$\sigma(f) = (t-\theta)^{-1} f^{(-1)}.$$

Let us also define the category $\mathcal{A}$ to be the full subcategory of $\mathcal{T}$ with objects consisting of Anderson t-motives. Then, by Proposition 2.2.9, there is a natural embedding $\mathcal{A} \hookrightarrow \mathcal{P}$.

Theorem 2.2.11 ([22, Theorem 3.4.9])**.** *The category $\mathcal{P}$ is a rigid abelian $\mathbb{F}_q(t)$-linear tensor category. If we write $\mathcal{P}$ as the category of pre t-motives, then $\mathcal{A} \hookrightarrow \mathcal{P}$ is fully faithful.* □

Proposition 2.2.12 ([25, Theorem 7.4.2])**.** *For all effective t-motives M and N,*

$$\mathrm{Ext}^1_{\mathcal{T}}(M,N) = \mathrm{Ext}^1_{\mathcal{T}^o}(M,N) \otimes_{\mathbb{F}_q[t]} \mathbb{F}_q(t)$$

$$\mathrm{Ext}^i_{\mathcal{T}}(M,N) = 0 \text{ for } i > 1.$$ □

2.3 Uniformizable t-motives

From now on till the end of the thesis, we will concentrate on the case $F = \overline{k}$. Let $\mathbb{T}$ be the Tate algebra of $\mathbb{C}_\infty[[t]]$, which is the subalgebra of $\mathbb{C}_\infty[[t]]$ consisting of all power series $\sum_{i\geq 0} c_i t^i$ in $\mathbb{C}_\infty[[t]]$ satisfying $|c_i| \to 0$ as $i \to \infty$. Also let $\mathbb{L}$ be the field of fractions of $\mathbb{T}$.

Definition 2.3.1. Let M be a pre t-motive over $\overline{k}$, and let $M^B := (M \otimes_{\overline{k}(t)} \mathbb{L})^\sigma$. Then M is *uniformizable* if the natural map

$$h_M : M^B \otimes_{\mathbb{F}_q(t)} \mathbb{L} \longrightarrow M \otimes_{\overline{k}(t)} \mathbb{L}$$

is an isomorphism.

The definition of uniformizability is a generalization of Anderson's notion of rigid analytic triviality, where he showed in [1] that the exponential map of an Anderson t-motive M is surjective if and only if M is uniformizable. Exponential maps are particular useful for us in the case of Carlitz motives as it gives us a computation of Carlitz's zeta values at "even" integers; see the discussion before Proposition 4.5.5.

Proposition 2.3.2 ([22, Proposition 3.3.9])**.** *Assume that the Anderson t-motive M has rank m, and fix a $F[t]$-basis* $\mathbf{m}$ *of M. If $\Phi \in \mathrm{Mat}_{m\times m}(\overline{k}[t])$ is the matrix corresponding to the σ-action of M with respect to* $\mathbf{m}$, *then M is uniformizable if*

$$\Psi^{(-1)} = \Phi\Psi$$

for some $\Psi \in \mathrm{GL}_m(\mathbb{L})$. Furthermore, the entries of $\Psi^{-1}\mathbf{m}$ forms a k-basis for M^B.

□

Remark. In the remainder of this thesis, if $\Psi^{(-1)} = \Phi\Psi$, we will say that Φ *is uniformizable by* Ψ, or that Ψ *is a uniformizer of* Φ.

Proposition 2.3.3 ([2, Proposition 3.1.3])**.** *Let $\mathbb{E}$ be the subring of the Tate algebra $\mathbb{T}$ consisting of all elements that is entire on $\mathbb{C}_\infty$. If M is a uniformizable t-motive,* *then there exists a* Ψ *as above such that* $\Psi \in \mathrm{GL}_m(\mathbb{E})$. *In particular,* $\Psi \in \mathrm{GL}_m(\mathbb{T})$.

$\square$

Definition 2.3.4. The Ψ in the above proposition is called a *period matrix* of a uniformizable t-motive M with respect to Φ. The *periods* of Ψ are the entries of $\Psi^{-1}|_{t=\theta}$.

Example 2.3.5. The trivial motive $\mathbb{1}$ is uniformizable with $\Psi = [1]$.

Example 2.3.6. The Carlitz motive C is uniformizable. Carlitz [6] indirectly constructed the $\mathbb{C}_\infty$-entire function

$$\Omega = \Omega(t) = (-\theta)^{-\frac{q}{q-1}} \prod_{i=1}^{\infty} \left(1 - \frac{t}{\theta^{(i)}}\right)$$

and showed that $\Omega^{(-1)} = (t-\theta)\Omega$. (Here we fix a choice of $(q-1)^{st}$ root for $-\theta$.) Thus the matrix $\Psi = [\Omega]$ uniformizes C, as $\Phi = [t-\theta]$ with respect to the basis $\{1\}$. Similarly, $\Psi = [\Omega^n]$ uniformizes $C^{\otimes n}$ as $\Phi = [(t-\theta)^n]$.

It is possible for a t-motive to be non-uniformizable, though the constructions of these are contrived; see [1, Section 2.2] for an example.

Here is a definition that will be useful later on.

Definition 2.3.7. The *Carlitz period* is defined to be

$$\tilde{\pi} := \frac{1}{\Omega(\theta)} = -\theta(-\theta)^{\frac{1}{q-1}} \prod_{i=1}^{\infty} \left(1 - \frac{\theta}{\theta^{(i)}}\right)^{-1},$$

where we fix a choice of $(q-1)^{st}$ root for $-\theta$.

In [32], it was proven that the Carlitz period is transcendental over $\mathbb{F}_q(\theta)$.

Theorem 2.3.8 ([22, Theorem 3.3.15])**.** *Let $\mathcal{R}$ be the category of uniformizable pre t-motives over $\overline{k}$. Then $\mathcal{R}$ is a neutral Tannakian category with fiber functor $\omega\colon P \longmapsto P^B$, where P^B is defined in Definition 2.3.1.* $\square$

Following [22], define $\mathcal{S}$ to be the strictly full Tannakian subcategory of $\mathcal{R}$ generated by the Anderson t-motives $\mathcal{A}$.

Definition 2.3.9. Let $M \in \mathcal{S}$, and let $\mathcal{S}_M$ be the strictly full Tannakian subcategory of $\mathcal{S}$ generated by M. (That is, $\mathcal{S}_M$ consists of all objects in $\mathcal{S}$ isomorphic to subquotients and finite direct sums of t-motives of the form $M^{\otimes\alpha} \otimes (M^\vee)^{\otimes\beta}$.) By the Tannakian formalism, the *motivic Galois group* $\Gamma = \Gamma_M$ of M is defined to be the affine group scheme over $\mathbb{F}_q(t)$ such that, for every commutative algebra R over $\mathbb{F}_q(t)$, the group of R-points of Γ_M is

$$\Gamma_M(R) = \mathrm{Aut}^{\otimes}_{\mathcal{S}_M}(\omega^{(R)}),$$

where $\omega^{(R)}$ is the base change of ω to R over $\mathbb{F}_q(t)$. This is a linear algebraic group over $\mathbb{F}_q(t)$.

The next section explain how one can compute Γ_M explicitly via difference equations.

2.4 The motivic Galois group

In this section we summarize the main facts on computing the motivic Galois group; details can be found in [22]. We fix the following notations:

- $J = \mathbb{F}_q(t)$ and $K = \overline{k}(t)$ and $L = \mathbb{L}$;
- M is a fixed effective t-motive over $\overline{k}$ of rank $r+1$;
- Γ_M is the motivic Galois group of M obtained via Tannakian formalism;
- Φ is the matrix defining the σ-action of M after fixing a basis;
- Ψ is a period matrix of M satisfying the uniformizability rule $\Psi^{(-1)} = \Phi\Psi$.

Let Ψ_{ij} be the $(i,j)^{th}$ entry of Ψ. We make the following two definitions.

- Consider the K-algebra map

$$\begin{aligned} v\colon K[X_{ij}, \frac{1}{\det(X)}] &\longrightarrow L \\ X_{ij} &\longmapsto \Psi_{ij}. \end{aligned}$$

  We define Z to be the schematic closure of the map $\mathrm{Spec}(v)\colon \mathrm{Spec}(L) \longrightarrow \mathrm{GL}_{r+1}\big/_K$. In other words, $Z = \mathrm{Spec}(K[X_{ij}, \frac{1}{\det(X)}]/\ker v)$, and is a closed subgroup scheme of $\mathrm{GL}_{r+1}\big/_K$.

- Consider the matrices $\Psi_1, \Psi_2 \in \mathrm{GL}_{r+1}(L \otimes_k L)$ defined by

$$(\Psi_1)_{ij} = \Psi_{ij} \otimes 1, \qquad (\Psi_2)_{ij} = 1 \otimes \Psi_{ij}.$$

Now let $\tilde{\Psi} = \Psi_1^{-1}\Psi_2$, and consider the J-algebra map

$$\mu\colon J[X_{ij}, \frac{1}{\det(X)}] \longrightarrow L \otimes_k L$$

$$X_{ij} \longmapsto \tilde{\Psi}_{ij}.$$

We define Γ to be the schematic closure of the map $\operatorname{Spec}(\mu)\colon \operatorname{Spec}(L\otimes_k L) \longrightarrow \mathrm{GL}_{r+1}/J$. In other words, $\Gamma = \operatorname{Spec}(J[X_{ij}, \frac{1}{\det(X)}]/\ker\mu)$, and is a closed subgroup scheme of GL_{r+1}/J.

Theorem 2.4.1 ([22])**.** *The following are true.*

- Γ *is isomorphic to* Γ_M *over* J*.*
- *For any positive integer* n*, there is a natural* J*-isomorphsm* $\varphi_n\colon \Gamma_{C^{\otimes n}} \xrightarrow{\sim} \mathbb{G}_m$*.*
- *Let* E *be the subfield of* $\mathbb{C}_\infty$ *generated by the entries of* $\Psi|_{t=\theta}$ *over* $\overline{k}$*. Then*

$$\operatorname{trdeg}_{\overline{k}} E = \dim \Gamma_M.$$

- Z *is stable under right multiplication by* $K \times_J \Gamma$*, and is a torsor for* $K \times_J \Gamma$ *over* K*.*
- Γ *is geometrically connected and smooth over* $\overline{J} = \overline{\mathbb{F}_q(t)}$*.* □

2.5 Aside: Some aspects of Hodge-Pink theory

The final section of the background will be devoted to summarizing some aspects of Hodge-Pink theory [20], applied to the case of t-motives. This will not be needed for the remainder of the thesis, and is here to indicate the existence of a framework generalizing t-motives and uniformization.

In this section, we will briefly discuss purity and mixedness, as well as the comparison theorem between Betti and de-Rham cohomology.

Definition 2.5.1. Let M be a pre t-motive over $\overline{k}$, and let

$$\hat{M} = M_{\mathbb{C}_\infty} \otimes_{A_{\mathbb{C}_\infty}} \mathbb{C}_\infty,$$

where $M_{\mathbb{C}_\infty} = \mathbb{C}_\infty \otimes_{\overline{k}[t]} M$ and $A_{\mathbb{C}_\infty} = \mathbb{C}_\infty \otimes_{\mathbb{F}_q} A$.

- M is *pure* if there exists integers d and r, with $r > 0$, and a $\mathbb{C}_\infty$-lattice L of $\hat{M}$ such that

$$\sigma^r L = t^d L.$$

  In this case, the *weight* of M is defined to be $\mathrm{wt}(M) = -d/r$.

- M is *mixed* if it possesses an increasing weight filtration by pre t-motives $W_\mu M$ indexed by $\mu \in \mathbb{Q}$, such that each graded pieced $\mathrm{Gr}_\mu M = W_\mu M / \bigcup_{\mu' < \mu} W_{\mu'} M$ is a pure pre t-motive of weight μ, and $\sum_\mu \mathrm{rank}_{\overline{k}(t)} \mathrm{Gr}_\mu M = \mathrm{rank}_{\overline{k}(t)} M$.

It is clear that tensor products of two pure pre t-motives M, M' (over the base ring $F[t]$) is still pure: if

$$\sigma^r L = t^d L \qquad \text{and} \qquad \sigma^{r'} L' = t^{d'} L',$$

then $L \otimes L'$ is a lattice of $M \otimes M'$ satisfying

$$\sigma^{dd'}(L \otimes L') = t^{rd' + r'd}(L \otimes L').$$

In particular $\mathrm{wt}(M \otimes M') = \mathrm{wt}(M) + \mathrm{wt}(M')$. Using this fact, the tensor product of two mixed pre t-motives N, N' is again mixed, with weight filtration

$$W_\mu(N \otimes N') = \sum_{\mu'+\mu''=\mu} (W_\mu N) \otimes (W_\mu N').$$

More properties on purity and mixedness can be found in [20, Proposition 4.10]. We highlight two important ones. Firstly, any morphism between mixed pre t-motives as $\overline{k}(t)[\sigma, \sigma^{-1}]$-modules actually respects the weight filtration, so we can use the same definition of morphisms in Section 2.2 for the mixed case. Secondly, the weight filtration of a mixed t-motive M is uniquely determined by M itself.

Example 2.5.2. By the uniformization theorem for Drinfeld modules (see [27, Theorem 2.4.2]), any Anderson t-motive M over $\overline{k}$ with $\operatorname{rank}_{\overline{k}[\sigma]} M = 1$ is pure of weight $-1/r$, where $r = \operatorname{rank}_{\overline{k}[t]} M$. In particular, by the above discussion, the Carlitz motive $C^{\otimes n}$ of Example 2.2.6 is pure of weight $-n$.

It turns out that mixed uniformizable pre t-motives also forms a neutral Tannakian category.

Theorem 2.5.3 ([20, Theorem 4.23])**.** *Let $\mathcal{MP}$ be the category of mixed uniformizable pre t-motives. Then $\mathcal{MP}$ is a neutral Tannakian category with fiber functor $\omega\colon P \longmapsto P^B$, where P^B is defined in Definition 2.3.1.* □

Let M be a mixed uniformizable pre t-motive. Then $\dim_{\mathbb{F}_q(t)} M^B = \dim_{\overline{k}(t)} M$ by [20, Lemma 4.16] or [22, Proposition 3.3.8]. In particular, M^B is of finite $\mathbb{F}_q(t)$-rank, and we can define Betti and de Rham cohomologies as follows.

Definition 2.5.4. For a mixed uniformizable pre t-motive M, its *Betti cohomology* and *de Rham cohomology* are defined to be

$$H_B(M, \mathbb{C}_\infty) := M^B \otimes_{\mathbb{F}_q(t)} \mathbb{C}_\infty \qquad \text{and} \qquad H_{dR}(M, \mathbb{C}_\infty) := M_{\mathbb{C}_\infty}/JM_{\mathbb{C}_\infty},$$

where $M_{\mathbb{C}_\infty} = \mathbb{C}_\infty \otimes_{\overline{k}[t]} M$, and J is the maximal ideal in A_R generated by $\theta \otimes 1 - 1 \otimes \theta$.

With this, the uniformization map h_M in Definition 2.3.1 can be reformulated in Hodge-Pink theory as an analog of the comparison theorem between Betti and de Rham cohomology.

Theorem 2.5.5 ([20, Lemma 4.18, Theorem 4.36])**.** *Let M is a mixed uniformizable pre t-motive. Then there is a canonical isomorphism*

$$h_{B,dR} : H_B(M, \mathbb{C}_\infty) \longrightarrow H_{dR}(M, \mathbb{C}_\infty)$$

defined by $h_{B,dR} := h_M \pmod{J}$, where h_M is the uniformization map in Definition 2.3.1. Furthermore, if we choose a basis for M with the matrix for its σ-action being Φ, and if Ψ is its period matrix, then $h_{B,dR}$ can be written in coordinates as Ψ^{-1}.

□

Chapter 3

Mixed Carlitz Motives

3.1 The category of mixed Carlitz motives

In this section we propose a definition for the function field counterpart of mixed Tate motives. This category is motivated from the examples in [3, 5, 9, 19, 22], and also includes our examples in Section 4.1.

Definition 3.1.1. Let M be an object in the category $\mathcal{P}$ satisfying the following condition: there exists $n \in \mathbb{Z}$ such that $M \otimes C^{\otimes n}$ is $\overline{k}$-isomorphic to the object in

$\mathcal{P}$ attached to an Anderson t-motive represented by a σ-matrix of the form

$$\Phi = \begin{bmatrix} (t-\theta)^{n_1} & & & & \\ a_{21}(t-\theta)^{n_1} & (t-\theta)^{n_2} & & & \\ a_{31}(t-\theta)^{n_1} & a_{32}(t-\theta)^{n_2} & (t-\theta)^{n_3} & & \\ \vdots & \vdots & \vdots & \ddots & \\ a_{r1}(t-\theta)^{n_1} & a_{r2}(t-\theta)^{n_2} & a_{r3}(t-\theta)^{n_3} & \cdots & (t-\theta)^{n_r} \end{bmatrix}$$

where $n_i \in \mathbb{Z}_{\geq 0}$ for all i, and $a_{ij} \in \overline{k}[t]$ for all i, j. Such a pre t-motive M is a *mixed Carlitz motive* if Φ further satisfies the following two conditions:

- the natural numbers n_i satisfy $n_1 \geq n_2 \geq \cdots \geq n_r \geq 0$;
- if $n_j = n_{j+1} = \cdots = n_{j+l}$, then $a_{uv} = 0$ for $j \leq v < u \leq j+l$.

Note that a mixed Carlitz motive is an object in the category of successive extensions of tensor products of Carlitz motives. Let $\mathcal{C}$ be the category of mixed Carlitz motives, with Hom sets defined by

$$\mathrm{Hom}_{F[t]}(-,-) \otimes_{\mathbb{F}_q[t]} \mathbb{F}_q(t).$$

Proposition 3.1.2. *A mixed Carlitz motive is mixed in the sense of Definition 2.5.1. Furthermore:*

- *the direct sum and tensor product of two mixed Carlitz motives is still a mixed Carlitz motive;*
- *$\mathcal{C}$ is a rigid $\mathbb{F}_q(t)$-linear tensor category;*
- *the map $\mathcal{C} \hookrightarrow \mathcal{MP}$ from $\mathcal{C}$ to the category $\mathcal{MP}$ of mixed uniformizable pre t-motives is fully faithful, and $\mathcal{C}$ is a full Tannakian subcategory of $\mathcal{MP}$.*

Proof. The first assertion is clear by the discussion in Section 2.5 and straightforward computation with the matrices defining mixed Carlitz motives. The proof of the second and third assertions are the same as the ones given in [22] for Proposition 2.2.9 and Theorem 2.2.11, since morphisms respects weight filtrations by [20, Proposition 4.10(g)]. □

3.2 Explicit period computations for mixed Carlitz motives

In this section, we explicitly compute the period matrix for a mixed Carlitz motive. This is motivated by the computations of [3].

Lemma 3.2.1. *Let $U \in \mathrm{Mat}_r(\overline{k}[t])$ be a square matrix with nonzero determinant. Consider the $\mathbb{F}_q[t]$-module*

$$S = \{V \in \mathrm{GL}_r(\mathbb{T}) : V^{(-1)} = UV\}.$$

If S is nonempty, then S is a $\mathrm{GL}_r(\mathbb{F}_q[t])$-torsor.

Proof. Fix $W \in S$. For any other T in S, consider

$$G = W^{-1}T.$$

Then, by invertibility of U in $\overline{k}(t)$, one gets

$$G^{(-1)} = G.$$

Note that the entries of G are still in $\mathbb{T}$, so the above is only possible if G has entries in $\mathbb{F}_q[t]$. Thus $T = WG$ with $G \in \mathrm{GL}_r(\mathbb{F}_q[t])$. $\square$

Theorem 3.2.2. *Mixed Carlitz motives over $\overline{k}$ are uniformizable.*

Remark. In fact, the period matrix of a mixed Carlitz motive can be explicitly determined from its σ-matrix. This will be demonstrated in the procedure described in the proof.

Proof of Theorem 3.2.2. Preserve the notations of Definition 3.1.1. We can replace M by $M \otimes C^{\otimes n}$ for $n >> 0$ without loss of generality, so as to ensure $n_i \geq 0$ in the matrix Φ. The proof proceed by induction on the $\overline{k}[t]$-rank r of our mixed Carlitz motive M.

As the case $r = 0$ is trivial, assume $r \geq 1$. Then M can be written in matrix form as

$$\left[\begin{array}{c|c} (t-\theta)^{n_1} & \\ \hline a_{21}(t-\theta)^{n_1} & \\ \vdots & \Phi_s \\ a_{r1}(t-\theta)^{n_1} & \end{array}\right]$$

where Φ_s is a mixed Carlitz submotive. By induction there exists $\Psi_s \in \mathrm{GL}_{r-1}$ with

$\Psi_s^{(-1)} = \Phi_s \Psi_s$. We now seek a matrix Ψ of the form

$$\left[\begin{array}{c|c} \beta_1 & \\ \hline \beta_2 & \\ \vdots & \Psi_s \\ \beta_r & \end{array}\right]$$

such that $\beta_i \in \mathbb{T}$ and $\Psi^{(-1)} = \Phi\Psi$. Thus we need

$$\beta_1^{(-1)} = (t-\theta)^{n_1}\beta,$$

and by Lemma 3.2.1 we can pick $\beta_1 = \Omega^{n_1}$. A computation tells us that necessarily

$$\begin{bmatrix} \beta_2^{(-1)} \\ \vdots \\ \beta_r^{(-1)} \end{bmatrix} = \Omega^{n_1} \begin{bmatrix} a_{21}(t-\theta)^{n_1} \\ \vdots \\ a_{r1}(t-\theta)^{n_1} \end{bmatrix} + \Phi_s \begin{bmatrix} \beta_2 \\ \vdots \\ \beta_r \end{bmatrix}.$$

Writing

$$\Phi_s = \begin{bmatrix} s_{2,1} & & \\ \vdots & \ddots & \\ s_{r,1} & \cdots & s_{r,r-1} \end{bmatrix},$$

with $s_{i,j} \in \overline{k}[t]$, we need to solve the equations

$$\begin{aligned} \beta_2^{(-1)} &= a_{21}\Omega^{n_1}(t-\theta)^{n_1} + s_{2,1}\beta_2 \\ \beta_3^{(-1)} &= a_{31}\Omega^{n_1}(t-\theta)^{n_1} + s_{3,1}\beta_2 + s_{3,2}\beta_3 \\ &\vdots \\ \beta_r^{(-1)} &= a_{r1}\Omega^{n_1}(t-\theta)^{n_1} + s_{r,1}\beta_2 + s_{r,r-1}\beta_r. \end{aligned}$$

Let us solve for β_2 first. Let $v = \max\{1, |s_{21}|\}$, where $|s_{21}|$ is the maximum among all the absolute values of its coefficients. Then, as Ω is an entire function in $\mathbb{T}$, we can write $a_{21}\Omega^{n_1}(t-\theta)^{n_1} = A(t) + B(t)$, where

- $A(t)$ is a polynomial in $\overline{k}[t]$;
- $B(t)$ is an element in $\mathbb{T}$ with coefficients all having valuation at most $1/v^2$.

Then a solution to β_2 is $\beta_2 = \beta_2^1 + \beta_2^2$, where

- β_2^1 is a solution to $x^{(-1)} = A(t) + s_{2,1}x$, which exists in $\overline{k}[t]$;
- β_2^2 is a solution to $x^{(-1)} = B(t) + s_{2,1}x$.

Solve for β_2^2 by doing a telescoping sum to obtain

$$\beta_2^2 = B^{(1)}(t) + s_{2,1}^{(1)}B^{(2)}(t) + s_{2,1}^{(1)}s_{2,1}^{(2)}B^{(3)}(t) + \cdots ,$$

which converges by assumption on $B(t)$. The rest of the the β_i are solved iteratively in the same fashion.

By Lemma 3.2.1, once we have computed such a uniformizer Ψ for Φ, we can obtain any other uniformizer via right multiplication with a matrix in $\mathrm{GL}_r(\mathbb{F}_q[t])$. If we choose another set of basis for M, the associated σ-matrix Φ transforms by $\Phi \longrightarrow S^{(-1)}\Phi S^{-1}$ for some matrix S, and so its associated uniformizer changes by $\Psi \longrightarrow S^{-1}\Psi$. $\square$

Chapter 4

Application to Colored Multizeta Values

4.1 Multizeta values and multipolylogarithms

We finally define multizeta values and multipolylogarithms. The latter has already been defined in [7] in relation to Thakur's multizeta values (i.e. for $\epsilon_i = 1$). For easier notation, we introduce the following product.

Definition 4.1.1. For any nonnegative integer i, define an element $l_i \in A[t] = \mathbb{F}_q[\theta][t]$ by

$$l_i := \begin{cases} \prod_{j=1}^{i}(t-\theta^{(j)}) & \text{if } i > 0; \\ 1 & \text{if } i = 0. \end{cases}$$

Also define $\mathcal{L}_i := l_i|_{t=\theta}$.

In the definition below, our absolute value $|\cdot|$ is normalized by $|\theta| = q$.

Definition 4.1.2. Fix a positive integer r, and let $\vec{s} = (s_1, \ldots, s_r)$ be a list of positive integers.

- For $\vec{x} \in (\overline{k}^{\times})^r$ satisfying $|x_i| = 1$ for all i, the *multizeta value* (or MZ) $\zeta_{\vec{s}}(\vec{x})$ is the element of $\mathbb{C}_\infty$ defined by

$$\zeta_{\vec{s}}(\vec{x}) := \sum_{\substack{\deg(a_1) > \cdots > \deg(a_r) \geq 0 \\ a_i \in A_+}} \frac{x_1^{\deg(a_1)} \cdots x_r^{\deg(a_r)}}{a_1^{s_1} \cdots a_r^{s_r}},$$

  where A_+ denotes the set of monic polynomials in $A = \mathbb{F}_q[\theta]$.

- For $\vec{\varepsilon} \in (\mathbb{F}_q^{\times})^r$ and $\vec{z} \in (\overline{k}^{\times})^r$ satisfying $|z_i| < q^{\frac{qs_i}{q-1}}$ for all i, define an element of the Tate algebra $\mathbb{T}$ by

$$\mathrm{L}_{\vec{s}}(\vec{\varepsilon}, \vec{z}) := \sum_{i_1 > \cdots > i_r \geq 0} \frac{\varepsilon_1^{i_1} z_1^{(i_1)} \cdots \varepsilon_r^{i_r} z_r^{(i_r)}}{l_{i_1}^{s_1} \cdots l_{i_r}^{s_r}},$$

  where the l_i's are defined in Definition 4.1.1. The *multipolylogarithm* (or MP) $\mathrm{Li}_{\vec{s}}(\vec{\varepsilon}, \vec{z})$ associated to $\vec{\varepsilon}$ and $\vec{z}$ is defined to be $\mathrm{Li}_{\vec{s}}(\vec{\varepsilon}, \vec{z}) := \mathrm{L}_{\vec{s}}(\vec{\varepsilon}, \vec{z})|_{t=\theta}$. In case $r = 1$, we will call an MP a *polylogarithm.*

For both the MZs and MPs, its *depth* is r, and its *weight* is $w := s_1 + \cdots + s_r$.

The condition imposed on $\vec{x}$ in the definition of our multizeta values is to ensure we have a well-defined period interpretation (Proposition 4.1.7). We have only defined MPs for $\vec{\varepsilon}$ a tuple of elements of $(q-1)^{st}$ roots of unity as this is sufficient for our applications.

Remark. Note that the expression $\mathrm{L}_{\vec{s}}(\vec{\varepsilon}, \vec{z})$ can be formally expressed as an element in $\mathbb{C}_\infty[[t]]$ by applying the identity

$$\frac{1}{t-\theta^{(j)}} = -\frac{1}{\theta^{(j)}}\left(1+\frac{t}{\theta^{(j)}}+\frac{t^2}{(\theta^{(j)})^2}+\cdots\right), \qquad i>0.$$

(The right hand side of this identity converges on the interval $|t| < |\theta^{(j)}| = q^{q^j}$.)

We now explain why $\mathrm{L}_{\vec{s}}(\vec{\varepsilon},\vec{z})$ converges whenever $|t| < |\theta^{(1)}| = q^q$; in particular, $\mathrm{L}_{\vec{s}}(\vec{\varepsilon},\vec{z}) \in \mathbb{T}$, and it makes sense to define MPs by evaluating $\mathrm{L}_{\vec{s}}(\vec{\varepsilon},\vec{z})$ at $t=\theta$. To see this, let $\tau \in \mathbb{C}_\infty$ be such that $|\tau| < q^q$, and let $l_{i,\tau} := l_i|_{t=\tau}$. Then

$$|l_{i,\tau}| = q^{\frac{qs}{q-1}(q^s-1)}.$$

Preserving the notations in the above definition, this implies

$$\left|\frac{\varepsilon_1^{i_1} z_1^{(i_1)}\cdots\varepsilon_r^{i_r} z_r^{(i_r)}}{l_{i_1,\tau}^{s_1}\cdots l_{i_r,\tau}^{s_r}}\right| = q^{\frac{q}{q-1}(s_1+\cdots+s_r)}\left(\frac{|z_1|}{q^{\frac{qs_1}{q-1}}}\right)^{(i_1)}\left(\frac{|z_r|}{q^{\frac{qs_1}{q-1}}}\right)^{(i_r)},$$

and the above expression approaches 0 as $i_1 > \cdots > i_r \geq 0$ approaches ∞.

Definition 4.1.2 generalizes the MZ and MP considered in [4], and are special values of Goss's analytic continuation in [18]. In particular, Thakur's multizeta values are the values

$$\zeta_{\vec{s}}(\vec{1}) = \sum_{\substack{\deg(a_1)>\cdots>\deg(a_r)\geq 0\\ a_i\in A_+}} \frac{1}{a_1^{s_1}\cdots a_r^{s_r}}.$$

Remark. No functional equation for the MZ is known. If we consider the MZ at negative integers, it turns out that the infinite sum reduces to a finite sum, and a study of the depth 1 case was done in [28].

We will study combinatorial properties of the MPs and MZs in the next section. For now we concern ourselves with realizing MPs and MZs as periods of special mixed Carlitz motives.

Polylogarithms as periods

For a family of polylogarithms $\mathrm{Li}_n(\varepsilon_1, z_1), \ldots, \mathrm{Li}_n(\varepsilon_m, z_m)$ of the same weight n, we can use an analogous construction in [22] to identify them as periods of mixed Carlitz motives. Let us first observe that, as ε_i is a $(q-1)^{st}$ root of unity, the power series defining the polylogarithms satisfy the functional equation

$$\mathrm{L}_n(\varepsilon_i, z_i)^{(-1)} = z_i^{(-1)} + \frac{\varepsilon_i}{(t-\theta)^n} \mathrm{L}_n(\varepsilon_i, z_i) \tag{4.1.1}$$

Hence, if we define γ_i to be a fixed $(q-1)^{st}$ root of ε_i, we see that $\gamma_i^{(-1)}\varepsilon_i = \gamma_i$, and one gets the following.

Proposition 4.1.3. *The function $\mathrm{L}_n(\varepsilon_i, z_i)$ above satisfies the functional equations*

$$(\gamma_i \mathrm{L}_n(\varepsilon_i, z_i))^{(-1)} = (\gamma_i z_i)^{(-1)} + \frac{\gamma_i}{(t-\theta)^n} \mathrm{L}_n(\varepsilon_i, z_i).$$

Consequently, the mixed Carlitz motive defined by

$$\Phi = \begin{bmatrix} (t-\theta)^n & & & \\ (\gamma_1 z_1)^{(-1)}(t-\theta)^n & 1 & & \\ \vdots & & \ddots & \\ (\gamma_m z_m)^{(-1)}(t-\theta)^n & & & 1 \end{bmatrix}$$

is uniformizable by

$$\Psi = \begin{bmatrix} \Omega^n & & & \\ \gamma_1 \mathrm{L}_n(\varepsilon_1, z_1)\Omega^n & 1 & & \\ \vdots & & \ddots & \\ \gamma_m \mathrm{L}_n(\varepsilon_m, z_m)\Omega^n & & & 1 \end{bmatrix}.$$

Proof. The equality $\Psi^{(-1)} = \Phi\Psi$ is immediate by the functional equations for polylogarithms (Equation 4.1.1) and Ω (Example 2.3.6). $\square$

Multipolylogarithms as periods

For an MP $\mathrm{Li}_{\vec{s}}(\vec{\varepsilon}, \vec{z})$, we can identify a mixed Carlitz motive such that this MP appears as the bottom right entry of its period matrix. Let us introduce the notation

$$\vec{s}_{ij} = (s_j, s_{j+1}, \ldots, s_{i-1}), \qquad 1 \leq j < i \leq r+1, \tag{4.1.2}$$

and similarly for $\vec{\varepsilon}_{ij}$ and $\vec{z}_{ij}$. Observe that, if we write $L_{i,j} = \mathrm{L}_{\vec{s}_{ij}}(\vec{\varepsilon}_{ij}, \vec{z}_{ij})$, then

$$L_{i,j}^{(-1)} = \frac{\varepsilon_j \cdots \varepsilon_{i-2} z_{i-1}^{(-1)}}{(t-\theta)^{s_j+\cdots+s_{i-2}}} L_{i-1,j} + \frac{\varepsilon_j \cdots \varepsilon_{i-1}}{(t-\theta)^{s_j+\cdots+s_{i-1}}} L_{i,j}.$$

Define γ_i to be a fixed $(q-1)^{st}$ root of ε_i as before

Proposition 4.1.4. *The $L_{i,j}$'s above satisfy the functional equations*

$$(\gamma_j \cdots \gamma_{i-1} L_{i,j})^{(-1)} = \frac{\gamma_j \cdots \gamma_{i-2} (\gamma_{i-1} z_{i-1})^{(-1)}}{(t-\theta)^{s_j+\cdots+s_{i-2}}} L_{i-1,j} + \frac{\gamma_j \cdots \gamma_{i-1}}{(t-\theta)^{s_j+\cdots+s_{i-1}}} L_{i,j},$$

Consequently, the mixed Carlitz motive defined by

$$\Phi = \begin{bmatrix} (t-\theta)^{s_1+\cdots+s_r} & & & & \\ (\gamma_1 z_1)^{(-1)}(t-\theta)^{s_1+\cdots+s_r} & (t-\theta)^{s_2+\cdots+s_r} & & & \\ & (\gamma_2 z_2)^{(-1)}(t-\theta)^{s_2+\cdots+s_r} & \ddots & & \\ & & \ddots & (t-\theta)^{s_r} & \\ & & & (\gamma_r z_r)^{(-1)}(t-\theta)^{s_r} & 1 \end{bmatrix}$$

is uniformizable by

$$\Psi = \begin{bmatrix} \Omega^{s_1+\cdots+s_r} & & & & \\ \gamma_1 L_{21}\Omega^{s_1+\cdots+s_r} & \Omega^{s_2+\cdots+s_r} & & & \\ \gamma_1\gamma_2 L_{31}\Omega^{s_1+\cdots+s_r} & \gamma_2 L_{32}\Omega^{s_2+\cdots+s_r} & & & \\ \vdots & \vdots & \ddots & & \\ \gamma_1\cdots\gamma_{r-1}L_{r,1}\Omega^{s_1+\cdots+s_r} & \gamma_2\cdots\gamma_{r-1}L_{r,2}\Omega^{s_2+\cdots+s_r} & \cdots & \Omega^{s_r} & \\ \gamma_1\cdots\gamma_r L_{r+1,1}\Omega^{s_1+\cdots+s_r} & \gamma_2\cdots\gamma_r L_{r+1,2}\Omega^{s_2+\cdots+s_r} & \cdots & \gamma_r L_{r+1}\Omega^{s_r} & 1 \end{bmatrix}$$

Proof. The equality $\Psi^{(-1)} = \Phi\Psi$ is immediate by the functional equations for multipolylogarithms (Equation 4.1.2) and Ω (Example 2.3.6). □

Multizeta values as periods

The realization of $\zeta_{\vec{s}}(\vec{x})$ as periods has essentially been done in [19]. We will recap it here. In order to do this, we need to make use of the following. For every nonnegative integer i, define

$$D_i = \begin{cases} \prod_{j=0}^{i-1}(\theta^{(i)} - \theta^{(j)}) & \text{if } i > 0; \\ 1 & \text{if } i = 0. \end{cases}$$

Definition 4.1.5. Let n be a non-negative integer. The *Carlitz gamma* is defined to be

$$\Gamma_{n+1} = \prod_i D_i^{n_i},$$

where $n = \sum_i n_i q^i$ is the q-adic expansion of n.

Theorem 4.1.6 ([4]). *There is a sequence of nonzero polynomials $H_n(t) \in A[t]$ satisfying*

$$(H_{s-1}\Omega^s)^{(d)}\Big|_{t=\theta} = \frac{\Gamma_s}{\tilde{\pi}^s} \sum_{\substack{\deg(a)=d \\ a \in A_+}} \frac{1}{a^s},$$

where $s \geq 1$, $d \geq 0$, $\Omega = \Omega(t)$ is the function in Example 2.3.6, and A_+ is the set of monic polynomials in A. Further, regarding H_n as an element in $\mathbb{F}_q[t][\theta]$,

$$\deg_\theta H_n \leq \frac{qn}{q-1}.$$

Because of the above theorem, the formal sums

$$L^{i,j} := \sum_{d_i > \cdots d_j \geq 0} x_i^{d_i} (H_{s_i-1}\Omega^{s_i})^{(d_i)} \cdots x_j^{d_j} (H_{s_j-1}\Omega^{s_j})^{(d_j)}$$

are the key to giving a period interpretation for multizeta values. If each $x_k \in \mathbb{C}_\infty$ satisfies $|x_k| = 1$, then these formal sums $L^{i,j}$ converges in $\mathbb{C}_\infty[t]$ by the bound on $\deg_\theta H_{s_k}$.

Proposition 4.1.7. *For each $x_i \in \overline{k}^\times$ satisfying $|x_i| = 1$, fix a solution $y_i \in \overline{k}$ to the equation*

$$y^q - \frac{x_i^{s_i+1}}{(x_i^{(i)})^{s_i}} y = 0.$$

Then the mixed Carlitz motive defined by

$$\Phi = \begin{bmatrix} (t-\theta)^{s_1+\cdots+s_r} & & & & \\ y_1^{(-1)} H_{s_1-1}^{(-1)} (t-\theta)^{s_1+\cdots+s_r} & (t-\theta)^{s_2+\cdots+s_r} & & & \\ & y_2^{(-1)} H_{s_2-1}^{(-1)} (t-\theta)^{s_2+\cdots+s_r} & \ddots & & \\ & & \ddots & (t-\theta)^{s_r} & \\ & & & y_r^{(-1)} H_{s_r-1}^{(-1)} (t-\theta)^{s_r} & 1 \end{bmatrix}$$

is uniformizable by

$$\Psi = \begin{bmatrix} \Omega^{s_1+\cdots+s_r} & & & & \\ y_1 L^{1,1}\Omega^{s_2+\cdots+s_r} & \Omega^{s_2+\cdots+s_r} & & & \\ \vdots & \vdots & \ddots & & \\ y_1\cdots y_{r-1}L^{1,r-1}\Omega^{s_r} & y_2\cdots y_{r-1}L^{2,r-1}\Omega^{s_r} & \cdots & \Omega^{s_r} & \\ y_1\cdots y_r L^{1,r} & y_2\cdots y_r L^{2,r} & \cdots & y_r L^{r,r} & 1 \end{bmatrix}.$$

Moreover, the nontrivial coefficients of Ψ *satisfy*

$$y_i\cdots y_j L^{i,j}\Omega^{s_{j+1}+\cdots+s_r}\Big|_{t=\theta} = \frac{y_i\cdots y_j\Gamma_{s_i}\cdots\Gamma_{s_j}}{\tilde{\pi}^{s_i+\cdots+s_r}}\zeta_{\vec{s}_{j+1,i}}(\vec{x}_{j+1,i}),$$

where $\vec{x}_{j+1,i} := (x_i,\ldots,x_j)$.

Proof. The first part of the proposition is a straightforward computation, and the second part is a consequence of Theorem 4.1.6, since

$$L^{i,j}\Big|_{t=\theta} = \frac{\Gamma_{s_i}\cdots\Gamma_{s_j}}{\tilde{\pi}^{s_i+\cdots+s_j}}\zeta_{\vec{s}_{j+1,i}}(\vec{x}_{j+1,i})$$

and $\Omega|_{t=\theta} = \tilde{\pi}^{-1}$. $\square$

4.2 Some combinatorial properties

In this section we discuss combinatorial properties of MZs and MPs. These are generalizations of various results in [7, 13, 19, 29].

Shuffle relations

As a consequence of the inclusion-exclusion principle, the product of two MPs is a linear combination of MPs, and the same holds for MZs.

Proposition 4.2.1. *The MPs satisfy shuffle relations of the form*

$$\mathrm{Li}_{\vec{s}}(\vec{\varepsilon}, \vec{z})\, \mathrm{Li}_{\vec{s}'}(\vec{\varepsilon}', \vec{z}') = \sum_{(\vec{v},\vec{v}')\in V} \mathrm{Li}_{\vec{v}\uplus\vec{v}'}(\vec{\varepsilon}_{v,v'}, \vec{z}_{v,v'}),$$

where the notations in the right hand side are defined as follows. For each positive integer r'' satisfying $\max\{r, r'\} \le r'' \le r + r'$, let $V_{r''}$ be the set of all tuples $(\vec{v}, \vec{v}')$ such that $\vec{v}$ (resp. $\vec{v}'$) can be obtained from $\vec{s}$ (resp. $\vec{s}'$) by inserting $r'' - r$ (resp. $r'' - r'$) zeros in all possible ways. Then

$$V := \bigcup_{\max\{r,r'\}\le r''\le r+r'} V_{r''}.$$

If $(\vec{v}, \vec{v}') \in V$, then $\vec{v} \uplus \vec{v}'$ is the result after removing all the zeros in the vector sum $\vec{v} + \vec{v}'$. For each $(\vec{v}, \vec{v}') \in V$, we then declare the

$$i^{th} \text{ coordinate of } \vec{\varepsilon}_{v,v'} := \begin{cases} \varepsilon_j & \text{if the } i^{th} \text{ coordinate of } \vec{v}\uplus\vec{v}' \text{ is } s_j; \\ \varepsilon'_k & \text{if the } i^{th} \text{ coordinate of } \vec{v}\uplus\vec{v}' \text{ is } s'_k; \\ \varepsilon_j\varepsilon'_k & \text{if the } i^{th} \text{ coordinate of } \vec{v}\uplus\vec{v}' \text{ is } s_j + s'_k. \end{cases}$$

The variables $z_{v,v'}$ obey the same rule as above, replacing all ε's by z's. □

Proposition 4.2.2. *Consider two MZs $\zeta_{\vec{s}}(\vec{x}), \zeta_{\vec{s}'}(\vec{x}')$ of depths r, r', such that all entries of $\vec{x}$ and $\vec{x}'$ have absolute values at most 1. Then the product $\zeta_{\vec{s}}(\vec{x})\zeta_{\vec{s}'}(\vec{x}')$ can be written as a "shuffle relation", i.e. an $\mathbb{F}_q$-linear combination of MZs.* □

It is not easy to explicitly write down the coefficients for an arbitrary MZ shuffle relation. However, [13] has computed the shuffle relation for the product of two MZs with depth 1:

$$\zeta_s(x)\zeta_{s'}(x') = \zeta_{s+s'}(xx') + \sum_{\substack{0<j<s+s' \\ q-1|j}} \left((-1)^{s-1}\binom{j-1}{s-1} + (-1)^{s'-1}\binom{j-1}{s'-1} \right) \zeta_{s+s'-j,j}(xx',1).$$

Explicit relations for colored multizeta values (see Definition 4.5.1) in low depths are also given in [19].

Multizeta values in terms of multipolylogarithms

For our eventual goal of proving some algebraic independence results on MZs, we will need the following result.

Proposition 4.2.3. *Let $\vec{\varepsilon} \in (\mathbb{F}_q^\times)^r$. For $i = 1, \dots, r$, also let C_i be the set of all coefficients for the polynomial $H_{s_i-1}(t)$ in Theorem 4.1.6, and let*

$$U := \{\vec{u} = (u_1, \dots, u_r) : u_i \in C_i \textit{ for all } i\}.$$

Then there exists $a_{\vec{u}} \in A$, indexed by $\vec{u} \in U$, such that

$$\zeta_{\vec{s}}(\vec{\varepsilon}) = \frac{1}{\Gamma_{s_1} \cdots \Gamma_{s_r}} \sum_{\vec{u}} a_{\vec{u}} \operatorname{Li}_{\vec{s}}(\vec{\varepsilon}, \vec{u}).$$

Furthermore, each $a_{\vec{u}}$ is a nonnegative power of θ.

Proof. Consider the convergent function

$$L(t) = \sum_{d_1 > \cdots d_r \geq 0} \varepsilon_1^{d_1}(H_{s_1-1}\Omega^{s_1})^{(d_1)} \cdots \varepsilon_r^{d_r}(H_{s_r-1}\Omega^{s_r})^{(d_r)}.$$

Then, by Theorem 4.1.6,

$$L(\theta)=\frac{\Gamma_{s_1}\cdots\Gamma_{s_r}}{\tilde{\pi}^{s_1+\cdots+s_r}}\zeta_{\vec{s}}(\vec{\varepsilon}).$$

On the other hand, as $\Omega^{(-1)}=(t-\theta)\Omega$,

$$\frac{L(t)}{\Omega^{s_1+\cdots+s_r}}=\sum_{d_1>\cdots d_r\geq 0}\frac{\varepsilon_1^{d_1}H_{s_1-1}^{(d_1)}\cdots\varepsilon_r^{d_r}H_{s_r-1}^{(d_r)}}{l_{d_1}^{s_1}\cdots l_{d_r}^{s_r}},$$

where

$$l_d=\begin{cases}\prod_{j=1}^{d}(t-\theta^{(j)}) & \text{if } d>0;\\ \\ 1 & \text{if } d=0.\end{cases}$$

Hence, letting $t=\theta$, one obtains the relation

$$\Gamma_{s_1}\cdots\Gamma_{s_r}\zeta_{\vec{s}}(\vec{\varepsilon})=\sum_{d_1>\cdots d_r\geq 0}\frac{\varepsilon_1^{d_1}H_{s_1-1}^{(d_1)}(\theta)\cdots\varepsilon_r^{d_r}H_{s_r-1}^{(d_r)}(\theta)}{\mathcal{L}_{d_1}^{s_1}\cdots\mathcal{L}_{d_r}^{s_r}}.$$

We now see that $a_{\vec{u}}$ is a power of θ from the power of t's in the $H_{s_j-1}^{(d_j)}$'s. $\square$

Non-vanishing properties

Before discussing algebraic independence properties, let us show the following non-vanishing properties to ensure that all colored multizeta values and colored multi-polylogarithms are nontrivial (see Definition 4.5.1 for definitions of these two terms).

Proposition 4.2.4. *Let $\vec{s}=(s_1,\ldots,s_r)$ be an arbitrary list of positive integers. For every $\vec{x}\in(\mathbb{C}_\infty^\times)^r$ satisfying $|x_i|=1$ for all i,*

$$\zeta_{\vec{s}}(\vec{x})\neq 0.$$

Proof. Write

$$\zeta_{\vec{s}}(\vec{x}) = \sum_{d_1 > \cdots > d_r \geq 0} x_1^{d_1} \cdots x_r^{d_r} S_{d_1}(s_1) \cdots S_{d_r}(s_r),$$

where

$$S_{d_i}(s_i) = \sum_{\substack{\deg(a)=d_i \\ a \in A_+}} \frac{1}{a^{s_i}}.$$

Then, as $\deg_\theta S_d(s) \geq \deg_\theta S_{d+1}(s) > 0$ by [29],

$$|\zeta_{\vec{s}}(\vec{x})| \geq |S_{r-1}(s_1) \cdots S_0(s_r)| > 0,$$

giving us what we want. □

Remark. By a similar proof strategy, for every $\vec{\varepsilon} \in (\mathbb{F}_q^\times)^r$ and $\vec{z} \in (\mathbb{C}_\infty^\times)^r$ satisfying $|z_i| = 1$ for all i,

$$\mathrm{Li}_{\vec{s}}(\vec{\varepsilon}, \vec{z}) \neq 0.$$

To see this, observe that $0 < \deg_\theta \mathcal{L}_i < \deg_\theta \mathcal{L}_{i+1}$, and so a direct estimate implies

$$|\mathrm{Li}_{\vec{s}}(\vec{\varepsilon}, \vec{z})| \geq \left| \frac{1}{\mathcal{L}_{r-1}^{s_1} \cdots \mathcal{L}_0^{s_r}} \right| > 0.$$

4.3 Linear relations on polylogarithms

Throughout this section, fix a positive integer m. For each $i = 1, \ldots, m$, also fix choices $\varepsilon_i \in \mathbb{F}_q^\times$ and $z_i \in k$ satisfying $|z_i| < q^{\frac{qs_i}{q-1}}$. In addition, fix a choice of $(q-1)^{st}$ root γ_i for each ε_i.

Let us consider a collection of polylogarithms

$$\mathcal{P} = \{\gamma_1 \,\mathrm{Li}_n(\varepsilon_1, z_1), \ldots, \gamma_m \,\mathrm{Li}_n(\varepsilon_m, z_m)\}$$

of the same weight n, multiplied by γ_i's. Recall that we gave a period interpretation for this collection of polylogarithms in Proposition 4.1.3. Let Φ and Ψ be as defined in this Proposition, and let Γ be its associated motivic Galois group. We now use the technique of [22, Section 6] to show that defining polynomials of Γ gives us all the k-linear relations on $\mathcal{P}$. We will use Theorem 2.4.1 throughout this argument without explicitly mentioning it.

Let $J = \mathbb{F}_q(t)$. By definition of Ψ and the construction of Γ, it is clear that we have an inclusion

$$\Gamma(R) \subset \left\{ \begin{bmatrix} * & 0 \\ * & I_m \end{bmatrix} \in \mathrm{Mat}_{(k+1)\times(k+1)}(R) \right\}$$

for any J-algebra R. Furthermore, as the mixed Carlitz motive Φ contains a tensor product of a Carlitz motive (corresponding to the top-left entry of Φ), there is a epimorphism over J

$$\Gamma \twoheadrightarrow \mathbb{G}_m$$

by the top-left entry, giving rise to an exact sequence

$$1 \longrightarrow V \longrightarrow \Gamma \longrightarrow \mathbb{G}_m \longrightarrow 1.$$

Notice that V is the unipotent subgroup of Γ, and is a vector group over $\overline{J}$ of dimension k. In fact,

$$V(\overline{J}) \subset \left\{ \begin{bmatrix} 1 & 0 \\ * & I_m \end{bmatrix} \in \mathrm{Mat}_{(k+1)\times(k+1)}(\overline{J}) \right\}.$$

Additionally, V is a vector group over $\overline{J}$: the product of two elements is linear on the non-trivial coordinates, and if $\alpha \in \Gamma(\overline{J})$ has image $a \in \mathbb{G}_m(\overline{J})$, then

$$\alpha^{-1} \begin{bmatrix} 1 & 0 \\ v & I_m \end{bmatrix} \alpha = \begin{bmatrix} 1 & 0 \\ av & I_m \end{bmatrix}.$$

Lemma 4.3.1. *V is a linear subspace of $\mathbb{G}_a^k$ over J.*

Proof (c.f. Lemma A.2 of [9]). Note that the induced map $d : \operatorname{Lie}\Gamma \longrightarrow \operatorname{Lie}\mathbb{G}_m$ is nonzero since the map is nontrivial when restricted to the upper-left corner of the matrix. Furthermore, $\ker(d) = \operatorname{Lie} V$. By smoothness of Γ and $\mathbb{G}_m$ over J,

$$\dim_J \operatorname{Lie}\Gamma = \dim \Gamma \qquad \text{and} \qquad \dim_J \operatorname{Lie}\mathbb{G}_m = \dim \mathbb{G}_m = 1,$$

and so

$$\dim_J \operatorname{Lie} V = \dim V.$$

Thus V is smooth and defined over J. Since V is also a vector group over $\overline{J}$, this means V is defined by linear forms over J. $\square$

If we now fix a matrix

$$\omega = \begin{bmatrix} b_0 & & & \\ b_1 & 1 & & \\ \vdots & & \ddots & \\ b_m & & & 1 \end{bmatrix} \in \Gamma(J), \qquad b_0 \in J^\times \setminus \mathbb{F}_q^\times,$$

the Zariski closure of the cyclic group generated by ω in Γ is the line L connecting ω to the identity matrix I_{k+1}. As Γ is absolutely irreducible and V is of codimension

1, we conclude that Γ is the linear space spanned by V and L. Thus, if $F_1, \ldots, F_l$ are linear polynomials in $J[X_1, \ldots, X_m]$ defining V, and ω is as above, then

$$G_i(X_0, \ldots, X_m) := (b_0 - 1)F_i(X_1, \ldots, X_m) - F_i(b_1, \ldots, b_m)(X_0 - 1)$$

is a set of linear polynomials defining Γ. Furthermore, using the fact the Z is a Γ-torsor by the map defined in 2.4.1, one also sees that the linear polynomials

$$H_i(X_0, \ldots, X_m) := G_i(X_0, \ldots, X_m) - f_i X_0$$

defines Z for some determined $f_i \in J$. These H_i are the polynomials giving us linear relations on polylogarithms as the construction of Z is based on the entries of Ψ. Recall that $\mathcal{P}$ is our collection of polylogarithms of the same weight n.

Proposition 4.3.2. *Fix one of the F_i's above, and write $F_i = c_1 X_1 + \cdots + c_m x_m$ with $c_i \in \mathbb{F}_q(t)^\times$.*

(a) Each polynomial G_i gives rise to the relation

$$(b_0(\theta) - 1)\sum_{i=1}^{k} c_i(\theta)\gamma_i \operatorname{Li}_n(\varepsilon_i, z_i) - \sum_{i=1}^{k} b_i(\theta)c_i(\theta)\tilde{\pi}^n = 0,$$

where the b_i are the nontrivial entries of ω.

(b) Every k-linear relation among $\{\tilde{\pi}^n\} \cup \mathcal{P}$ is a linear combination of the relations $G_1, \ldots, G_l$ above. In fact,

$$\dim \Gamma = \dim_k \operatorname{Span}_k\{\{\tilde{\pi}^n\} \cup \mathcal{P}\}.$$

Proof. (a) By definition of Z, substituting the first column of Ψ into $H_i = G_i - fX_0$ gives

$$G_i\left(\Omega^n, \Omega^n \gamma_1 \operatorname{L}_n(\varepsilon_1, z_1), \ldots, \Omega^n \operatorname{L}_n(\varepsilon_m, z_m)\right) = f\Omega^n.$$

Using the functional equation for polylogarithms (Proposition 4.1.3) and the definition of G_i gives

$$\begin{aligned} f^{(-1)}(\Omega^n)^{(-1)} &= f_i\Omega^n - F_i(b_1,\dots,b_n)\Omega^n \\ &\quad + \Omega^n G_i((t-\theta)^n - 1, (\gamma_1 z_1)^{(-1)}(t-\theta)^n, \cdots, (\gamma_m z_m)^{(-1)}(t-\theta)^n). \end{aligned}$$

Since $\Omega^{(-1)} = (t-\theta)\Omega$, the above gives

$$\begin{aligned} (t-\theta)f^{(-1)} - f_i = G_i((t-\theta)^n - 1, (\gamma_1 z_1)^{(-1)}(t-\theta)^n, \cdots, (\gamma_m z_m)^{(-1)}(t-\theta)^n) \\ - F_i(b_1,\dots,b_n). \end{aligned}$$

Notice this implies that f_i has no pole at $t=\theta$. Otherwise $f^{(-1)}$ would have a pole at $t=\theta^{(-1)}$, and the relation above shows that f would have a pole here as well. Iterating shows f has poles at $\theta^(-i)$ for all positive integers i, contradicting the fact that f is a rational function. The same argument shows that $f_i^(-1)$ has no poles at $t=\theta$. This implies that we can evaluate the relation above at $t=\theta$ to get

$$\begin{aligned} f(\theta) &= -G_i(-1,0,\dots,0) + F_i(b_1,\dots,b_n)|_{t=\theta} \\ &= -\sum_{i=1}^m c_i(\theta)b_i(\theta). \end{aligned}$$

Therefore, the first relation we started with now becomes

$$G_i\left(\Omega^n, \Omega^n\gamma_1 \operatorname{L}_n(\varepsilon_1, z_1), \dots, \Omega^n\gamma_m \operatorname{L}_n(\varepsilon_m, z_m)\right)|_{t=\theta} = -\sum_{i=1}^m c_i(\theta)b_i(\theta)\tilde{\pi}^{-n}.$$

An elementary manipulation implies the desired relation.

(b) Let $N = \operatorname{Span}_k\{\{\tilde{\pi}^n\} \cup \mathcal{P}\}$. Since defining polynomials of Γ gives linear relations on $\{\tilde{\pi}\} \cup \mathcal{P}$,

$$\operatorname{codim}\Gamma \geq \operatorname{codim}_k N,$$

implying $\dim_k N \le \dim \Gamma$. On the other hand,

$$\dim_k N \ge \operatorname{trdeg}_{\overline{k}} \overline{k}(\{\tilde{\pi}\} \cup \mathcal{P}) = \dim \Gamma,$$

implying $\dim \Gamma = \dim_k N$. $\square$

Corollary 4.3.3. *Preserve the notations of the above Proposition. If $\mathcal{P}$ is linearly independent over k, then $\mathcal{P}$ is algebraically independent over $\overline{k}$. In this case,*

$$\operatorname{trdeg}_{\overline{k}} \overline{k}(\operatorname{Li}_n(\varepsilon_1, z_1), \ldots, \operatorname{Li}_n(\varepsilon_m, z_m)) = |\mathcal{P}|,$$

and each $\operatorname{Li}_n(\varepsilon_i, z_i)$ is transcendental over $\overline{k}$.

Proof. Let $N = \operatorname{Span}_k\{\{\tilde{\pi}^n\} \cup \mathcal{P}\}$, so that $m \le \dim_k N \le m+1$. Also let P be the field extension of $\overline{k}$ by adjoining the elements of $\{\tilde{\pi}^n\} \cup \mathcal{P}$. Since $\gamma_i \in \overline{k}$,

$$P = \overline{k}(\tilde{\pi}^n, \operatorname{Li}_n(\varepsilon_1, z_1), \ldots, \operatorname{Li}_n(\varepsilon_m, z_m)).$$

Now, recall that $\dim_k N = \dim \Gamma = \operatorname{trdeg}_{\overline{k}} P$. Thus we are done if $\{\tilde{\pi}^n\} \cup \mathcal{P}$ is a linearly independent set over k. If $\tilde{\pi}^n$ is a k-linear combination of elements in $\mathcal{P}$, then

$$P = \overline{k}(\operatorname{Li}_n(\varepsilon_1, z_1), \ldots, \operatorname{Li}_n(\varepsilon_m, z_m))$$

and $\operatorname{trdeg}_{\overline{k}} P \ge m$, and we are done once again. $\square$

Corollary 4.3.4. *Let $F \in \overline{k}[X_1, \ldots, X_m]$ be a degree 1 polynomial, and suppose*

$$f = F(\gamma_1 \operatorname{Li}_n(\varepsilon_1, z_1), \ldots, \gamma_m \operatorname{Li}_n(\varepsilon_m, z_m))$$

is nonzero. Assume $\mathcal{P} = \{\gamma_1 \operatorname{Li}_n(\varepsilon_1, z_1), \ldots, \gamma_m \operatorname{Li}_n(\varepsilon_m, z_m)\}$ is linearly independent over k. Then f is transcendental over $\overline{k}$.

Proof. Immediate by the previous Corollary, after replacing any element of $\mathcal{P}$ by f. □

We have only displayed the existence of linear relations between polylogarithms of the same weight in this section. The next section tells us that we should not expect to get nontrivial linear relations between polylogarithms of different weights.

4.4 Algebraic independence of multipolylogarithms

The purpose of this section is to prove some results on MPs. The entirety of this section consists of Lemmas that we will use for the next section.

Lemma 4.4.1. *Let $V \subset \mathbb{G}_a^m$ be an algebraic group of dimension zero, and assume V is stable under the $\mathbb{G}_m$-action defined by*

$$a \cdot (x_1, \dots, x_m) = (a^{n_1} x_1, \dots, a^{n_m} x_m) \qquad \forall a \in \overline{k}^{\times}.$$

Then $V \equiv 1$ over $\overline{k}$.

Proof. $\mathbb{G}_m(\overline{k})$ is an infinite set. □

Lemma 4.4.2. *Consider distinct positive integers $n_1, \dots, n_d$ not divisible by p. For each n_i, fix distinct $\varepsilon_1^i, \dots, \varepsilon_{l_i}^i \in \mathbb{F}_q^{\times}$ and their $(q-1)^{st}$ roots $\gamma_1^i, \dots, \gamma_{l_i}^i$. Consider a family $f_{n_i,1}, \dots, f_{n_i,l_i}$ of MPs:*

$$f_{n_i,j} = \gamma_j^i \, \mathrm{L}_{n_i}(\varepsilon_j^i, z_j^i), \qquad z_j^i \in k.$$

If $\tilde{\pi}^{n_i}, f_{n_i,1}|_{t=\theta}, \ldots, f_{n_i,l_i}|_{t=\theta}$ *is* k*-linearly independent for each* i*, then the set*

$$\{\tilde{\pi}, f_{n_i,j}|_{t=\theta} : 1 \leq i \leq d, 1 \leq j \leq l_i\}$$

is algebraically independent over $\overline{k}$*.*

Proof. Let the indexing set be $I = \{(i,j) : 1 \leq i \leq d, 1 \leq j \leq l_i\}$ with lexicographic ordering $\leq$. Any tuple $(i.j)$ in this proof will be assumed to be in I.

Consider the t-motive associated to each $f_{n_i,j}$

$$\Phi_{i,j} = \begin{bmatrix} (t-\theta)^{n_i} & 0 \\ (\gamma_j^i z_j^i)^{(-1)}(t-\theta)^{n_i} & 1 \end{bmatrix}$$

with uniformization

$$\Psi_{i,j} = \begin{bmatrix} \Omega^{n_i} & 0 \\ f_{n_i,j}\Omega^{n_i} & 1 \end{bmatrix}.$$

Now, for any tuple (k,l), define

$$M(k,l) = C \oplus \bigoplus_{(i,j)\leq(k,l)} \Phi_{i,j} \qquad \text{and} \qquad M_k(l) = C \oplus \bigoplus_{j \leq l} \Phi_{k,j}$$

where C is the Carlitz t-motive. (Here $M_k(l)$ is the slice of $M(k,l)$ corresponding to $f_{n_k,1}, \ldots, f_{n_k,l_k}$.) Also define $\Gamma(k,l)$ and $\Gamma_k(l)$ to be the motivic Galois groups of $M(k,l)$ and $M_k(l)$. Then $\Gamma(k,l)$ and $\Gamma_k(l)$ are constructed from

$$[\Omega] \oplus \bigoplus_{(i,j)\leq(k,l)} \Psi_{i,j} \qquad \text{and} \qquad [\Omega] \oplus \bigoplus_{j \leq l} \Psi_{k,j}.$$

By construction,

$$\Gamma(k,l) \subset \left\{ [a] \oplus \bigoplus_{(i,j)\leq(k,l)} \begin{bmatrix} a^{n_i} & 0 \\ x_{ij} & 1 \end{bmatrix} : a, x_{ij} \in \overline{k} \right\}$$

and

$$\Gamma_k(l) \subset \left\{ [a] \oplus \bigoplus_{j \leq l} \begin{bmatrix} a^{n_k} & 0 \\ x_{kj} & 1 \end{bmatrix} : a, x_{ij} \in \overline{k} \right\}.$$

Due to Theorem 2.4.1, it suffices to show that the left inclusion is an equality. We will achieve this by doing induction on (k, l) with respect to the ordering on I.

By assumption, the inclusions above are equalities for $\Gamma(1, j)$ and $\Gamma_k(j)$ across all possible j's (and a fixed k). Let $(k, l) \geq (2, 1)$, and let (k', l') to be the element preceding (k, l) under the ordering $\leq$. Then

$$(k', l') = \begin{cases} (k, l-1) & \text{if } l \neq 1; \\ (k-1, l_{k-1}) & \text{if l=1.} \end{cases}$$

We now have injections

$$\begin{tikzcd} & M(k', l') \arrow[dr, hook] & \\ C \arrow[ur, hook] \arrow[rr, hook] \arrow[dr, hook] & & M(k, l) \\ & M_k(l) \arrow[ur, hook] & \end{tikzcd}$$

where the inclusion of C is to every mixed Carlitz motive in the direct sum (not just the natural inclusion). By Tannakian duality, we get surjections

$$\begin{tikzcd} & \Gamma(k', l') \arrow[dl, two heads, "\pi'"'] & \\ \mathbb{G}_m & & \Gamma(k, l) \arrow[ll, two heads, "\pi"'] \arrow[ul, two heads, "\psi"'] \arrow[dl, two heads, "\psi_k"] \\ & \Gamma_k(l) \arrow[ul, two heads, "\pi''"] & \end{tikzcd}$$

where the kernels of π, π', π'' lie in the unipotent radical of the respective groups. Letting $V = \ker \pi$, and similarly for V' and V'', we have a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & V'' \arrow[r] & \Gamma_k(l) \arrow[r] & \mathbb{G}_m \arrow[r] & 1 \\
1 \arrow[r] & V \arrow[r] \arrow[u, two heads, "\psi_k|_V"'] \arrow[d, two heads, "\psi|_V"] & \Gamma(k,l) \arrow[r] \arrow[u, two heads, "\psi_k"'] \arrow[d, two heads, "\psi"] & \mathbb{G}_m \arrow[r] \arrow[u, equal] \arrow[d, equal] & 1 \\
1 \arrow[r] & V' \arrow[r] & \Gamma(k',l') \arrow[r] & \mathbb{G}_m \arrow[r] & 1
\end{tikzcd}$$

By assumption $V'' = \prod_{j \le l} \mathbb{G}_a$, and by induction $V' = \prod_{(i,j) \le (k,l)} \mathbb{G}_a$, and these isomorphisms are via the coordinates x_{ij}.

The action of $\mathbb{G}_m$ on V (and similarly for V' and V'') via the short exact sequence above is via conjugation:

$$a \cdot v = \tilde{a}^{-1} v \tilde{a}, \text{ where } \tilde{a} \text{ is a lift of } a \text{ to } \Gamma(k,l).$$

A computation tells us that, on the coordinates x_{ij}, the action is

$$a \cdot x_{ij} = a^{n_i} x_{ij}.$$

Now, notice that the difference between the coordinates defining $\Gamma(k,l)$ and $\Gamma(k',l')$ is just x_{kl}, so it follows that

$$\dim \Gamma(k',l') \le \dim \Gamma(k,l) \le \Gamma(k',l') + 1.$$

Hence it suffices to show that $\dim \Gamma(k',l') \ne \dim \Gamma(k,l)$. The rest of the proof follows the strategy of [21, Theorem 4.2].

Assume that $\dim \Gamma(k',l') = \dim \Gamma(k,l)$. Then the commutative diagram above implies $\dim \ker \psi|_V = 0$, whence $\ker \psi|_V \equiv 1$ by Lemma 4.4.1. Hence $\psi|_V$ is a bijection, and there is a surjective $\mathbb{G}_m$-homomorphism

$$\varphi\colon V' \xrightarrow{\psi|_V^{-1}} V \xrightarrow{\psi_k|_V} V''.$$

For each $(i,j) \neq (k,l)$ with $i \neq k$, let V_{ij} be the subvariety of V defined by

$$x_{i',j'} = 0 \text{ if } (i',j') \neq (i,j),(k,l),$$

and define $V'_{ij} \subset V'$ using the same equations. Then

$$V_{ij} \subset \mathbb{G}_a^{\oplus 2} \text{ and } V'_{ij} \subset \mathbb{G}_a.$$

Via the bijection, this implies that $\dim V_{ij} = 1$. Hence [15, Corollary 1.8] tells us that V_{ij} is defined by a polynomial of the form

$$p(x_{ij}, x_{kl}) = \sum_{\alpha=0}^{d_1} f_\alpha x_{kl}^{p^{e_\alpha}} - \sum_{\beta=0}^{d_2} f_\beta x_{ij}^{p^{e_\beta}} \in \overline{k}[x_{ij}, x_{kl}].$$

If $d_1 > 0$, then by normality of V_{ij} any point $(x_{ij}, x_{kl}) \in V_{ij}$ must also satisfy the polynomial

$$p(a \cdot x_{ij}, a \cdot x_{kl}) - a^{n_k p^{e_{d_1}}} p(x_{ij}, x_{kl})$$

of lower x_{kl}-degree. Hence, by iterating, every point in V_{ij} satisfies a polynomial of the form

$$f(x_{ij}, x_{kl}) = x_{kl}^{p^e} - \sum_{\beta=0}^{d_2} f_\beta x_{ij}^{p^{e_\beta}}.$$

We now contend that $\varphi|_{V_{ij}} \equiv 0$. If not, there is some (x_{ij}, x_{kl}) on V_{ij} such that $\varphi(x_{ij}, x_{kl}) = x_{kl} \neq 0$. Hence using the polynomial f to write

$$x_{kl} = \left(\sum_{\beta=0}^{d_2} f_\beta x_{ij}^{p^{e_\beta}} \right)^{p^{-e}}$$

and using the fact that φ is equivariant under $\mathbb{G}_m$, we get the equality

$$\left(\sum_{\beta=0}^{d_2} f_\beta (a^{n_i} x_{ij})^{p^{e_\beta}}\right)^{p^{-e}} = a^{n_k}\left(\sum_{\beta=0}^{d_2} f_\beta x_{ij}^{p^{e_\beta}}\right)^{p^{-e}}.$$

Comparing coefficients gives us $n_i p^{e_{d_2}-e} = n_k$, a contradiction by assumptions on the n_i's. Hence $\varphi|_{V_{ij}} \equiv 0$.

Recall V_{ij} was defined for $(i,j) \neq (k,l)$ with $i \neq k$, and (k',l') was the tuple preceding (k,l) with respect to $\leq$. Hence $\varphi_{V_{ij}} \equiv 0$ for all such i implies

$$\varphi\left(\prod_{\substack{(i,j)\leq(k',l')\\ i=k}} \mathbb{G}_a\right) = \varphi(\mathbb{G}_a^{l-1}) = V''.$$

But $V'' = \prod_{j\leq l} \mathbb{G}_a$ has dimension l, a contradiction. □

Lemma 4.4.3 (c.f. [21, Theorem 4.3])**.** *Let $s_1, \ldots, s_r$ be positive integers, and let $\varepsilon_1, \ldots, \varepsilon_r \in \mathbb{F}_q^\times$. If the set*

$$\{\tilde{\pi}, \mathrm{Li}_{s_1}(\varepsilon_1, z_1), \ldots, \mathrm{Li}_{s_r}(\varepsilon_r, z_r)\}$$

is algebraically independent over $\overline{k}$, then so is the set

$$\{\tilde{\pi}, \mathrm{Li}_{\vec{s}_{ij}}(\vec{\varepsilon}_{ij}, \vec{z}_{ij}) : 1 \leq j < i \leq r+1\},$$

where we write $\vec{\varepsilon}_{ij} = (\varepsilon_j, \varepsilon_{j+1}, \ldots, \varepsilon_{i-1})$ and $\vec{z}_{ij} = (z_j, z_{j+1}, \ldots, z_{i-1})$.

Proof. By assumption the set

$$\{\tilde{\pi}, \gamma_1 \mathrm{Li}_{s_1}(\varepsilon_1, z_1), \ldots, \gamma_r \mathrm{Li}_{s_r}(\varepsilon_r, z_r)\}$$

is algebraically independent over $\overline{k}$, where γ_i is a fixed $(q-1)^{st}$ root of ε_i. Then, by considering the mixed Carlitz motives of Proposition 4.1.4, and using the same proof as in [21, Theorem 4.3], we get algebraic independence of

$$\{\tilde{\pi}, \gamma_j \cdots \gamma_{i-1} \operatorname{Li}_{\vec{s}_{ij}}(\vec{\varepsilon}_{ij}, \vec{z}_{ij}) : 1 \leq j < i \leq r+1\},$$

as desired. $\square$

4.5 Implications on colored multizeta values

In this section we study colored multizeta values. Let us preserve the notations in Section 4.1.

Definition 4.5.1. Fix a positive integer r, and let $\vec{s} = (s_1, \ldots, s_r)$ be a list of positive integers. Also consider a list $\vec{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_r)$ of elements in $\mathbb{F}_q^\times$.

- The *colored multizeta value* (or CMZ) associated to $\vec{s}$ and $\vec{\varepsilon}$ is defined to be $\zeta_{\vec{s}}(\vec{\varepsilon})$.
- Let $\vec{u} = (u_1, \ldots, u_r)$ be a list of elements in A. The *colored multipolylogarithm* (or CMP) associated to $\vec{s}$ and $\vec{u}$ is defined to be $\operatorname{Li}_{\vec{s}}(\vec{\varepsilon}, \vec{u})$.

There are two points of view for this definition. One is that colored multizeta values are special values of Goss's analytic continuation of $\zeta_{\vec{s}}(\vec{1})$ in [18]. The other is that these are multizeta values twisted by $\mathbb{C}_\infty$-Hecke characters $\mathbb{A}^\times/k^\times \longrightarrow \mathbb{C}_\infty^\times$ of the form

$$(a)_v \longmapsto \prod_v \varepsilon^{\log |a|_v}, \qquad \varepsilon \in \mathbb{F}_q^\times.$$

Note that the CMZs includes the Thakur multizeta values $\zeta_{\vec{s}}(\vec{1})$ as special cases, and that both the CMZs and CMPs lie in k_∞. The definition of CMP is also motivated by the relation in Proposition 4.2.3. We now list some known results related to the CMZs.

- In [34], transcendence of Carlitz zeta values (i.e. depth 1 Thakur multizeta values) was established by using elements of [4].
- In [9], algebraic independence of Carlitz zeta values was shown by extending methods of [22].
- In [7], transcendence of Thakur multizeta values was established by extending methods of [2].
- In [24, 30, 31], some linear relations between Thakur multizeta values are given. Of particular note is that the list $\{\vec{s}\}$ in such relations satisfy $q-1|s_i$ or $p|s_i$ for some entry in a tuple $\vec{s}$.
- In [8], some computations on linear relations between zeta values of the form $\zeta_{(s,t)}(1,1)$ were done. In particular, the k-linear space spanned by such zeta values of a fixed weight was shown to be related to special points of certain t-modules.
- In [21], algebraic independence of large subsets of Thakur multizeta values was shown by extending methods of [9].
- In [19], transcendence of CMZs was established by extending methods of [7].

The final goal of this thesis is to add the following bullet point in the above list:

- Find large subsets of CMZs that are algebraically independent.

Along the way, we obtain a few results listed in the papers above as corollaries. Let us first make three easy observations.

Lemma 4.5.2. *Let w be a positive integer not divisible by $q-1$. Let $\mathcal{P}_w = \{\gamma_i \operatorname{Li}_w(\varepsilon_i, u_i)\}$ be a finite collection of CMPs of weight w that are k-linearly independent. Then $\{\tilde{\pi}^w\} \cup \mathcal{P}_w$ is k-linearly independent.*

Proof. If $|\mathcal{P}_w| = 1$ then the result follows as CMPs are in k_∞ by definition but not $\gamma^{-1}\pi^w$ (due to the $(-\varepsilon\theta)^{\frac{w}{q-1}}$ term). If $|\mathcal{P}_w| > 1$, assume there is a nontrivial relation

$$c_0\tilde{\pi}_w + \sum_i c_i\gamma_i \operatorname{Li}_w(\varepsilon_i, u_i) = 0, \qquad c_i \in k.$$

Fix any α, and consider

$$c_\alpha \operatorname{Li}_w(\varepsilon_\alpha, u_\alpha) = -\gamma_\alpha^{-1}\left(c_0\tilde{\pi}_w + \sum_{i \neq \alpha} c_i\gamma_i \operatorname{Li}_w(\varepsilon_i, u_i)\right).$$

Then both sides of the equation equals 0 since the left hand side is in k_∞ but not the right hand side. The result follows by induction. □

Lemma 4.5.3. *Let $\zeta_w(\varepsilon)$ be a CMZ of weight w not divisible by $q-1$, and let γ be a $(q-1)^{st}$ root of ε. Then $\gamma\zeta_w(\varepsilon)$ is linearly independent with $\tilde{\pi}^w$ over k.*

Proof. Any such CMZ lies in k_∞, but not $\gamma^{-1}\tilde{\pi}^w$ due to the $(-\varepsilon^{-1}\theta)^{\frac{w}{q-1}}$ term. □

Lemma 4.5.4. *Let ε_i run through all elements of $\mathbb{F}_q^\times$, and let γ_i be a fixed $(q-1)^{st}$ root of ε_i. Then the set $\{\gamma_1, \ldots, \gamma_{q-1}\}$ is linearly independent over k_∞.*

Proof. Let $G = \mathrm{Gal}(\mathbb{F}_{q^{q-1}}/\mathbb{F}_q)$, and consider a minimal dependence $\sum_\alpha c_\alpha \gamma_\alpha = 0$ with $c_\alpha \in k_\infty$ of length at least two (length one is trivial). Pick any two γ_i, γ_j appearing in this linear dependence. As $\varepsilon_i/\varepsilon_j \neq 1$ for $i \neq j$, there exists $\sigma \in G$ such that

$$\frac{\sigma(\gamma_i)}{\sigma(\gamma_j)} \neq \frac{\gamma_i}{\gamma_j}.$$

Also, as $\sigma(\gamma_\alpha) = \omega_\alpha \gamma_\alpha$, where ω_α is a $(q-1)^{st}$ root of unity, the quotient $\sigma(\gamma_\alpha)/\gamma_\alpha$ lies in $\mathbb{F}_q^\times$. Now, after lifting σ to the unique element in $\mathrm{Gal}(\mathbb{F}_{q^{q-1}}((\frac{1}{\theta})), k_\infty)$, we compute

$$0 = \frac{\sigma(\gamma_i)}{\gamma_i} \sum_\alpha c_\alpha \gamma_\alpha - \sigma\left(\sum_\alpha c_\alpha \gamma_\alpha\right) = \sum_\alpha c_\alpha \left(\frac{\sigma(\gamma_i)}{\gamma_i} - \frac{\sigma(\gamma_\alpha)}{\gamma_\alpha}\right)\gamma_\alpha$$

This is a shorter relation among the γ_α's, a contradiction. □

Depth 1 CMZs

Let us recall a consequence of Carlitz's work in [6], which says that

$$\zeta_{(q-1)n}(1) = \frac{B_{(q-1)n}}{\Gamma_{(q-1)n+1}} \tilde{\pi}^{(q-1)n},$$

where the B_n's are the Bernoulli-Carlitz numbers defined by the Carlitz exponential series:

$$\frac{z}{\exp_C(z)} = \sum_{n=0}^{\infty} B_n \frac{z_n}{\Gamma_{n+1}}.$$

(A explicit computation of this can be found in [27, Theorem 5.2.1].) In the general setting of CMZs however, we do not know how to compute these values. We now show algebraic independence between them and $\tilde{\pi}$ instead. Most interesting results we obtain will have the assumption that the weight w is not divisible by $q-1$ and p. This is due to the observation above, the fact that $\tilde{\pi}^{q-1}$ lies in k_∞, and that

$$\zeta_{pn}(\varepsilon) = \zeta_n(\varepsilon^{q/p})^p.$$

Proposition 4.5.5. *Each $\zeta_n(\varepsilon)$ is transcendental over $\overline{k}$.*

Proof. Each $\zeta_n(\varepsilon)$ is nonzero by Proposition 4.2.4, and $\zeta_n(\varepsilon)$ can be written as a k-linear combination of CMPs by Proposition 4.2.3. We are now done by Corollary 4.3.4. □

Proposition 4.5.6. *If n is not divisible by $q-1$, then $\zeta_n(\varepsilon)$ and $\tilde{\pi}$ are algebraically independent over $\overline{k}$.*

Proof. Consider the set $\mathcal{P}$ of all CMPs appearing the expression in Proposition 4.2.3. Then

$$\mathrm{Span}_k\{\{\tilde{\pi}^n\} \cup \gamma\mathcal{P}\} = \mathrm{Span}_k\{\{\tilde{\pi}^n\} \cup \gamma\mathcal{P} \cup \{\gamma\zeta_n(\varepsilon)\}\},$$

where γ is a $(q-1)^{st}$ root of ε. Choose a maximal subset $\mathcal{S}$ of $\mathcal{P}$ such that $\{\tilde{\pi}^n\} \cup \mathcal{S}$ is linearly independent, which is possible by Lemma 4.5.2. Then Lemma 4.4.2 implies $\{\tilde{\pi}^n\} \cup \mathcal{S}$ is algebraically independent over $\overline{k}$. By using Proposition 4.2.3, we can replace any element of $\mathcal{S}$ by $\gamma\zeta_n(\varepsilon)$ to form $\mathcal{S}'$. Then $\{\tilde{\pi}^n\} \cup \mathcal{S}'$ is still a transcendence basis over $\overline{k}$, implying what we want. □

Lemma 4.5.7. *Fix a positive integer n not divisible by $q-1$, and fix distinct $\varepsilon_1, \ldots, \varepsilon_m \in \mathbb{F}_q^\times$. For each i, consider a nonzero linear sum of CMPs of same* *weight n and ε_i:*

$$f_i = a_{i1}\gamma_i \operatorname{Li}_n(\varepsilon_i, u_1) + \cdots + a_{im}\gamma_i \operatorname{Li}_n(\varepsilon_i, u_m), \qquad a_{ij} \in k \text{ for all } j.$$

Then $\tilde{\pi}^n, f_1, \ldots, f_m$ is linearly independent over k. In particular, $\{\tilde{\pi}^n, \gamma\zeta_n(\varepsilon) : \varepsilon \in \mathbb{F}_q^\times\}$ is linearly independent over k.

Proof. By Lemma 4.5.2, it suffices to show that the f_i's are linearly independent over k. This is immediate by Lemma 4.5.4, as $f_i = \gamma_i g_i$ with $g_i \in k_\infty$. □

Using the same notation as the above Lemma, the same proof can be used to show the following.

Lemma 4.5.8. *If n is a positive integer divisible by $q-1$, and $\varepsilon_1, \ldots, \varepsilon_m \in \mathbb{F}_q^\times$ are distinct, then $f_1, \ldots, f_m$ is linearly independent over k. In particular, $\{\pi^n, \gamma\zeta_n(\varepsilon) : \varepsilon \in \mathbb{F}_q^\times \setminus \{1\}\}$ is linearly independent over k.* □

We now consider depth 1 CMZs of different weights.

Theorem 4.5.9. *Let n be a positive integer. The following set is algebraically independent over $\overline{k}$:*

$$\{\tilde{\pi}\} \cup \left\{\zeta_s(\varepsilon) : \begin{matrix} 1 \le s \le n \text{ with } q-1 \nmid s \text{ and } p \nmid s, \\ \varepsilon \in \mathbb{F}_q^\times \end{matrix}\right\}$$
$$\cup \left\{\zeta_{(q-1)s}(\varepsilon) : \begin{matrix} 1 \le s \le n \text{ with } p \nmid s, \\ \varepsilon \in \mathbb{F}_q^\times \setminus \{1\} \end{matrix}\right\}.$$

Proof. Let $\mathcal{P}'_{s,\varepsilon}$ be the set of all CMPs of weight at most n appearing in the expression in Proposition 4.2.3 for $\zeta_{s,\varepsilon}$, and consider a maximal subset of $\mathcal{P}_{s,\varepsilon}$ such that

$\{\pi^s\} \cup \mathcal{P}_{s,\varepsilon}$ is linearly independent over k. By applying Lemmas 4.5.7 or 4.5.8, the elements within each of the subsets

$$\{\tilde{\pi}^s\} \cup \bigcup_{\substack{q-1\nmid s \text{ and } p\nmid s,\\ \varepsilon\in\mathbb{F}_q^\times}} \mathcal{P}_{s,\varepsilon} \qquad \text{and} \qquad \{\tilde{\pi}^{(q-1)s}\} \cup \bigcup_{\substack{p\nmid s,\\ \varepsilon\in\mathbb{F}_q^\times\setminus\{1\}}} \mathcal{P}_{(q-1)s,\varepsilon}$$

are linearly independent over k for all $1 \leq s \leq n$. Hence the union

$$\{\tilde{\pi}\} \cup \bigcup_{1\leq s\leq n} \left(\bigcup_{\substack{q-1\nmid s \text{ and } p\nmid s,\\ \varepsilon\in\mathbb{F}_q^\times}} \mathcal{P}_{s,\varepsilon} \right) \cup \bigcup_{1\leq s\leq n} \left(\bigcup_{\substack{p\nmid s,\\ \varepsilon\in\mathbb{F}_q^\times\setminus\{1\}}} \mathcal{P}_{(q-1)s,\varepsilon} \right)$$

is algebraically independent over $\overline{k}$ by Lemma 4.4.2. The theorem follows after replacing any element of $\mathcal{P}_{s,\varepsilon}$ by $\gamma\zeta_{s,\varepsilon}$, and any element of $\mathcal{P}_{(q-1)s,\varepsilon}$ by $\gamma\zeta_{(q-1)s,\varepsilon}$. □

Corollary 4.5.10. *Let $\mathcal{Z}_n = \{\tilde{\pi}, \zeta_s(\varepsilon) : 1 \leq s \leq n$ and $\varepsilon \in \mathbb{F}_q^\times\}$. Then*

$$\operatorname{trdeg}_{\overline{k}} \overline{k}(\mathcal{Z}_n) = 1 - \left\lfloor \frac{n}{q-1} \right\rfloor + \left\lfloor \frac{n}{p(q-1)} \right\rfloor + (q-1)\left(n - \left\lfloor \frac{n}{p} \right\rfloor \right).$$

Proof. Counting using inclusion-exclusion principle. □

The main point of Theorem 4.5.9 is the assertion that attaching any nontrivial C_∞-Hecke character of the sort described at the start of this section to a Carlitz zeta value adds another algebraically independent number to the set

$$\{\tilde{\pi}, \zeta_s(1) : 1 \leq s \leq n \text{ with } q-1 \nmid s \text{ and } p \nmid s\},$$

which was first shown in [9] to be the largest possible algebraically independent set among Carlitz's zeta values with bounded weight.

General CMZs

We start with an example.

Example 4.5.11. Let $s_1, s_2 \geq 1$ not divisible by $q-1$ and p, and let $\varepsilon_1, \varepsilon_2 \in \mathbb{F}_q^\times$ such that, if $s_1 = s_2$, then $\varepsilon_1 \neq \varepsilon_2$. For each ε_i let γ_i be a $(q-1)^{st}$ root of ε_i. Furthermore, let $H(i)$ be the set consisting of all coefficients of the Anderson-Thakur polynomials H_{s_i-1}.

For each $i \in \{1, 2\}$, Proposition 4.2.4 implies that $\zeta_{s_i}(\varepsilon_i)$ is nonzero. Using Proposition 4.2.3, this implies the existence of some $u_i \in H(i)$ such that $\mathrm{Li}_{s_i}(\varepsilon_i, u_i) \neq 0$. Identify such a u_i, and consider the set

$$S'' = \begin{cases} \{\tilde{\pi}^s, \gamma_1 \mathrm{Li}_s(\varepsilon_1, u_1), \gamma_2 \mathrm{Li}_s(\varepsilon_2, u_2)\} & \text{if } s_1 = s_2 = s; \\ \{\tilde{\pi}^{s_1}, \tilde{\pi}^{s_2}, \gamma_1 \mathrm{Li}_{s_1}(\varepsilon_1, u_1), \gamma_2 \mathrm{Li}_{s_2}(\varepsilon_2, u_2)\} & \text{if } s_1 \neq s_2. \end{cases}$$

In case $s_1 = s_2 = s$, notice that S'' is linearly independent over k by Lemmas 4.5.2 and 4.5.7. Now, by Lemma 4.4.2, the following set is algebraically independent over $\overline{k}$:

$$S' = \{\tilde{\pi}, \gamma_1 \mathrm{Li}_{s_1}(\varepsilon_1, u_1), \gamma_2 \mathrm{Li}_{s_2}(\varepsilon_2, u_2)\}.$$

Furthermore, by Lemma 4.4.3, the set

$$S = \{\tilde{\pi}, \mathrm{Li}_{s_1}(\varepsilon_1, u_1), \mathrm{Li}_{s_2}(\varepsilon_2, u_2), \mathrm{Li}_{s_1 s_2}(\varepsilon_1, \varepsilon_2, u_1, u_2)\}$$

is algebraically independent over $\overline{k}$. Consider the collection

$$T' = \{\tilde{\pi}, \mathrm{Li}_{s_1}(\varepsilon_1, \alpha), \mathrm{Li}_{s_2}(\varepsilon_2, \beta), \mathrm{Li}_{s_1 s_2}(\varepsilon_1, \varepsilon_2, \gamma, \delta) : \alpha, \gamma \in H(1), \beta, \delta \in H(2)\}.$$

This set might not be algebraically independent, but we can pick a largest algebraically independent subset $T \subset T'$ such that T contains S. After that, we can replace the last three elements of T using Proposition 4.2.3 such that

$$\{\pi, \zeta_{s_1}(\varepsilon_1), \zeta_{s_2}(\varepsilon_2), \zeta_{s_1,s_2}(\varepsilon_1, \varepsilon_2)\} \subset T,$$

implying algebraic independence of the CMZs above.

Using the ideas in the example, let us give a recipe to generate a subset of CMZ.

Recipe to generate algebraically independent CMZs. This is five-step process to generate a set $\mathcal{MZ}$ of algebraically independent CMZs over $\overline{k}$, if we are given a list of positive integers and elements of $\mathbb{F}_q^\times$.

I. Choose distinct positive integers $s_1, \ldots, s_r$ not divisible by p. Fix the ordering.

II. For each $i = 1, \ldots, r$, choose distinct elements $\varepsilon_{i1}, \ldots, \varepsilon_{im_i}$ of $\mathbb{F}_q^\times$, such that if $q-1|s_i$ then none of the ε_{ij} equals 1. Fix the ordering.

III. Define the ordered string

$$S = (s^1, \ldots, s^{m_1+\cdots+m_r})$$

which is some permutation of the multiset $\{m_1 \cdot s_1, \ldots, m_r \cdot s_r\}$.

IV. Define another ordered string

$$E = (\varepsilon^1, \ldots, \varepsilon^{m_1+\cdots+m_r})$$

as follows. For each $i = 1, \ldots, r$, let $\alpha_1, \ldots, \alpha_{m_i}$ be the indices of S with $s^{\alpha_j} = s_i$ for all j. Then let $\varepsilon^{\alpha_1}, \ldots, \varepsilon^{\alpha_{m_i}}$ be a some permutation of the elements $\varepsilon_{i1}, \ldots, \varepsilon_{im_i}$.

V. Consider the set

$$\mathcal{MZ} = \{\tilde{\pi}\} \cup \{\zeta_{\vec{s}^{ij}}(\vec{\varepsilon}^{ij}) : 1 \leq i \leq j \leq m_1 + \cdots + m_r\}.$$

where $\vec{s}^{ij} = (s^i, \ldots, s^j)$ and $\vec{\varepsilon}^{ij} = (\varepsilon^i, \ldots, \varepsilon^j)$. Then this set is algebraically independent over $\overline{k}$.

The set $\mathcal{MZ}$ given in the recipe above will be algebraically independent over $\overline{k}$, and this can be easily shown using a similar proof as given in Theorem 4.5.9. Let $\mathcal{P}'_{s^i,\varepsilon^i}$ be the set of all CMPs of weight at most n appearing in the expression in Proposition 4.2.3 for $\zeta_{s^i}(\varepsilon^i)$, and consider a maximal subset of $\mathcal{P}_{s^i,\varepsilon^i}$ such that $\{\pi^s\} \cup \mathcal{P}_{s^i,\varepsilon^i}$ is linearly independent over k. By applying Lemmas 4.5.7 or 4.5.8, each subset

$$\{\tilde{\pi}^s\} \cup \bigcup_{s^i=s} \mathcal{P}_{s^i,\varepsilon^i}$$

is linearly independent over k for all $s \in \{s_1, \ldots, s_r\}$. Hence the union

$$\{\tilde{\pi}\} \cup \bigcup_i \mathcal{P}_{s^i,\varepsilon^i}$$

is algebraically independent over $\overline{k}$ by Lemma 4.4.2. The theorem follows by applying Lemma 4.4.3, and using Proposition 4.2.3 again to replace relevant CMPs by CMZs.

We single out two potentially interesting such $\mathcal{MZ}$. The second one generalizes [21, Theorem 1.1] (on finding large algebraically independent subsets of Thakur's multizeta values).

Theorem 4.5.12. *Fix a positive integer r with $r \le q-1$. Let $s_1, \dots, s_r$ be distinct positive integers not divisible by p, and let $\varepsilon_{i,1}, \dots, \varepsilon_{i,r} \in \mathbb{F}_q^\times$ be distinct, such that if $q-1|s_i$ then none of the ε_{ij} equals 1. Then the union of the following sets is algebraically independent over $\overline{k}$:*

$$\{\tilde{\pi}\},$$

$$\{\zeta_{s_1}(\varepsilon_{1,i}), \dots, \zeta_{s_r}(\varepsilon_{r,i}) : 1 \le i \le r\},$$

$$\{\zeta_{s_j,s_{j+1}}(\varepsilon_{j,i}, \varepsilon_{j+1,i}) : 1 \le j \le r-1, 1 \le i \le r\},$$

$$\{\zeta_{s_j,s_{j+1},s_{j+2}}(\varepsilon_{j,i}, \varepsilon_{j+1,i}, \varepsilon_{j+2,i}) : 1 \le j \le r-2, 1 \le i \le r\},$$

$$\vdots$$

$$\{\zeta_{s_1,\dots,s_r}(\varepsilon_{1,i}, \dots, \varepsilon_{r,i}) : 1 \le i \le r\}.$$

Proof. Set

$$S = (s_1, \dots, s_r, s_1, \dots, s_r, \dots, s_1, \dots, s_r),$$

$$E = (\varepsilon_{11}, \dots, \varepsilon_{r1}, \varepsilon_{12}, \dots, \varepsilon_{r2}, \dots, \varepsilon_{1r}, \dots, \varepsilon_{rr}),$$

where S and E are the strings appearing in the recipe above. $\square$

Theorem 4.5.13. *Consider $s_i, \varepsilon_{ij}, S, E$ as in the recipe above. Then the set $\{\tilde{\pi}\} \cup \{\zeta_{\vec{s}^{ij}}(\vec{\varepsilon}^{ij})\}$ is algebraically independent over $\overline{k}$, where*

$$S = (s_1, \dots, s_1, s_2, \dots, s_2, \dots, s_r, \dots, s_r),$$

$$E = (\varepsilon_{11}, \dots, \varepsilon_{1m_1}, \varepsilon_{21}, \dots, \varepsilon_{2m_2}, \dots, \varepsilon_{r1}, \dots, \varepsilon_{rm_r}).$$

Proof. Immediate by the recipe above. $\square$

Corollary 4.5.14. *Let $q \neq 2$ and $r \geq 2$, and let*

- *k_1 be $\overline{k}$ adjoining all CMZs of depth 1;*
- *k_r be k_1 adjoining all CMZs of depth r.*

Then $\operatorname{trdeg}_{k_1} k_r = \infty$.

Proof. As $q \neq 2$, there are infinitely many positive integers not divisible by $q-1$ and p. Let $\mathbb{Z}'$ be this set. For any positive integer $l \geq r$, let $s_1, \ldots, s_l$ be the first l terms of $\mathbb{Z}'$. Then Theorem 4.5.13 gives rise to $l-r$ CMZs of depth r that is algebraically independent with one another and all the CMZs of depth 1. We are done as l can be made arbitrarily large. □

4.6 Remarks on linear relations

Note that, if we let $k_{\leq w}$ be the field $\overline{k}$ adjoining all CMZs of weight at most w, then the algebraic independence Theorems above gives us a crude bound

$$\operatorname{trdeg}_{\overline{k}}(k_{\leq w}) \geq 1 + \frac{r(q-1)(r(q-1)+1)}{2}, \qquad r = \left\lfloor \sqrt{w + \frac{1}{4}} - \frac{1}{2} \right\rfloor.$$

This restricts the number of relations between the CMZs. In order to cut down the transcendence degree of $k_{\leq w}$, we will need to write down explicit algebraic relations. For example, a trivial one mentioned before is

$$\zeta_{pn}(\varepsilon) = \zeta_n(\varepsilon^{q/p})^p,$$

and a non-trivial one is the shuffle relation (Proposition 4.2.2).

Work of Rodríguez-Thakur

In [24, 30], nontrivial linear relations between Thakur's multizeta values has been written down. As $\varepsilon^{q-1} = 1$ for any $\varepsilon \in \mathbb{F}_q^\times$, many of these relations applies to the CMZ case as well. We record some of the linear relations here.

- For $m \leq q$,

$$\zeta_{m,m(q-1)}(\varepsilon_1, 1) = \frac{1}{\mathcal{L}_1^m}\zeta_{mq}(\varepsilon_1).$$

- If $u = q^n - \sum_{i=1}^s q^{k_i}$ and $v = (q-1)q^n$, then

$$\zeta_{u,v}(\varepsilon_1, 1) = \frac{(-1)^s}{\mathcal{L}_1^{q^n}} \prod_{i=1}^s (t^{(n-k_i)} - t)^{(k_i)} \zeta_{q^{n+1} - \sum_{i=1}^s q^{k_i}}(\varepsilon_1)$$

- Writing $[k] = \theta^{(k)} - \theta$, here are four relations with no constraints:

$$\zeta_{1,q^2-1}(\varepsilon_1, 1) = \left(\frac{1}{\mathcal{L}_1} + \frac{1}{\mathcal{L}_2}\right)\zeta_{q^2}(\varepsilon_1),$$

$$\zeta_{2q-1,(q-1)(q^2+q-1)}(\varepsilon_1, 1) = \frac{1 - (t^{(2)} - t)^{(1)}}{\mathcal{L}_1^{q+1}\mathcal{L}_2^{q-1}}\zeta_{q^3}(\varepsilon_1),$$

$$\zeta_{q^2-(q-1),(q-1)(q^2+1)}(\varepsilon_1, 1) = \frac{1 - (t^{(2)} - t)^{(1)})}{\mathcal{L}_1^{q^2-1}\mathcal{L}_2}\zeta_{q^3}(\varepsilon_1),$$

$$\zeta_{1,q-1,(q-1)q,\dots,(q-1)q^n}(\varepsilon_1, 1, \dots, 1) = \frac{(-1)^{n+1}}{[1]^{(n)}[2]^{(n-1)}\cdots[n+1]^{(0)}}\zeta_{q^{n+1}}(\varepsilon_1).$$

Before discussing the next example, the following definitions are needed. For any positive integer d, and two strings $\vec{s} = (s_1, \dots, s_r) \in \mathbb{Z}_{\geq 1}$ and $\vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_r) \in \mathbb{F}_q^\times$, define two finite sums

$$S_{d;\vec{s}}(\vec{\varepsilon}) := \sum_{\substack{d=\deg(a_1)>\cdots>\deg(a_r)\geq 0 \\ a_i \in A_+}} \frac{\varepsilon_1^{\deg(a_1)} \cdots \varepsilon_r^{\deg(a_r)}}{a_1^{s_1} \cdots a_r^{s_r}},$$

$$S_{<d;\vec{s}}(\vec{\varepsilon}) := \sum_{d'<d} S_{d';\vec{s}}(\vec{\varepsilon})$$

We will call the sums above *power sums*. The power sums are related to CMZs by

$$\zeta_{\vec{s}}(\vec{\varepsilon}) = \sum_{d=0}^{\infty} S_{d,\vec{s}}(\vec{\varepsilon}).$$

After communications with Thakur, and with computational help by Rodríguez, the following linear relations between CMZs and Thakur's multizeta values were also discovered.

Example 4.6.1. Concentrate on the depth two case. Implicit in the proof of the main theorems in [24] is the computation of various linear relations between power sums. In particular, one can use the results of this paper to verify the following.

- If $a = q^n - \sum_{i=1}^{m} q^{k_i}$ and $b = (q-1)q^n$, with $1 \le m < q$ and $1 \le k_i < n$, then

$$S_{d;a,b}(1,1) = \frac{(-1)^m}{\mathcal{L}_1^{q^n}} \prod_{i=1}^{m} [n-k_i]^{q^{k_i}} S_{d-1;a+b}(1).$$

- If $a' = m_1 q^n$ and $b' = m_1(q^{n+1} - q^n) + \sum_{i=1}^{m_2}(q^{n+1} - q^{k_i})$, with $0 \le k_i \le n+1$, $1 \le m_1 < q$, $0 \le m_2 \le q - m_1$, then

$$S_{d;a',b'}(1,1) = \frac{1}{\mathcal{L}_1^{q^n m_1}} S_{d-1;a'+b'}(1).$$

By observing that

$$\frac{S_{d;\alpha,\beta}(-1,1)}{S_{d-1;\alpha+\beta}(-1)} = -\frac{S_{d;\alpha,\beta}(1,1)}{S_{d-1;\alpha+\beta}(1)},$$

summing the above equalities over d gives us

$$\zeta_{a,b}(-1,1) = -\frac{(-1)^m}{\mathcal{L}_1^{q^n}} \prod_{i=1}^{m} [n-k_i]^{q^{k_i}} \zeta_{a+b}(-1);$$

$$\zeta_{a',b'}(-1,1) = -\frac{1}{\mathcal{L}_1^{q^n m_1}} \zeta_{a'+b'}(-1).$$

If we set $q = 3$, this shows that

$$\frac{\zeta_{\alpha,\beta}(-1,1)}{\zeta_{\alpha+\beta}(-1)}$$

is rational for $(\alpha, \beta) \in \{(1,2), (1,4), (1,6), (1,8), (2,4), (2,6)\}$. Interestingly enough, computer calculations tells us that these are the only values in the range $1 \leq \alpha, \beta \leq 9$ for which the above quotient is rational.

Todd's method in the colored case

In [31], Todd explained how one can use the shuffle relation to generate new linear relations among Thakur's multizeta values from known ones. This can be extended to the case of CMZs. To do this, we need a more refined version of the shuffle relation presented in Proposition 4.2.2. In the remainder of this section, for every integer l, we will always fix an ordering of all compositions of l.

Definition 4.6.2. Fix positive integers d and l, and let $\mathcal{V} = (V_1, \ldots, V_{2^{l-1}})$ be the fixed ordering of the 2^{k-1} compositions of l. Define the space of *binary relations*

$$B_l := \Bigg\{ (\vec{a}_1, \ldots, \vec{a}_{2^{l-1}}, \vec{b}_1, \ldots, \vec{b}_{2^{l-1}}) \in k^{|V_1|+\cdots+|V_{2^{l-1}}|+|V_1|+\cdots+|V_{2^{l-1}}|} : \\ \sum_{i=1}^{2^{l-1}} \sum_{\vec{\varepsilon} \in (\mathbb{F}_q^\times)^{|V_i|}} (a_{i,\vec{\varepsilon}} S_{d;V_i}(\vec{\varepsilon}) + b_{i,\vec{\varepsilon}} S_{d+1;V_i}(\vec{\varepsilon})) = 0 \text{ for all } d \in \mathbb{Z}_{\geq 1} \Bigg\},$$

where each $\vec{a}_i$ above is a vector of length $(q-1)^{|V_i|}$ with entries indexed by elements of $(\mathbb{F}_q^\times)^{|V_i|}$.

The above space is a colored analog of Todd's spaces of linear relations, and is motivated by the work of Rodriguez-Thakur in the previous section. For example, by [30] or a direct computation, if $m \leq q$, one has the binary relation

$$S_{d;mq}(\varepsilon) - \mathcal{L}_1^m S_{d+1;m,m(q-1)}(\varepsilon, 1) = 0,$$

which also gives rise to the relation

$$\zeta_{m,m(q-1)}(\varepsilon, 1) = \frac{1}{\mathcal{L}_1^m} \zeta_{mq}(\varepsilon).$$

Elements in the space B_l of binary relations do not necessarily give rise to a linear relation among CMZs, and vice versa. Nevertheless, the main goal of this section is to show that B_l is a source of producing linear relations among CMZs.

Lemma 4.6.3 ([19, Lemma 2.5 and Theorem 2.6])**.** *Let $\vec{s} = (v_1, \ldots, v_r)$ and $\vec{s}' = (v'_1, \ldots, v'_s)$ be strings of positive integers, and let $\vec{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_r)$ and $\vec{\varepsilon}' = (\varepsilon'_1, \ldots, \varepsilon'_s)$ be strings of elements in $\mathbb{F}_q^\times$. Then*

$$S_{<d;\vec{s}}(\vec{\varepsilon}) S_{<d;\vec{s}'}(\vec{\varepsilon}') = \sum_{\vec{s}'', \vec{\varepsilon}''} f_{\vec{s}'', \vec{\varepsilon}''} S_{<d;\vec{s}''}(\vec{\varepsilon}'')$$

$$S_{d;\vec{s}}(\vec{\varepsilon}) S_{d;\vec{s}'}(\vec{\varepsilon}') = \sum_{\vec{s}'', \vec{\varepsilon}''} g_{\vec{s}'', \vec{\varepsilon}''} S_{d;\vec{s}''}(\vec{\varepsilon}'')$$

where the sums above are finite, and the coefficients $f_{\vec{s}'', \vec{\varepsilon}''}, g_{\vec{s}'', \vec{\varepsilon}''} \in \mathbb{F}_q$ are independent of d. In particular, writing

$$\Delta^j_{s_1, s_2} = \left((-1)^{s-1} \binom{j-1}{s-1} + (-1)^{s'-1} \binom{j-1}{s'-1} \right),$$

one has

$$S_{d;s}(\varepsilon) S_{d;s'}(\varepsilon') = S_{d;s+s'}(\varepsilon\varepsilon') + \sum_{\substack{0<j<s+s' \\ q-1|j}} \Delta^j_{s,s'} S_{d;s+s'-j,j}(\varepsilon\varepsilon', 1). \qquad \square$$

Note that the Lemma above implies Proposition 4.2.2, and is an exercise on the inclusion-exclusion principle. This Lemma also implies the following. Suppose we have a binary relation

$$\sum_{i=1}^{2^{l-1}} \sum_{\vec{\varepsilon} \in (\mathbb{F}_q^\times)^{|V_i|}} (a_{i,\vec{\varepsilon}} S_{d;V_i}(\vec{\varepsilon}) + b_{i,\vec{\varepsilon}} S_{d+1;V_i}(\vec{\varepsilon})) = 0.$$

Choose a positive integer w and a composition W of w. Also fix a positive integer D and $\mathcal{E} \in {\mathbb{F}_q^\times}^{|W|}$. Then

$$\begin{aligned}
&S_{D;W}(\mathcal{E}) \sum_{d<D} \left(\sum_{i=1}^{2^{l-1}} \sum_{\vec{\varepsilon} \in (\mathbb{F}_q^\times)^{|V_i|}} (a_{i,\vec{\varepsilon}} S_{d;V_i}(\vec{\varepsilon}) + b_{i,\vec{\varepsilon}} S_{d+1;V_i}(\vec{\varepsilon})) \right) \\
&= S_{D;W}(\mathcal{E}) \left(\sum_{i=1}^{2^{l-1}} \sum_{\vec{\varepsilon} \in (\mathbb{F}_q^\times)^{|V_i|}} (a_{i,\vec{\varepsilon}} S_{<D;V_i}(\vec{\varepsilon}) + b_{i,\vec{\varepsilon}} S_{D;V_i}(\vec{\varepsilon})) \right) \\
&= \sum_{i,j} c_{X_i,\vec{\varepsilon}_{ij}} S_{D;X_i}(\vec{\varepsilon}_{ij}),
\end{aligned}$$

where the coefficients $c_{X_j,\vec{\varepsilon}_j} \in k$ are independent of the choice of D by the shuffle relation. Hence, by summing over D, one gets new linear relations among CMZs. In general the new relations produced are complicated to describe, but we highlight the following very special case as an example.

Proposition 4.6.4. *Choose the composition $W = (w)$ and a $(q-1)^{st}$ root of unity $\mathcal{E} = (\omega)$. Suppose we have a binary relation*

$$\sum_{i=1}^{2^{l-1}} \sum_{\vec{\varepsilon} \in (\mathbb{F}_q^\times)^{|V_i|}} (a_{i,\vec{\varepsilon}} S_{d;V_i}(\vec{\varepsilon}) + b_{i,\vec{\varepsilon}} S_{d+1;V_i}(\vec{\varepsilon})) = 0,$$

Write each $V_i = (v_i, V_i')$, and assume q is large enough so that $w + v_i \leq q$ for all i.

Then the relation produced by applying the procedure described above is

$$\sum_{i=1}^{2^{l-1}} \sum_{\vec{\varepsilon}=(\varepsilon,\vec{\varepsilon}')\in(\mathbb{F}_q^\times)^{|V_i|}} \big((a_{i,\vec{\varepsilon}} + b_{i,\vec{\varepsilon}})S_{D;w,V_i}(w,\vec{\varepsilon}) + b_{i,\vec{\varepsilon}}S_{D;w+v_i,V_i'}(w\varepsilon,\vec{\varepsilon}')\big) = 0.$$

In particular, one obtains the linear relation

$$\sum_{i=1}^{2^{l-1}} \sum_{\vec{\varepsilon}=(\varepsilon,\vec{\varepsilon}')\in(\mathbb{F}_q^\times)^{|V_i|}} \big((a_{i,\vec{\varepsilon}} + b_{i,\vec{\varepsilon}})\zeta_{w,V_i}(w,\vec{\varepsilon}) + b_{i,\vec{\varepsilon}}\zeta_{w+v_i,V_i'}(w\varepsilon,\vec{\varepsilon}')\big) = 0.$$

Proof. This follows by the computations

$$\begin{aligned} S_{D;w}(\omega)\sum_{d<D} S_{d;V_i}(\vec{\varepsilon}) &= S_{D;w}(\omega)S_{<D;V_i}(\vec{\varepsilon}) \\ &= S_{D;w,V_i}(\omega,\vec{\varepsilon}) \end{aligned}$$

and

$$\begin{aligned} S_{D;w}(\omega)\sum_{d<D} S_{d+1;V_i}(\vec{\varepsilon}) &= S_{D;w}(\omega)\left(S_{<D;V_i}(\vec{\varepsilon}) + S_{D,v_i}(\varepsilon)S_{<D,V_i'}(\varepsilon')\right) \\ &= S_{D;w,V_i}(\omega,\vec{\varepsilon}) + S_{D;w+v_i}(\omega\varepsilon)S_{<D,V_i'}(\varepsilon') \\ &= S_{D;w,V_i}(\omega,\vec{\varepsilon}) + S_{D;w+v_i,V_i}(\omega\varepsilon,\varepsilon'). \end{aligned}$$

The second-last equality comes from the well-known equality

$$S_{d,a}(1)S_{d,b}(1) = S_{d,a+b}(1) \text{ if } a+b \le q,$$

which is a special case of Lemma 4.6.3 that is known since Carlitz's work. $\square$

Remark. One can remove the restriction on q completely in the proposition above by using the second equality of Lemma 4.6.3, but at the expense of producing a longer binary relation involving $\Delta^j_{w,\epsilon}$.

Chapter 5

Future Directions

5.1 Adelic multizeta values

We have studied properties of the colored multizeta values $\zeta_{\vec{s}}(\vec{\varepsilon})$, and these are all elements in k_∞. Let us now consider the case of Thakur multizeta values $\zeta_{\vec{s}}(\vec{1}) = \zeta(\vec{s})$.

In [10], the definition for multizeta values $\zeta(\vec{s})_v$ at every finite place v of A was defined by realizing Carlitz multipolylogarithms as coordinates of a special point under the logarithm map of a certain t-module. This is an element in the completion k_v of k at v, and one can ask about algebraic relations on these v-adic multizeta values. An answer has been given very recently.

Theorem 5.1.1 ([11]). *For any finite place v, the multizeta values $\zeta(\vec{s})_v$ satisfy the algebraic relations over $\overline{k}$ that the $\zeta(\vec{s})$ satisfy. In particular, they all satisfy the* *same shuffle relations.*

Theorem 5.1.2 ([14]). *For a Thakur multizeta value $\zeta(\vec{s})$, its v-adic counterpart $\zeta(\vec{s})_v$ is a v-adic integer for almost all finite place v.*

Using this, we can define the adelic (Thakur) multizeta values. Let S be a finite collection of finite places of A, let Σ_S be the set of places of A that are not in S, and let $\mathbb{A}_S$ be the ring of adeles for A with respect to Σ_S. (If [11, Conjecture 5.4.1] is true, we can allow S to include the infinite place of A as well.)

Definition 5.1.3. Given a tuple of positive integers $\vec{s} = (s_1, \ldots, s_r)$, the *finite adelic multizeta value* is

$$\zeta^{\mathbb{A}_S}(\vec{s}) := (\zeta(\vec{s})_v)_{v \in \Sigma_S},$$

which is an element of $\mathbb{A}_S$.

We want to define a k-algebra using these finite adelic multizeta values. Consider the set of finite adelic multizeta values

$$\mathrm{FAM} := \left\{ \zeta^{\mathbb{A}_S}(\vec{s}) : \vec{s} \text{ is a tuple of positive integers} \right\}.$$

By Theorem 5.1.1 the finite adelic multizeta values $\zeta^{\mathbb{A}_S}(\vec{s})$ satisfy the same linear relations as $\zeta(\vec{s})_v$ for each place v. Hence many linear algebraic relations carry over to the adelic case. For example, let $\mathcal{AM}_{\mathrm{FAM}}$ be the k-algebra generated by the elements of FAM. For $w \geq 1$, also let $\mathcal{AM}_{\mathrm{FAM},w}$ be the k-linear space spanned by elements of FAM of weight w. Then there is a grading

$$\mathcal{AM}_{\mathrm{FAM}} = k \oplus \bigoplus_{w=1}^{\infty} \mathcal{AM}_{\mathrm{FAM},w}.$$

This is because the k-algebra generated by all Carlitz multizeta values possesses a weight grading (by [7] or [19]). Furthermore, by Theorem 4.5.13, we can find arbitrarily large families of elements in $\mathcal{AM}_{\mathrm{FAM}}$ that are algebraically independent, and if $\mathcal{F}$ is such a family, then there is an injection

$$k[x_1, \ldots, x_{|\mathcal{F}|}] \hookrightarrow \mathbb{A}_S.$$

Question 5.1.4. Can we construct adelic colored multizeta values and obtain similar results?

At present we do not know how to do this. A main obstruction is the following. The construction of the v-adic multizeta values in [10] makes use of the fact that we can realize Carlitz multipolylogarithms as coordinates of a special point under the logarithm map of a carefully written-down t-module. In the colored case, our multipolylogarithms (Definition 4.1.2) do not seem to obey this due to the extra terms $\varepsilon_j^{i_j}$.

Question 5.1.5. Can we write down more linear relations between colored multizeta values that is not implied by those from the noncolored case?

5.2 Some other classes of multizeta values

Everything discussed in this thesis is for multizeta values over A, or in other words, multizeta values on the function field of $\mathbb{P}^1_{\mathbb{F}_q}$. A natural question arises.

Question 5.2.1. Can we obtain results for multizeta values over function fields for curves of higher genera?

The definition of such multizeta values is given in [27, Section 5.1]. A possible starting point is to do some tests on function fields of class number one, of which there are only finitely many.

Here is another question that does not seem too tractable due to loss of symmetry. If we return back to the $\mathbb{P}^1$ case, we have indicated that colored multizeta values are obtained by twisting multizeta values with special $\mathbb{C}_\infty$-Hecke characters (degree-preserving q-finite characters $\mathbb{A}^\times/k^\times \longrightarrow \mathbb{C}_\infty^\times$ with trivial conductor).

Question 5.2.2. Can we generalize the results in this thesis to multizeta values twisted by other kinds of $\mathbb{C}_\infty$-Hecke characters, or to coefficients of Eisenstein series (as defined in [13])?

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