LIFTINGS OF ELEMENTARY ABELIAN COVERS OF CURVES

Jianing Yang

A DISSERTATION

in

Mathematics

Presented to the Faculties of the University of Pennsylvania

in

Partial Fulfillment of the Requirements for the

Degree of Doctor of Philosophy

2023

Supervisor of Dissertation

David Harbater, Christopher H. Browne Distinguished Professor in the School of Arts and Sciences

Graduate Group Chairperson

Ron Donagi, Thomas A. Scott Professor of Mathematics and Graduate Chair

Dissertation Committee

David Harbater, Christopher H. Browne Distinguished Professor in the School of Arts and Sciences
James Haglund, Professor of Mathematics
Julia Hartmann, Professor of Mathematics

ABSTRACT

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Jianing Yang

David Harbater

Given a Galois cover of curves f over a field of characteristic p, the lifting problem asks whether there exists a Galois cover over a complete mixed characteristic discrete valuation ring whose reduction is f. In this thesis, we try to answer this question in the case of elementary abelian p-groups. We prove a combinatorial criterion for lifting an elementary abelian p-cover, dependent on the branch loci of its p-cyclic subcovers. Moreover, we study how branch points of a lift coalesce on the special fiber. Finally, we construct lifts for several families of $(\mathbb{Z}/2)^3$-covers of various conductor types, both with equidistant branch locus geometry and non-equidistant branch locus geometry.

TABLE OF CONTENTS

ABSTRACT . . . ii

CHAPTER 1 : INTRODUCTION . . . 1

1.1 The Lifting Problem and Oort Groups . . . 2

1.2 Outline of the Chapters . . . 3

CHAPTER 2 : BACKGROUND AND NOTATIONS . . . 6

2.1 The Local Lifting Problem . . . 6

2.2 Ramification Theory and Conductors . . . 8

2.3 Artin-Schreier Covers . . . 10

2.4 Branch Locus Geometry and Hurwitz Trees . . . 12

2.5 Liftings of $\mathbb{Z}/p$ and $\mathbb{Z}/p \times \mathbb{Z}/p$-Covers . . . 15

CHAPTER 3 : BRANCH CYCLE CRITERION FOR $(\mathbb{Z}/p)^n$-COVERS . . . 18

3.1 Ramification Jumps . . . 18

3.2 Conductor Type . . . 20

3.3 Key Lemmas . . . 21

3.4 Main Theorem . . . 26

CHAPTER 4 : COALESCING OF BRANCH POINTS . . . 29

4.1 Stratification of the Space of Artin-Schreier Covers . . . 29

4.2 Coalescing of Branch Points of a Lift . . . 30

CHAPTER 5 : LIFTINGS OF $(\mathbb{Z}/2)^3$-COVERS . . . 33

5.1 Hurwitz Trees for $(\mathbb{Z}/2)^3$-Covers of Type $(4, 4, 4)$ . . . 33

5.2 Lifting $(\mathbb{Z}/2)^3$-Covers of Type $(4, 4, 2r), r \geq 3$ . . . 38

BIBLIOGRAPHY . . . 42

CHAPTER 1

INTRODUCTION

Given a smooth curve over a field k of characteristic p, e.g. $\mathbb{F}_p$, we can study its lift to characteristic 0, which is a smooth (relative) curve over a mixed characteristic complete discrete valuation ring R with residue field k, e.g. the p-adic integers $\mathbb{Z}_p$. Moreover, if we let a finite group act on the curve and take the quotient, we obtain a Galois cover of such curves. The Lifting Problem asks: given a Galois cover of smooth curves in characteristic p, $X \xrightarrow{G} \mathbb{P}^1_k$, when can we lift it to characteristic 0? Which groups can be realized as Galois groups of covers that lift? One famous result in the area is the Oort conjecture, which states that all cyclic covers lift. This topic is also related to the Inverse Galois Problem, deformation theory, étale fundamental groups, and patching, etc..

The focus of my thesis is on the elementary abelian case, i.e. $(\mathbb{Z}/p)^n$-covers of smooth projective curves. It is known that some of them lift, while some of them don't, but results about when they lift are very incomplete. My main result, which generalizes Barry Green and Michel Matignon's criterion for lifting $\mathbb{Z}/p \times \mathbb{Z}/p$-covers [GM98], applies to all elementary abelian p-covers of $\mathbb{P}^1_k$, where k is an algebraically closed field of characteristic p. I have shown the following branch cycle criterion, a precise version of which will be stated in section 3 (Theorem 3.4.1, see also Section 1.2).

Theorem 1.0.1 (Imprecise version). *Let $C : X \to \mathbb{P}^1_k$ be a $(Z/p)^n$-Galois cover, and $m_1+1 \leq \cdots \leq m_n+1$ be the conductors of its n generating $\mathbb{Z}/p$-subcovers. Then C can be lifted to characteristic 0 if and only if $m_i \equiv -1$ mod p^{n-i} for $1 \leq i \leq n-1$ and these $\mathbb{Z}/p$-subcovers can be respectively lifted with branch loci $B_1, \ldots, B_n$ that satisfy a certain combinatorial criterion.*

Moreover, I construct explicit lifts for several new families of $(\mathbb{Z}/2)^3$-covers, including the first known lifts for elementary abelian covers with non-equidistant geometry beyond $(\mathbb{Z}/p)^2$. Finally, I classify all admissible Hurwitz trees for certain types of $(\mathbb{Z}/2)^3$-covers. I also translate the p-rank stratification of the Artin-Schreier space to a stratification of the characteristic 0 Hurwitz space by the branch locus coalescing behavior of p-cyclic covers in characteristic 0.

1.1 The Lifting Problem and Oort Groups

Let R be a complete mixed characteristic p discrete valuation ring with field of fractions K and algebraically closed residue field k. Then we can state the lifting problem as follows:

Question 1.1.1 (**The global lifting problem**). *Let $f : X_k \xrightarrow{G} \mathbb{P}^1_k$ be a Galois branched cover of smooth projective curves. Does there exist some choice of R, and a smooth projective R-curve X_R such that the special fiber of $X_R \xrightarrow{G} \mathbb{P}^1_{R'}$ is f? If the answer is yes, we say that f lifts.*

Remark 1.1.2. A smooth projective curve always lifts over any complete discrete valuation ring R with residue field k [SGA03, III, Corollaire 6.10 and Proposition 7.2]. However, simply taking the equation defining X_k, lifting its coefficients to R does not always work, since there may not be a G-action on X_R that reduces to the one on X_k.

There are various obstructions to lifting. First of all, the Hurwitz bound [Har77, IV.2] tells us that, if $|G| > 84(g(X) - 1)$, then $X_k \xrightarrow{G} \mathbb{P}^1_k$ does not lift. For abelian groups, however, Hurwitz bounds do not give obstructions.

A key statement concerning the lifting problem is the Oort conjecture.

Theorem 1.1.3 (**Oort conjecture**). *The answer to the lifting problem is positive if G is cyclic.*

In the case of prime to p groups, it was proven by Grothendieck, using the "tame Riemann existence converse" [Obu17, Theorem 1.5], . The $\mathbb{Z}/p$ case was proven by Oort-Sekiguchi-Suwa [OSS] in 1989, using Artin-Schreier theory. The $\mathbb{Z}/p^2$ case was proven by Green-Matignon [GM98] in 1998, by reducing to the local lifting problem and using Artin-Schreier-Witt theory. Finally the Oort conjecture was proven for general cyclic groups by Obus-Wewers [OW14] and Pop [Pop14] in 2014.

This result motivates the natural question: for which finite groups G do *all* G-covers lifts? For which finite groups G do *some* G-covers lift? we define the following:

Definition 1.1.4. A finite group G for which every G-Galois cover $X \to \mathbb{P}^1_k$ lifts to characteristic 0 is called an **Oort group** for k. If there exists a G-Galois cover that lifts, G is called a **weak Oort group**.

In particular, all Oort groups are local Oort groups. The Oort conjecture states that cyclic groups are Oort groups. A more detailed classification of Oort groups and weak Oort groups will be discussed in the next chapter.

Example 1.1.5. *Let $X_k = \mathbb{P}^1_k$, and $G = (\mathbb{Z}/p)^n$. Then G embeds into the additive group of k and has an additive action on X_k. Suppose that the G-Galois cover $X_k \to \mathbb{P}^1_k$ lifts to R. Then G acts on the generic fiber X_K. However, since the genus of X_K is 0, the group of automorphisms of X_K embeds into $\mathrm{PGL}_2(\bar{K})$, which does not contain $(\mathbb{Z}/p)^n$ for $n > 1$ except for $(\mathbb{Z}/2)^2$. Therefore, elementary abelian p-groups, apart from p-cyclic groups and the Klein-four group, are not Oort groups.*

Meanwhile, they are shown to be be weak Oort groups in [Mat99].

1.2 Outline of the Chapters

In Chapter 2, we introduce the notations that we will use throughout this thesis. In Section 2.1, we state the local-global principle, which reduces the global lifting problem to the local lifting problem. Then we define the local Oort groups and weak local Oort groups, as well as summarize known classifications of these groups. In Section 2.2, we discuss some results from Chapter 4 of [Ser] on ramification groups, and degree of the different, which will be used in Chapter 3. We also define the conductor as in [GM98]. In Section 2.3, we define a local G-cover of rings, and show that an Artin-Schreier extension of local rings can be written in certain standard form. In Section 2.4, we define the branch locus of a cover, and give a brief construction of a Hurwitz tree associated with a branched cover. In Section 2.5, we states the Different Criterion, and a branch cycle criterion for lifting $\mathbb{Z}/p \times \mathbb{Z}/p$-covers, which we generalize in Chapter 3. We also list some lifting results for p-cyclic covers due to Oort-Sekiguchi-Suwa and for $(\mathbb{Z}/p)^2$-covers due to Green-Matignon and Pagot.

In Chapter 3, we prove the Branch Cycle Criterion for lifting general elementary abelian p-covers. In Section 3.1, we define ramification jumps for a $(\mathbb{Z}/p)^n$-extension, and compute them in terms of conductors of intermediate extensions. In Section 3.2, we define the conductor type of a $(\mathbb{Z}/p)^n$-cover, and show that these covers can be defined by a n-tuple of Artin-Schreier equations of certain form. In Section 3.3, we prove some lemmas regarding the degrees of the special different and the generic different. In Section 3.4, we prove the following main theorem (Theorem 3.4.1).

Theorem 1.2.1 (Branch Cycle Criterion). *Let $G = (\mathbb{Z}/p)^n$. Suppose $k[[z]]/k[[t]]$ is a G-extension of conductor type $(m_1+1, \ldots, m_n+1)$. Then there is a lifting of G to a group of automorphisms of $R[[Z]]$ if and only if the following two conditions hold:*

1. $m_i \equiv -1 \mod p^{n-i}$ *for* $1 \leq i \leq n-1$,
2. $k[[z]]^{G_1}, \ldots, k[[z]]^{G_n}$ *can be lifted with branch loci* $B_1, \ldots, B_n$ *such that for any subset of* k *branch points* $\{B_{i_1}, \ldots, B_{i_k}\}$, $|\cap_{1\leq j \leq k} B_{i_j}| = \dfrac{(\min_j(m_{i_j})+1)(p-1)^{k-1}}{p^{k-1}}$.

In Chapter 4, we discuss the way in which branch points of a lift coalesce on the special fiber. In Section 4.1, we state a result by Pries-Zhu on stratification of the space of Artin-Schreier covers. In Section 4.2, we prove a condition on the branch locus geometry of a lift of an Artin-Schreier cover (Proposition 4.2.1).

Proposition 1.2.2. *Consider the component of $AS_{g,s}$ containing an Artin-Schreier cover $f: X \to \mathbb{P}^1_k$ with p-rank s, which corresponds to the partition $[e_1, ..., e_{r+1}]$ of $d+2$, with each $e_j \not\equiv 1 \mod p$. Suppose f is branched at $\{c_1, \ldots, c_{r+1}\}$. As above, f is given by an equation of the form $y^p - y = \sum_{i=1}^{r+1} f_i(\frac{1}{x-c_i})$, where $e_i = \deg(f_i)+1$. Then there exists a lift of f to R whose generic fiber is a degree p Kummer cover with $d+2$ branch points, e_i of which coalesce to c_i on $\mathbb{P}^1_k$ for $1 \leq i \leq r+1$.*

Conversely, any lift of f is a $\mathbb{Z}/p$-cover with $d+2$ branch points, e_i of which coalesce to c_i on $\mathbb{P}^1_k$ for $1 \leq i \leq r+1$.

We then give an interpretation of the Pries-Zhu result in the characteristic 0 setting.

In Chapter 5, we apply results from Chapters 3 and 4 to the case $(\mathbb{Z}/2)^3$ in characteristic 2. In Section 5.1, we prove that a $(\mathbb{Z}/2)^3$-cover of type $(4, 4, 4)$ can only be lifted equidistantly (Proposition 5.1.4), and the Hurwitz tree for the lift has branch partition $(1, 1, 1, 1, 1, 1, 1)$, see Figure 5.2. We also show that a codimension 1 subspace of these covers lift (Proposition 5.1.6), generalizing the construction in [Mat99]. In Section 5.2, we construct lifts for all $(\mathbb{Z}/2)^3$-covers of type $(4, 4, 2r), r \geq 3$ (Proposition 5.2.2). The construction uses a generalized version of a Klein-four lift in [Mit] (Lemma 5.2.1), and results in [Pag] (Proposition 2.5.4). The resulting Hurwitz tree has branch partition $(3, 3, 3, 2, \ldots, 2)$, see Figure 5.3. They are the first known non-equidistant lifts for a $(\mathbb{Z}/p)^n$-cover with $n > 2$.

CHAPTER 2

BACKGROUND AND NOTATIONS

Now we give precise definitions for the relevant objects, and state the questions being studied more carefully. Throughout this thesis, we will use the following notations and assumptions:

- Let k be an algebraically closed field of characteristic p.
- Let R be a finite extension of $W(k)$, the ring of Witt vectors over k ([Ser], Section 2.6), i.e. R is a complete discrete valuation ring of characteristic 0 with residue field k. Let K be the fraction field of R. We always allow extension of R if necessary.
- Let π be the uniformizer of R, and v be the valuation on R with respect to π.
- A curve is assumed to be reduced, connected, and projective unless stated otherwise.
- A G-(Galois) cover of curves $X \to Y$ is a finite, generically separable morphism such that the group of automorphisms $\mathrm{Aut}_Y(X)$ is isomorphic to G, and acts transitively on each fiber.

2.1 The Local Lifting Problem

The following Local-global principle reduces the lifting problem to one of local nature.

Theorem 2.1.1 (**Local-global principle** [Gar96])**.** *Let $f : Y \to X$ be a G-cover of smooth projective curves over k. For each closed point $y \in Y$, let $I_y \leq G$ be the inertia group. If, for all y, the I_y-extension $\hat{\mathcal{O}}_{Y,y}/\hat{\mathcal{O}}_{X,f(y)}$ lifts over R, then f lifts over R.*

Before stating the local lifting problem, let us take a closer look at the p-adic open disk $\mathrm{Spec}R[[T]]$. By the Weierstrass preparation theorem [B1], the points on the open disk, i.e. the prime ideals of $R[[T]]$ are:

1. Height 0: (0).

2. Height 1: (π) (vertical divisor), and ideals generated by distinguished polynomials over R (horizontal divisors).

3. Height 2: (π, T) (the unique maximal ideal).

Remark 2.1.2. For the horizontal divisors of Spec$R[[T]]$, since the generator f is a monic polynomial, $R[T]/(f)$ is a finite extension of R. Then the divisor is defined over $R[T]/(f)$. Since we allow finite extensions of R, we can just consider the horizontal divisors as elements of R, after possible extension. These consists of a closed point on the generic fiber Spec$K[[T]]$, and a closed point on the special fiber Spec$k[[t]]$.

With the above notations, we can now ask:

Question 2.1.3 (**The local lifting problem**). *Suppose G is a finite group, and $k[[z]]/k[[t]]$ is a (possibly ramified) G-Galois extension. Does there exist some R, and a G-Galois extension $R[[Z]]/R[[T]]$ such that the G action on $R[[Z]]$ reduces to the given G action on $k[[z]]$?*

Definition 2.1.4. If the local lifting problem has a solution for a G-extension $k[[z]]/k[[t]]$. We say that the extension *lifts to characteristic* 0, and $R[[Z]]/R[[T]]$ is a lift of the extension.

We then have the corresponding definitions for local Oort groups and weak local Oort groups for k.

In fact, the classification of Oort groups are determined by that of local Oort groups.

Theorem 2.1.5 ([CGH08]). *A finite group G is an Oort group for p, if and only if every cyclic-by-p subgroup ($P \rtimes \mathbb{Z}/m$) of G is a local Oort group.*

Moreover, a finite cyclic-by-p group is an Oort group if and only if it is a local Oort group [CGH17].

Hence from now on, we are only going to look at the local lifting problem, and when we say (weak) Oort groups, we mean (weak) local Oort groups.

The following obstruction to the local lifting problem is due to Chinburg, Guralnick and Harbater.

Theorem 2.1.6 ([CGH08]). *All local Oort groups must be one of the following: cyclic groups, dihedral groups D_{p^n} for any n, the group A_4 (for* $\mathrm{char}(k) = 2$*), and $Q_{2^n}, n \geq 4$ (for* $\mathrm{char}(k) = 2$*).*

Furthermore, the Hurwitz tree obstruction [BW09] rules out all the quaternion groups. All these possible candidates are known to be Oort groups, apart from dihedral groups with $n > 1$. As discussed in chapter 1 and by Theorem 2.1.5, all cyclic groups are local Oort groups. Bouw and Wewers [BW06] show that D_p is a local Oort group for all odd p. Pagot [Pag] proves that the Klein-four group, $D_2 = \mathbb{Z}/2 \times \mathbb{Z}/2$ is a local Oort group for $p = 2$. Obus proves that A_4 [Obu16] and D_9 [Obu15] are local Oort groups. Weaver [Wea17] proves that D_4 is a local Oort group. Finally, Dang [Dan20] proves that D_{25} and D_{27} are local Oort groups.

Meanwhile, the question of whether a finite group G is a weak Oort groups is sometimes called the Inverse Galois Problem for lifting. Some known weak Oort groups are $(\mathbb{Z}/p)^n$ for all n [Mat99] and $G = \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ whenever G is center-free [Obu15]. In [CGH08] groups that are not weak Oort groups are called anti-Oort groups, i.e. no G-Galois cover lifts, for example, $(\mathbb{Z}/p)^2 \times \mathbb{Z}/n$.

For a weak Oort group G that is not an Oort group, we can look more closely and ask which G-covers lift. The subjects of this study are the elementary abelian p-covers in particular. Matignon, in proving elementary abelian p-groups are weak Oort groups, constructs lifts for a special family of $(\mathbb{Z}/p)^n$-covers of type $(p^{n-1}, \ldots, p^{n-1})$.

Theorem 2.1.7 ([Mat99]). *$(\mathbb{Z}/p)^n$ is a weak Oort group for all $n \geq 1$.*

However, no other lifts were previously known for $(\mathbb{Z}/p)^n$-covers for $n \geq 3$.

2.2 Ramification Theory and Conductors

Definition 2.2.1. Let L/K be a Galois extension of complete dicretely valued fields and G be its Galois group. For each integer $i \geq -1$, let G_i be the set of $s \in G$ satisfying $v_L(s(x) - x) \geq i + 1 \ \forall x \in \mathcal{O}_L$, where $\mathcal{O}_L$ is the corresponding valuation ring. Then G_i is a normal subgroup of G, called the *i-th ramification group* of L/K.

Definition 2.2.2. With the above notation, let A be the discrete valuation ring in K with maximal ideal $\mathfrak{m}$, and B its integral closure in L. Let $\mathcal{D}_{B/A}$ be the different of B/A ([Ser], Section 3.3). Then the *degree of the different* $d_{B/A}$ is the length of $B/\mathcal{D}_{B/A}$ as an $A/\mathfrak{m}$-module.

In the case where B/A is $k[[z]]/k[[t]]$, $B/\mathfrak{P} = A/\mathfrak{m}$ for any prime ideal $\mathcal{P}$ of B, since k is algebraically closed. Then the degree of the different equals to $v_B(\mathcal{D}_{B/A})$.

Proposition 2.2.3 ([Ser] IV.2, Proposition 4)**.** *If $\mathcal{D}_{L/K} := \mathcal{D}_{B/A}$ denotes the different of L/K, then*

$$v_L(\mathcal{D}_{L/K}) = \sum_{i=0}^{i=\infty}(|G_i| - 1).$$

Here, $|G_i| = 1$ for i sufficiently large, so this is a finite sum.

In the case where B/A is tamely ramified, for example, $K[[Z]]/K[[T]]$, $\mathrm{char}(K) = 0$, the degree of the different equals to $(|G_0| - 1)$ times the number of ramification points.

Definition 2.2.4. For a $\mathbb{Z}/p$-extension $k((z))/k((t))$ given by the Artin-Schreier equation

$$z^p - z = f(\frac{1}{t}),$$

where $f(\frac{1}{t}) \in k[t^{-1}]$, we call $m + 1 := \deg(f) + 1$ the *conductor* of the extension.

Remark 2.2.5. By [Ser] IV.2, Exercise 5b, if we let $G = \mathrm{Gal}(k((z))/k((t)))$ as above, then $G_m = G$ and $G_{m+1} = 1$.

Now we will give some information on the upper numbering of ramification groups, the Herbrand function, and some properties.

Definition 2.2.6. If u is a real number ≥ -1, G_u denotes the ramification group G_i, where i is the smallest integer $\geq u$. Then define the *Herbrand function* for L/K to be

$$\varphi(u) = \int_0^u \frac{dt}{(G_0 : G_t)}.$$

When we need to specify the extension, write $\varphi_{L/K}$.

By [Ser], page 73, we have that for a positive integer m,

$$\varphi(m) + 1 = \frac{1}{g_0} \sum_{i=0}^{i=m} g_i, \tag{2.1}$$

where $g_i := |G_i|$.

Now we can define the *upper numbering* of the ramification groups:

$$G^{\varphi(u)} = G_u.$$

Then the lower numbering is compatible with subgroups while the upper numbering is compatible with quotient groups. Namely,

Proposition 2.2.7 ([Ser] Chapter IV, Propositions 2 and 14)**.** *Let H be a subgroup of G. Then $H_i = G_i \cap H$ for all i. Let H be a normal subgroup of G. Then $(G/H)^v = G^v H/H$ for all v.*

Finally, we state Herbrands's theorem, which is used to prove the above proposition.

Lemma 2.2.8. *Let H be a normal subgroup of G, and $K' \subset L$ be the fixed field of L. If $v = \varphi_{L/K'}(u)$ then $G_u H/H = (G/H)_v$.*

2.3 Artin-Schreier Covers

Definition 2.3.1. Suppose a domain A is integrally closed in its field of fractions. A *G-extension* of rings B/A is a ring extension such that Frac(B)/Frac(A) is a G-Galois field extension, and B is the integral closure of A in Frac(B). Then the action of G on Frac(B) restricts to an action on B, and $B^G = A$.

If B/A is a G-extension, then Spec$B \to$ Spec$A$ is a G-Galois cover of schemes, and we sometimes also call B/A a G-cover.

Remark 2.3.2. In particular, we can consider local G-extensions of the form $R[[Z]]/R[[T]]$ or $k[[z]]/k[[t]]$. Since $R[[T]]$ and $k[[t]]$ are regular local rings, they are integrally closed in $K((T))$ and $k((t))$, thus satisfying the assumption in the above definition.

A $\mathbb{Z}/p$-extension $k[[z]]/k[[t]]$ is called an Artin-Schreier extension of local rings. Since Spec$k[[z]] \to$ Spec$k[[t]]$ is a $\mathbb{Z}/p$-cover of schemes, we sometimes also call $k[[z]]/k[[t]]$ an Artin-Schreier cover. By Artin-Schreier theory, it can be defined by an equation of the form $z^p - z = f(t)$, called an Artin-Schreier equation, where $f(t) \in k((t))$.

Proposition 2.3.3. *Every Artin-Schreier extension $k[[z]]/k[[t]]$ can be defined by an Artin-Schreier equation in the standard form:*

$$z^p - z = \sum_{i=1}^{m} c_i t^{-i},$$

where $p \nmid m$, $c_m \neq 0$, and $c_i = 0$ if p divides i.

Proof. Assume that $k[[z]]/k[[t]]$ is defined by the equation

$$z^p - z = \sum_{i=-m}^{i} nftye_i t^i = f(t^{-1}) + g(t),$$

where $f(t^{-1}) = \sum_{i=1}^{m} e_{-i} \frac{1}{t^i} \in t^{-1}k[t^{-1}]$, $p \nmid m$ and $g(t) \in k[[t]]$. We will show that after a sequence of changes of variables, this extension can be defined by an equation in the standard form.

First observe that any element $u \in k[[t]]$ can be written as $x^p - x$ for some $x \in k((t))$, by taking $x = -u - u^p - u^{p^2} - \cdots$. Thus taking $z' = z + g(t) + g(t)^p + g(t)^{p^2} + \cdots$, $k[[z]]/k[[t]]$ can be given by $z'^p - z' = f(t^{-1})$.

Similarly, for each i divisible by p, $e_{-i}^{1/p} \in k$ because k is algebraically closed. Let $z" = z' - e_{-i}^{1/p} t^{-i/p}$. Then $z"^p - z" = z'^p - z' - e_{-i} t^{-i} + e_{-i}^{1/p} t^{-i/p}$. Proceed this way, we can "absorb" all the terms in $f(t^{-1})$ with exponent divisible by p. Thus we arrive at the standard form. ◻

Remark 2.3.4. If we are allowed to choose a uniformizer of $k[[t]]$, we can simplify the equation further. We can write $f(t^{-1}) = t^{-m}h(t)$, where $h(t) \in k[t]$ has a non-zero constant term. Then $h(t)$ is a m-th power in $k[[t]]$, since k is algebraically closed. Thus replacing t with a m-th root t' of $1/f(t^{-1})$, which is a uniformizer of $k[[t]]$, $k[[z]]/k[[t]]$ can be defined by

$$z^p - z = \frac{1}{t'^m},$$

with $p \nmid m$.

2.4 Branch Locus Geometry and Hurwitz Trees

2.4.1. *Branch locus of a lift*

In order to discuss the geometry of the branch locus, we first give a precise definition of what we mean by branch points of a cover in characteristic 0, both in the global scenario and in the local scenario.

Definition 2.4.1. Let $f : C \to \mathbb{P}^1_R$ be a G-Galois branched cover of smooth projective (relative) curves over R. Assume that it is unramified at the prime ideal (π). Then the *branch points* of f are the étale divisors b of $\mathbb{P}^1_R$ such that f is ramified at $f^{-1}(b) \subset C$. Enlarge R so that all the branch points are R-rational. The set of branch points is called the *branch locus* of f.

Remark 2.4.2. Note that a branch point consists of a closed point on the generic fiber $\mathbb{P}^1_K$ and a closed point on the special fiber $\mathbb{P}^1_k$. In some other literatures, branch points of a lift refer to closed points on the generic fiber, whereas here they refer to the étale (horizontal) divisors of $\mathbb{P}^1_R$. Since they are R-rational, they can be considered as elements of R plus ∞.

Definition 2.4.3. Let $R[[Z]]/R[[T]]$ be a G-extension. Assume that the cover $f : \mathrm{Spec}R[[Z]] \to \mathrm{Spec}R[[T]]$ is unramified at the prime ideal (π). Then the *branch points* of $R[[Z]]/R[[T]]$ are the divisors b of $\mathrm{Spec}R[[T]]$ such that f is ramified at $f^{-1}(b)$. Enlarge R so that all branch points are R-rational. The set of branch points is called the *branch locus* of $R[[Z]]/R[[T]]$.

Again, for local covers, we can consider branch points b_i of $R[[Z]]/R[[T]]$ as elements of R, assuming it is not ramified at infinity. When we say $v(b_i - b_j)$, it is the valuation of the corresponding element of R we shall mean.

2.4.2. *Hurwitz trees*

A $(\mathbb{Z}/p)^n$-cover over R is determined by its $\mathbb{Z}/p$-subcovers, and $\mathbb{Z}/p$-cover in characteristic 0 is in turn determined by its branch locus. Therefore, we want to study these covers by studying the geometry of its branch locus. One way to describe that geometry is through Hurwitz trees. The concept of Hurwitz trees was first introduced in [GM99], and they were first precisely defined in Henrio's thesis [Hen00].

A Hurwitz tree consists of combinatorial data that tells us which branch points are closer to each other than the other branch points, how close they are, among other information. We will also use Hurwitz trees to construct lifts of Artin-Schreier covers, and combine the them to construct lifts of elementary abelian covers in certain cases. Finally, the Hurwitz tree obstruction [BW09] Theorem 4.2 essentially states that, if a local G-cover lifts to characteristic 0, then there exists a Hurwitz tree of corresponding type. This enables us to classify possible branch locus geometry for lifting an elementary abelian cover of a given conductor type, and thus refine our search for lifts.

For a precise definition of a Hurwitz tree as a rooted metric tree, see [BW09]. Since for the purpose of the discussion in this thesis, only the configuration of branch points as leaf nodes of the Hurwitz tree is needed, we will provide the construction of the Hurwitz tree associated to a branched cover, given in Section 4.2.1 of Obus' exposition [Obu17], instead.

Definition 2.4.4 (Models)**.** Given a smooth projective curve C over K, a R-*model* of C is a normal arithmetic surface [Liu, Chapter 9] $\mathcal{C} \to \mathrm{Spec}R$ together with an isomorphism $\mathcal{C} \otimes_R K \cong C$.

Definition 2.4.5. A *semi-stable model* $\mathcal{C}$ of C is a R-model of C whose special fiber is semi-stable, i.e. $\mathcal{C} \otimes_R k$ is reduced and its only singularities are ordinary double points [Liu, Definition 10.3.1].

Let $R[[Z]]/R[[T]]$ be a G-extension with branch locus $B = \{b_1, \ldots, b_r\}$, and ramification locus (preimage of B) $B' = \{a_1, \ldots, a_s\}$. Let Y (X) be the minimal semi-stable model of $\mathbb{P}^1_K$ such that elements of B' (B) and ∞ do not coalesce on the special fiber. Then G acts on each irreducible component of Y_k, and its "quotient" is an irreducble component of X_k.

The underlying tree of the Hurwitz tree for $R[[Z]]/R[[T]]$ is built from the dual graph Γ of the semi-stable curve X_k. Namely,

- Vertices and edges of Γ correspond to irreducible components and nodes of X_k, and there is an edge between two vertices if and only if their corresponding irreducible components intersect.

- We append a vertex v_0, connected via an edge e_0, to the vertex v_1 corresponding to the component ∞ specializes to. Call this the *root node* of the Hurwitz tree. Then v_0 correspond to the open unit disk $D_0 := \text{Spec}R[[T]]$, and v_1 correspond to the smallest open disk $D_1 \subset D_0$ such that $B \subset D_1$.

- For each $b_i \in B$, append a vertex x_i, via an edge e_i, to the vertex w_j, corresponding to the component b_i specializes to. Call these the *leaf nodes* of the Hurwitz tree. Then each w_j correspond to the smallest open disk containing all b_i such that x_i is connected to w_j.

- Finally, note that there is a partial ordering on the vertices of the Hurwitz tree as a rooted tree, given by its distance to v_0. For each edge e_{ij} between $v_i \leq v_j$, the open disk corresponding to v_i is contained in the open disk corresponding to v_j. Define the *thickness* of e_{ij} to be the radius of this annulus. Edges connected to leaf nodes have thickness 0.

Definition 2.4.6. We say that a branch locus B has *equidistant geometry* if $v(b_i - b_j) = \rho$, $\rho \geq 0$ fixed, for all pairs of distinct branch points $b_i, b_j \in B$. Otherwise, i.e. if some branch points are closer to each other than they are to the others, we say the branch locus has *non-equidistant geometry.*

Remark 2.4.7. The Hurwitz tree of a cover with equidistant branch locus has all branches of size 1 (single leaves), whereas the Hurwitz tree of a cover with non-equidistant branch locus has branches with multiple leaves. Further restrictions on the sizes of branches will be discussed in Chapter 4.

2.5 Liftings of $\mathbb{Z}/p$ and $\mathbb{Z}/p \times \mathbb{Z}/p$-Covers

It is easy to use Artin-Schreier theory and Kummer theory to find birational lifts ([Mit] Definition 2.4.1) of a G-extension in terms of explicit equation. We often use the following different criterion for verifying that a birational lift is an actual lift in the sense of Definition 2.1.4.

Theorem 2.5.1 (**The different criterion** [GM98] Section 3.4)**.** *Suppose $B/A := /R[[T]]$ is a local G-extension. Let $B_k = B/\pi B$, $A_k = A/\pi A$, and $\tilde{B_k}$ be the integral closure of B_k. Define $\delta_k(B) = \dim_k(\tilde{B_k}/B_k)$. Let d_η be the degree of the generic different, i.e. $\mathcal{D}_{B\otimes_R K/A\otimes_R K}$, and let d_s be the degree of the special different, i.e. $\mathcal{D}_{B_k/A_k}$. Then $d_\eta = d_s + 2\delta_k(B)$, and $d_\eta = d_s$ if and only if $B/R[[T]]$ is a lift of $k[[z]]/k[[t]] := B_k/A_k$ and consequently $B = K[[Z]]$.*

For a local $\mathbb{Z}/p$-cover $k[[z]]/k[[t]]$, we can write down explicit equations for its lift to R, with $\zeta_p \in R$, where ζ_p is a primitive p-th root of unity. Denote $\lambda := \zeta_p - 1$.

Theorem 2.5.2 ([GM98] Theorem 4.1, [OSS] Theorem 2.2)**.** *The equation*

$$((\lambda X + 1)^p)/\lambda^p = T^{-m}$$

defines a p-cyclic cover $\mathcal{C}$ of $\mathbb{P}^1_R$ which after normalization is étale outside the disc $|T| < 1$ (i.e. outside $\{x \in \mathbb{P}^1_K : |T(x)| < 1\}$). The special fiber is smooth and induces the extension of $k[[t]]$ defined by the equation

$$x^p - x = t^{-m}.$$

In this way we cover all p-cyclic extensions of $k[[t]]$. Moreover the set $\{a \in \mathcal{C}_\eta : |T(a)| < 1\}$ is an open disc and $X^{-1/m}$ is a parameter.

We can show that $d_\eta = (m+1)(p-1) = d_s$, and the different criterion gives the first part of the statement. For the second part, from the equation

$$X^p + \frac{p}{\lambda}X^{p-1} + \cdots + \frac{p}{\lambda^{p-1}}X = T^{-m},$$

we see that mod π, x^{-m} is a uniformizer in $k((t))[x]$, and applying the Weierstrass Preparation Theorem we get that $X^{1/m}$ is a parameter for the open disc.

Using the different criterion, Green and Matignon proved a criterion for lifting local G-covers in the case where $G = \mathbb{Z}/p \times \mathbb{Z}/p$. In Chapter 3, we will generalize this criterion to $G = (\mathbb{Z}/p)^n$ for an arbitrary n. Here we state the original theorem more succinctly, combining the two cases into one.

Theorem 2.5.3 ([GM98] Theorem 5.1). *Let G be an abelian group isomorphic to $(Z/p)^2$. Let $G_i, 1 \leq i \leq p+1$, be the $p+1$ subgroups of order p. Assume that G is a group of automorphisms of $k[[z]]$ and arrange G_i in such a way that the extensions $k[[z]]^{G_i}/k[[z]]^G$ have conductors $m_i + 1$, with $m_1 \leq m_2 \leq \cdots \leq m_{p+1}$. Denote the conductor of the extension $k[[z]]/k[[z]]^{G_i}$ by $m_i' + 1$. Then the following holds:*

If there is a lifting of G to a group of automorphisms of $R[[Z]]$ then $m_1 \equiv -1 \bmod p$, $m_1' = m_2 p - m_1(p-1)$, $m_i = m_2$, and $m_i' = m_1$, for $2 \leq i \leq p+1$. In this case the two covers $R[[Z]]^{G_i}/R[[Z]]^G$ for $i = 1, 2$ have $(p-1)\dfrac{m_1+1}{p}$ common geometric branch points.

Conversely, if $m_1 \equiv -1 \bmod p$ and if one can lift $k[[z]]^{G_i}/k[[z]]^G$ for $i = 1, 2$ in such a way that the corresponding covers have $(p-1)\dfrac{m_1+1}{p}$ common geometric branch points, then the normalization of the compositum of these two covers lifts $k[[z]]/k[[z]]^G$.

Applying this criterion to the case of $p = 2$, Pagot [Pag] constructs explicit lifts of $(\mathbb{Z}/2)^2$-covers of type (m, m), both with equidistant geometry and with non-equidistant geometry (Definition 2.4.6). He also constructs lifts of $(\mathbb{Z}/2)^2$-covers of type $(m_1, m_2), m_1 \neq m_2$, with non-equidistant geometry. The key ingredient of his constructions of lifts whose branch locus has non-equidistant geometry is the following construction of a non-equidistant $\mathbb{Z}/2$-cover, whose Hurwitz tree has branches of size 2. Here we state the version in Matignon's notes on Pagot's thesis.

Proposition 2.5.4 ([MatNotes] Proposition 3.3). *Let $\mathrm{char}(k) = 2$, and $\rho \in W(k)^{alg}$ such that $v(2)/2 \leq v(\rho) < v(2)$. Let $T_1, \ldots, T_r \in R$ with $v(T_i - T_j) = 0$ for $i < j$ and $Q(X) := \prod_{1 \leq i \leq r}(X - T_i)$.*

Let $\alpha \in R$ with $v(\alpha) = 0$. Let

$$F(X) := Q(X) \prod_{1 \le i \le r} \Big(X - T_i - 2\rho\alpha \frac{T_i^{1/2}}{Q'(T_i)} + \rho^2 \alpha^2 \frac{1}{Q'(T_i)^2} \Big).$$

Then we can write $F(X) = \prod_{1 \le i \le r} (X - T_i)(X - \tilde{T}_i)$, where $v(T_i - \tilde{T}_i) = v(\rho^2)$ and the cover $Y^2 = F(X)$ has good reduction over $R[(4/\rho^2)^{1/(2r-1)}]$ relatively to the coordinate $T : (\frac{2}{\rho})^{\frac{2}{2r-1}} X$. An equation for the special fiber of the cover is

$$w^2 - w = \bar{\alpha}^2 \frac{1}{t^{2r-1}},$$

where $\bar{\alpha}$ is the image of α mod π.

Pagot uses a corollary of this proposition to prove that $(\mathbb{Z}/2)^2$ are local Oort groups for k of characteristic 2.

Finally, Mitchell [Mit] constructs a new type of non-equidistant lift for certain $(\mathbb{Z}/2)^2$-covers, whose branch locus geometry is different from that of Pagot's covers. We will discuss this in detail in Chapter 5.

CHAPTER 3

BRANCH CYCLE CRITERION FOR $(\mathbb{Z}/p)^n$-COVERS

In this chapter, we first prove the necessary lemmas on the degree of the *special different*, i.e. the degree of different of $k((z))/k((t))$, related to ramification jumps, and the degree of the *generic different*, i.e. the degree of different of $K((Z))/K((T))$. Then we arrive at the main result of this thesis (Theorem 3.4.1), which is a combinatorial criterion for lifting elementary abelian p-covers.

3.1 Ramification Jumps

Definition 3.1.1. Let L/K be a $G = (\mathbb{Z}/p)^n$-Galois totally ramified extension of local fields in characteristic p. Let I_m (I^l) be the m-th ramification group in lower numbering (upper numbering). For $0 \leq i \leq n-1$, define the *i-th lower (upper) ramification jump* be the positive integer m such that the p-rank of I_m (I^l) is at least $n-i$ and the p-rank of I_{m+1} (I^{l+1}) is at most $n-i-1$.

In this section, the ramification jumps are always with respect to the lower numbering unless specified otherwise. Note that the ramification jumps can coincide when the quotient I_m/I_{m+1} has order greater than p.

Lemma 3.1.2. *Let L/K be a $G = (\mathbb{Z}/p)^n$-Galois totally ramified extension of complete dicretely valued fields with residue characteristic p. Suppose L/K can be written as a tower of $\mathbb{Z}/p$-extensions $L = K_n/K_{n-1}/\dots/K_1/K_0 = K$, where K_{i+1}/K_i has conductor $m^{(i)}+1$ (Definition 2.2.4, such that $m^{(0)} \leq m^{(1)} \leq \cdots \leq m^{(n-1)}$. Then the l-th lower ramification jump of L/K is $m^{(l)}$. Moreover, the degree of the different of L/K (Definition 2.2.3) is $\sum_{l=0}^{n-1}(m^{(l)}+1)p^{n-l-1}(p-1)$.*

Proof. First we use induction on n to compute the ramification jumps. When $n = 1$, $G = \mathbb{Z}/p$. Let $m+1$ be the conductor of $L = K_1/K_0 = K$. Then by [Ser], Chapter IV, exercise 2.5, $G_m = \mathbb{Z}/p$ and $G_{m+1} = 1$. Thus the unique ramification jump is one less than the conductor.

Suppose the statement about the ramification jumps is true for $n-1$. Consider L/K as in the hypothesis, and let $H = (\mathbb{Z}/p)^{n-1}$ be the Galois group of L/K_1. Then $m^{(l)}$, for $1 \leq l \leq n-1$, is the $(l-1)$-th ramification jump of H. Let H_i be the i-th ramification group of L/K_1. First note that by Equation 2.1,

$$\varphi_{L/K_1}(m^{(0)}) + 1 = \frac{1}{|H_0|}\sum_{i=0}^{m^{(0)}} |H_i| = \frac{1}{p^{n-1}}(m^{(0)}+1)p^{n-1} = m^{(0)} + 1,$$

where φ is the Herbrand function (Definition 2.2.6), and $\varphi_{L/K_1}(m) > m^{(0)}$ for $m > m^{(0)}$. Since $m^{(0)} \leq m^{(1)}$, $H = H_{m^{(0)}} = I_{m^{(0)}} \cap H$ by Proposition 2.2.7, and $H \subseteq I_{m^{(0)}}$. Thus $I_{m^{(0)}}/H = I_{m^{(0)}}H/H = (G/H)_{\varphi_{L/K_1}(m^{(0)})} = (G/H)_{m^{(0)}} = \mathbb{Z}/p$ by Herbrand's theorem (Lemma 2.2.8), hence $I_{m^{(0)}} = (\mathbb{Z}/p)^n$.

Now, let l be the largest integer such that $m^{(l)} = m^{(0)}$. We have

$$I_{m^{(l)}+1}H/H = (G/H)_{\phi_{L/K_1}(m^{(l)}+1)} = 1,$$

so $I_{m^{(l)}+1} \subseteq H$. Then $I_{m^{(l)}+1} = I_{m_{(l)}+1} \cap H = H_{m^{(l)}+1} = (\mathbb{Z}/p)^{n-l-1}$. Therefore $m^{(i)}$ is the i-th ramification jump of L/K for all $0 \leq i \leq l$. Similarly, for all $i > l$, $I_{m^{(i)}} = I_{m^{(i)}} \cap H = H_{m^{(i)}}$, and $I_{m^{(i)}+1} = I_{m^{(i)}+1} \cap H = H_{m^{(i)}+1}$, so $m^{(i)}$ is the i-th lower ramification jump.

Finally, by 2.2.3, we get that the degree of the different of L/K is

$$\begin{aligned} d_s &= \sum_{j=0}^{\infty}(|I_j| - 1) \\ &= \sum_{l=0}^{n-1}(m^{(l)} - m^{(l+1)})(p^{n-l} - 1) \\ &= \sum_{l=0}^{n-2} m^{(l)}(p^{n-l} - p^{n-l-1}) + m^{(n-1)}(p-1) + p^{n-1} - 1 \\ &= \sum_{l=0}^{n-1} m^{(l)}p^{n-l-1}(p-1) + \sum_{l=0}^{n-1} p^{n-l-1}(p-1) \\ &= \sum_{l=0}^{n-1}(m^{(l)}+1)p^{n-l-1}(p-1). \end{aligned}$$

□

Remark 3.1.3. One can check that if we take the tower of extensions $L/L^{G_{j_{n-1}}}/\cdots/L^{G_{j_0}}$, where j_i is the i-th lower ramification jump, then the ascending sequence of conductors in the hypothesis of the above lemma can always be achieved.

3.2 Conductor Type

For an elementary abelian p-cover $k[[z]]/k[[t]]$ where $G = (\mathbb{Z}/p)^n$, whether a G can be lifted to characteristic 0 often depends on the conductors of its $\mathbb{Z}/p$-subcovers. For ease of notation, we define a $(\mathbb{Z}/p)^n$-cover of certain (conductor) type.

Definition 3.2.1. Let $G = (\mathbb{Z}/p)^n$, and $\{G_i\}$ be the set of $(\mathbb{Z}/p)^{n-1}$-subgroups of G. Let k be an algebraically closed field of characteristic p. Suppose that G is a group of automorphisms of $k[[z]]$ as a k-algebra, and the extensions $k[[z]]^{G_i}/k[[z]]^G$ have conductors $m_i + 1$. Suppose that $(m_1 + 1, \ldots, m_n + 1)$ is the lexicographically smallest n-tuple of conductors such that the following conditions hold:

1. $k[[z]]^{G_1}, \ldots, k[[z]]^{G_n}$ are linearly disjoint over $k[[z]]^G$,

2. $m_1 \leq m_2 \leq \cdots \leq m_n$.

Then we say that $k[[z]]/k[[z]]^G$ is a *cover of type* $(m_1 + 1, \ldots, m_n + 1)$.

Remark 3.2.2. Note that this is different from the notations in Mitchell's thesis [Mit], where he calls such covers of type $(m_1, \ldots, m_n)$.

Proposition 3.2.3. *With the notations in the above definition, let $K_0 := k((t))$ and $K_i = k((z))^{G_i}$ for $1 \leq i \leq n$. Then $K_1, K_i, i \geq 2$ over K_0 be defined by the Artin-Schreier equations:*

$$w_1^p - w_1 = f_1(\frac{1}{t}) = \frac{1}{t^{m_1}}$$
$$w_i^p - w_i = f_i(\frac{1}{t}) = \sum_{1 \leq j \leq m_i,\ p \nmid m'} \frac{c_{i,j}}{t^j},$$

where the leading coefficients of f_i and f_j are $\mathbb{F}_p$-linearly independent if $m_i = m_j$ and $c_{i,m_i}^p \neq c_{i,m_i}$ for $i \geq 2$.

Proof. First, by 2.3.3, for some uniformizer t of K_0, K_1/K_0 can be defined by $w_1^p - w_1 = \frac{1}{t^{m_1}}$, and with that same uniformizer t, K_i/K_0 can be defined by Artin-Schreier equations as above.

Suppose that $m_i = m_j$ for some $i < j$. Then (m_i, m_j) must also be the lexicographically the smallest tuple of conductors satisfying the conditions in Definition 3.2.1 for the extension K_iK_j/K_0. Suppose $ac_{i,m_i} + bc_{j,m_j} = 0$ for some $a, b \in k$. Then there is a $\mathbb{Z}/p$-subextension of K_iK_j/K_0 defined by $w^p - w = af_i(\frac{1}{t}) + bf_j(\frac{1}{t})$, the right-hand-side of which has degree strictly less than m_i, i.e. its conductor is strictly less than m_i, giving a contradiction. Thus c_{i,m_1} and c_{j,m_j} are linearly independent over k.

Finally, suppose $c_{i,m_i}^p = c_{i,m_i}$ for some $i \geq 2$. Then a $\mathbb{F}_p$-linear combination of w_1 and w_i generates $\mathbb{Z}/p$-subextension of K_1K_i/K_0 having conductor strictly less than m_i, again a contradiction. Therefore, $c_{i,m_i}^p \neq c_{i,m_i}$ for $i \geq 2$. □

3.3 Key Lemmas

Lemma 3.3.1. *Let $G = (\mathbb{Z}/p)^n$, and $k[[z]]/k[[t]]$ be a G-cover of type $(m_1 + 1, \ldots, m_n + 1)$, where $k[[t]] = k[[z]]^G$. Then for $0 \leq l \leq n-1$, the l-th lower ramification jump of $k((z))/k((t))$ is*

$$p^l m_{l+1} - (p-1) \sum_{1 \leq i \leq l} p^{i-1} m_i.$$

I will include two proofs for this lemma. This first one is more reminiscent of Green and Matignon's original proof for the case $\mathbb{Z}/p \times \mathbb{Z}/p$ [Mat99, Theorem 5.1].

Proof. Using notations in Definition 3.2.1, let $K_0 = k((t))$, and $K_i := k((z))^{G_i}$ for $1 \leq i \leq n$. By Lemma 3.1.2, it suffices to show that the conductor of the extension $K_0 \cdots K_{l+1}/K_0 \cdots K_l$, denoted $m^{(l)} + 1$, is $p^l m_{l+1} - (p-1) \sum_{1 \leq i \leq l} p^{i-1} m_i + 1$.

For the base case $l = 0$, it follows from the hypothesis that the conductor of K_1/K_0 is $m_1 + 1$.

Now assume that the conductor of $K_0 \cdots K_{l-1}K_l/K_0 \cdots K_{l-1}$ is

$$m + 1 := p^{l-1}m_l - (p-1) \sum_{1 \le i \le l-1} p^{i-1}m_i + 1,$$

and similarly the extension $K_0 \cdots K_{l-1}K_{l+1}/K_0 \cdots K_{l-1}$ has conductor

$$m' + 1 := p^{l-1}m_{l+1} - (p-1) \sum_{0 \le i \le l-1} p^{i-1}m_i + 1.$$

Then by the assumption $m_{l+1} \ge m_l$, we have that $m' \ge m$.

By Proposition 2.3.3, after a change of variables in $K_0 \cdots K_{l-1}$, we can write, with respect to a uniformizer $z_{l-1} \in K_0 \cdots K_{l-1}$, the extension $K_0 \cdots K_{l-1}K_l/K_0 \cdots K_{l-1}$ as

$$w_l^p - w_l = \frac{1}{z_{l-1}^m}. \tag{3.1}$$

With respect to the same uniformizer z_{l-1}, the extension $K_0 \cdots K_{l-1}K_{l+1}/K_0 \cdots K_{l-1}$ can be written as

$$w_{l+1}^p - w_{l+1} = \sum_{0 \le i \le m',\ p \nmid i} \frac{e_i}{z_{l-1}^i}, \tag{3.2}$$

where $e_{m'} \not\equiv 1 \bmod \pi$ if $m = m'$, and $e_{m'}^p \ne e_{m'}$ by Proposition 3.2.3.

Note that $v_{K_0 \cdots K_l}(w_l^{-1}) = m$, so $w_l^{-1} = uz'^m$ for some uniformizer z' of $K_0 \cdots K_l$ and unit $u \in k$. Because the residue field k is algebraically closed, $u^{1/m} \in k$ and $z_l := w_l^{-\frac{1}{m}} \in K_0 \cdots K_l$ is a uniformizer in $K_0 \cdots K_l$. By Equation 3.1, we have

$$\frac{1}{z'} = \left(w_l^p(1 - w_l^{1-p})\right)^{\frac{1}{m}} = z_l^{-p}(1 - z_l^{m(p-1)})^{\frac{1}{m}}.$$

Substituting this into Equation 3.2 we get:

$$
\begin{aligned}
w_{l+1}^p - w_{l+1} &= \sum_{1\le i\le m'} \frac{e_i}{z_l^{pi}} \left(1 - \frac{i}{m} z_l^{m(p-1)} + \cdots\right) \\
&= \sum_{1\le i\le m'} \frac{e_i}{z_l^{pi}} - \sum_{1\le i\le m'} \frac{i}{m} \cdot \frac{e_i}{z_l^{pi-m(p-1)}} + \cdots .
\end{aligned}
\tag{3.3}
$$

Denote the conductor of $K_0 \cdots K_l K_{l+1}/K_0 \cdots K_l$ by $m^{(l)} + 1$. After an Artin-Schreier change of variables as in the proof of Proposition 2.3.3, the highest exponent of $\frac{1}{z_l}$ is m' in the first sum, and $pm' - m(p-1)$ in the second sum. Recall that $m' \ge m$, so $pm' - m(p-1) \ge m'$. Then the highest exponent of $\frac{1}{z_l}$ on the right hand side of Equation 3.3 is $pm' - m(p-1)$. Therefore, by the induction hypothesis, the conductor of $K_0 \cdots K_{l+1}/K_0 \cdots K_l$ minus one is

$$
\begin{aligned}
m^{(l)} &= pm' - m(p-1) \\
&= p\left(p^{l-1}m_{l+1} - (p-1)\sum_{1\le i\le l-1} p^{i-1}m_i\right) - \left(p^{l-1}m_l - (p-1)\sum_{1\le i\le l-1} p^{i-1}m_i\right)(p-1) \\
&= p^l m_{l+1} - (p-1)\left(p\sum_{1\le i\le l-1} p^{i-1}m_i + p^{l-1}m_l - (p-1)\sum_{1\le i\le l-1} p^{i-1}m_i\right) \\
&= p^l m_{l+1} - (p-1)\sum_{1\le i\le l} p^{i-1}m_i.
\end{aligned}
$$

This concludes the inductive proof that the l-th ramification jump is

$$m^{(l)} = p^l m_{l+1} - (p-1)\sum_{1\le i\le l} p^{i-1}m_i.$$

□

The second proof uses Herbrand's Formula and upper numbering for ramification groups, and is somewhat simpler. I would like to thank Andrew Obus for suggesting the idea.

Proof. Let $m^{(l)}$ denote the l-th lower ramification jump of L/K. For the base case $l = 0$, it follows from the hypothesis and Lemma 3.1.2 that $m^{(0)} = m_1$.

For the induction step, assume that $m^{(j)} = p^j m_{j+1} - (p-1)\sum_{1\le i\le j} p^{i-1} m_i$ for all $j \le l-1$. Let M/K be the $(\mathbb{Z}/p)^{l+1}$-extension $K_0 \cdots K_{l+1}/K_0$, and $\Gamma := \mathrm{Gal}(M/K)$. Recall that K_i/K_0 is a $\mathbb{Z}/p$-extension with conductor $m_i + 1$. Let Γ_j be the j-th ramification group of M/K with lower numbering, and $\varphi_{M/K}(j)$ be the Herbrand function [Ser]. Then $\Gamma_j = \Gamma^{\varphi_{M/K}(j)}$, where Γ^i is the i-th ramification group of M/K with upper numbering. Let H be the subgroup of Γ such that $M^H = K_{l+1}$. By Proposition IV.14 in [Ser],

$$\Gamma^i H/H = (\Gamma/H)^i = \begin{cases} \mathbb{Z}/p, \ 0 \le i \le m_{l+1} \\ 1, \ i > m_{l+1}, \end{cases}$$

the last equality due to $\Gamma/H \cong \mathbb{Z}/p$ and the unique upper jump of K_{l+1}/K equals to its unique lower jump, which is one less than its conductor. Therefore, the l-th upper ramification jump of M/K is m_{l+1}. Since the upper numbering for ramification groups is compatible with quotients (Proposition 2.2.7), so are the upper ramification jumps. Thus m_{l+1} is also the l-th upper ramification jump of L/K. Moreover, $\varphi_{L/K}(m^{(l)}) = m_{l+1}$, since $m^{(l)}$ is the l-th lower ramification jump of M/K.

Now, let $g_j = |I_j|$, where I_j is the j-th ramification group of L/K with lower numbering. Observe that $g_j = p^{n-i-1}$ for $m^{(i)} < j \le m^{(i+1)}$. By Formula 2.1 and the induction hypothesis, we have

$$\begin{aligned}
&m_{l+1} + 1 \\
=&1 + \varphi_{L/K}(m^{(l)}) = \frac{1}{|G|}\sum_{j=0}^{m^{(l)}} g_j \\
=&\frac{1}{p^n}\left((m^{(0)}+1)p^n + \sum_{i=0}^{l-1}(m^{(i+1)} - m^{(i)})p^{n-i-1}\right) \\
=&m_1 + 1 + \sum_{i=0}^{l-2}\left(p^{i+1}m_{i+2} - (p-1)\sum_{1\le j\le i+1} p^{j-1}m_j - p^i m_{i+1} + (p-1)\sum_{1\le j\le i} p^{j-1}m_j\right)p^{-i-1} \\
&+\left(m^{(l)} - p^{l-1}m_l + (p-1)\sum_{1\le j\le l-1} p^{j-1}m_j\right)p^{-l}
\end{aligned}$$

$$
\begin{aligned}
=&m_1 + 1 + \sum_{i=0}^{l-2}(m_{i+2} - m_{i+1}) + m^{(l)}p^{-l} - p^{-1}m_l + (p-1)p^{-l} \sum_{1\le j\le l-1} p^{j-1}m_j \\
=&m^{(l)}p^{-l} + (p-1)p^{-1}m_l + 1 + (p-1)p^{-l} \sum_{1\le j\le l-1} p^{j-1}m_j \\
=&m^{(l)}p^{-l} + 1 + (p-1)p^{-l} \sum_{1\le j\le l} p^{j-1}m_j.
\end{aligned}
$$

Therefore, the l-th ramification jump of L/K is

$$
m^{(l)} = p^l m_{l+1} - (p-1)\sum_{1\le i\le l} p^{i-1}m_i.
$$

□

Lemma 3.3.2. *Let $R[[Z]]/R[[T]]$ be a local G-cover, and $G_1, \dots, G_n$ be defined as in Lemma 3.3.1. Suppose $R[[Z]]^{G_1}, \dots, R[[z]]^{G_n}$ have branch loci $B_1, \dots, B_n$ (Definition 2.4.3), such that for any k with $1 \le k \le n$ and any subset of k branch points $\{B_{i_1}, \dots, B_{i_k}\}$, the cardinality of the set intersection $|\cap_{1\le j\le k} B_{i_j}| = \dfrac{(\min_j(m_{i_j})+1)(p-1)^{k-1}}{p^{k-1}}$. Then the generic different of $R[[z]]/R[[t]]$ is $\sum_{l=0}^{n-1}(p-1)p^l(m_{l+1}+1)$.*

Proof. Since the generic fiber of the lift $R[[Z]]/R[[T]]$ is in characteristic 0, it is tamely ramified at all branch points, each having p^{n-1} ramification points above it. Thus the generic different is $(p-1)p^{n-1}$ times the total number of branch points, counted without repeat.

Let $B = B_1 \cup \dots \cup B_n$ be the branch locus of $K((Z))/K((T))$. We use the inclusion-exclusion principle to count the number of branch points. For each $1 \le i \le n$, $\min_j(m_{i_j}) + 1$ is $m_i + 1$ for all $\{B_i, B_{i_2}, \dots, B_{i_k}\}$ such that $i_j \ge i$ for all j. There are $\binom{n-i}{k-1}$ such k-subsets. Therefore

$$
\begin{aligned}
d_\eta &= (p-1)p^{n-1}|B| \\
&= (p-1)p^{n-1}\sum_{k=1}^{n}(-1)^{k-1}\sum_{i=1}^{n-k+1}\sum_{i_j\ge i \forall j}|B_i \cap B_{i_2} \cap \dots \cap B_{i_k}|
\end{aligned}
$$

$$= (p-1)p^{n-1}\sum_{k=1}^{n}(-1)^{k-1}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}(p-1)^{k-1}p^{1-k}(m_i+1)$$
$$= (p-1)\sum_{k=0}^{n-1}(-1)^{k}(p-1)^{k}p^{n-k-1}\sum_{i=1}^{n-k}\binom{n-i}{k}(m_i+1).$$

Observe that for $1 \leq i \leq n$, the coefficient of $m_i + 1$ is

$$\begin{aligned}
&(p-1)\sum_{0\leq k\leq n-i}(-1)^{k}(p-1)^{k}p^{n-k-1}\binom{n-i}{k} \\
=&(p-1)p^{i-1}\sum_{0\leq k\leq n-i}(1-p)^{k}p^{n-i-k}\binom{n-i}{k} \\
=&(p-1)p^{i-1}(1-p+p)^{n-i} = (p-1)p^{i-1}.
\end{aligned}$$

Thus we have that $d_\eta = \sum_{i=1}^{n}(p-1)p^{i-1}(m_i+1) = \sum_{l=0}^{n-1}(p-1)p^{l}(m_{l+1}+1).$ □

3.4 Main Theorem

We now state our main result, which generalizes Theorem 5.1 of [GM98].

Theorem 3.4.1. *Let $G = (\mathbb{Z}/p)^n$. Suppose $k[[z]]/k[[t]]$ is a G-extension of conductor type $(m_1 + 1, \dots, m_n + 1)$. Then there is a lifting of G to a group of automorphisms of $R[[Z]]$ if and only if the following two conditions hold:*

1. *$m_i \equiv -1 \mod p^{n-i}$ for $1 \leq i \leq n-1$,*
2. *$k[[z]]^{G_1}, \dots, k[[z]]^{G_n}$ can be lifted with branch loci $B_1, \dots, B_n$ such that for any subset of k branch points $\{B_{i_1}, \dots, B_{i_k}\}$, $|\cap_{1\leq j\leq k} B_{i_j}| = \dfrac{(\min_j(m_{i_j})+1)(p-1)^{k-1}}{p^{k-1}}$.*

Proof. First we show that the combinatorial conditions on the branch loci of lifts of K_i are necessary. Suppose $k[[z]]/k[[t]]$ can be lifted to $R[[Z]]/R[[T]]$. Then so can all the intermediate extensions. We show that for any subset of k branch loci $\{B_{i_1}, \dots, B_{i_k}\}$, $|\cap_{1\leq j\leq k} B_{i_j}| = \dfrac{\min_j(|B_{i_j}|)(p-1)^{k-1}}{p^{k-1}}$. The base case $k = 2$ is shown in [GM98] (see Theorem 2.5.3). Suppose this is true for $k \leq l$, and consider the extension $K_0 \cdots K_{l+1}/K_0$. Then the number of branch points in the lift of $K_0 \cdots K_{l+1}/K_0 \cdots K_l$ is $(p-1)p^l$ times the number of branch points in the lift of K_{l+1}/K_0 that are not in that of any K_i/K_0 for $1 \leq i \leq l$. Write $d = |B_1 \cap \ldots \cap B_{l+1}|$. Using Lemma 3.3.2, the degree of the generic different of $K_0 \cdots K_{l+1}/K_0 \cdots K_l$ is given by

$$\begin{aligned} d_\eta &= (p-1)p^l \left(|B_{l+1}| + \sum_{k=1}^{l} (-1)^k \sum_{i=1}^{l-k+1} \sum_{i_j \geq i, \forall 2 \leq j \leq k} |B_i \cap B_{i_2} \cap \ldots \cap B_{i_k} \cap B_{l+1}| \right) \\ &= (p-1)p^l \left(|B_{l+1}| - (p-1)p^{-1} \sum_{k=1}^{l-1} (-1)^{k-1} \sum_{i=1}^{l-k+1} \sum_{i_j \geq i, \forall j} |B_i \cap B_{i_2} \cap \ldots \cap B_{i_k}| + (-1)^l d \right) \\ &= (p-1)\Big(p^l(m_{l+1}+1) - \Big(d_{\eta, K_0 \cdots K_l/K_0} - (p-1)(-p)^{l-1}(m_1+1)(p-1)^{l-1}p^{1-l} \Big) + p^l(-1)^l d \Big) \\ &= (p-1) \left(p^l(m_{l+1}+1) - \sum_{i=1}^{l} (p-1)p^{i-1}(m_i+1) + (-1)^{l-1}(p-1)^l(m_1+1) + p^l(-1)^l d \right). \end{aligned}$$

By the different criterion (Theorem 2.5.1), this equals to the degree of special different, which in this case equals to $p-1$ times the conductor of $K_0 \cdots K_{l+1}/K_0 \cdots K_l$. Recall from Lemma 3.1.2 this is

$$d_{s, K_0 \cdots K_{l+1}/K_0 \cdots K_l} = (p-1)\Big(p^l m_{l+1} - (p-1) \sum_{i=1}^{l} p^{i-1} m_i + 1 \Big).$$

Dividing both by $p-1$, we get

$$\begin{aligned} & p^l(m_{l+1}+1) - \sum_{i=1}^{l} (p-1)p^{i-1}(m_i+1) + (-1)^{l-1}(p-1)^l(m_1+1) + p^l(-1)^l d \\ =& d_{s, K_0 \cdots K_{l+1}/K_0 \cdots K_l}/(p-1) = p^l m_{l+1} - (p-1) \sum_{i=1}^{l} p^{i-1} m_i + 1, \end{aligned}$$

which implies

$$(-1)^{l-1}(p-1)^l(m_1+1) + p^l(-1)^l d = 0.$$

Hence the $l+1$ lifts share $|B_1 \cap \ldots \cap B_{l+1}| = d = \dfrac{(m_1+1)(p-1)^l}{p^l} = \dfrac{\min_i(|B_i|)(p-1)^{l+1-1}}{p^{l+1-1}}$ common branch points, proving that the conditions on the sets B_i are necessary. Furthermore, since the number of common branch points is an integer, this also shows that the congruence conditions $m_i \equiv -1 \bmod p^{n-i}$ for $1 \leq i \leq n-1$ are necessary.

Finally, we show that the conditions on the branch loci are sufficient.

We have from the beginning of the proof that the l-th lower ramification jump is

$$p^l m_{l+1} - (p-1) \sum_{1 \leq i \leq l} p^{i-1} m_i.$$

Therefore, by Lemma 3.1.2 the degree of the different of $k[[z]]/k[[t]]$ is

$$\begin{aligned}
d_s &= \sum_{l=0}^{n-1} (m_1^{(l)} + 1)(p-1)p^{n-l-1} \\
&= \sum_{l=0}^{n-1} (p-1)p^{n-l-1} \left(p^l m_{l+1} - (p-1) \sum_{i=1}^{l} p^{i-1} m_i + 1 \right) \\
&= (p-1) \left(\sum_{l=0}^{n-1} p^{n-1} m_{l+1} - (p-1) \sum_{l=1}^{n-1} \sum_{i=1}^{l} p^{n-l+i-2} m_i + \sum_{l=0}^{n-1} p^{n-l-1} \right) \\
&= (p-1) \left(\sum_{l=0}^{n-1} p^{n-1} m_{l+1} - (p-1) \sum_{i=1}^{n-1} \sum_{l=i}^{n-1} p^{n-l+i-2} m_i + \sum_{l=0}^{n-1} p^l \right) \\
&= (p-1) \left(p^{n-1} m_n + \sum_{j=1}^{n-1} \Big(p^{n-1} - \sum_{l=j-1}^{n-2} p^l \Big) m_j + \sum_{l=0}^{n-1} p^l \right) \\
&= (p-1) \sum_{l=0}^{n-1} p^l (m_{l+1} + 1).
\end{aligned}$$

The degree of the generic different of $k[[z]]/k[[t]]$ is

$$d_\eta = \sum_{l=0}^{n-1} (p-1) p^l (m_{l+1} + 1) = d_s.$$

It thus follows from the different criterion (Theorem 2.5.1) that G lifts to a group of automorphisms of $R[[Z]]$. □

CHAPTER 4

COALESCING OF BRANCH POINTS

In this chapter, we first introduce a result in Pries-Zhu [PZ], on stratification of the space of Artin-Schreier covers. We then give an interpretation of the result in terms of branch loci of lifts of these covers. We give a description of the coalescing behavior of the branch points of the lifts, which will be used in Chapter 5.

4.1 Stratification of the Space of Artin-Schreier Covers

Consider a smooth projective curve X over k of genus g. The p-rank of X is the integer s such that the cardinality of $\mathrm{Jac}(X)[p](k)$ is p^s. We have that $0 \leq s \leq g$. For $g = 1$, the p-rank is also called the Hasse invariant.

Now let $X \to \mathbb{P}^1_k$ be an Artin-Schreier cover in characteristic p. Then $s = r(p-1)$ for some integer $r \geq 0$ [PZ]. We can study the stratification of AS_g, the moduli space of Artin-Schreier covers of genus g, by p-rank into strata $AS_{g,s}$ consisting of covers with p-rank s. By the Riemann-Hurwitz formula, $2g-2 = p(-2) + \deg(D)$, where D is the ramification divisor. Thus $g = d(p-1)/2$ for some integer d. Assume $g \geq 1$. Then we have the following result:

Theorem 4.1.1 (Pries-Zhu, 2010).

1. *The set of irreducible components of $AS_{g,s}$ is in bijection with the set of partitions $[e_1, ..., e_{r+1}]$ of $d+2$ into $r+1$ positive integers such that each $e_j \not\equiv 1$ mod p.*

2. *The irreducible component of $AS_{g,s}$ for the partition $[e_1, ..., e_{r+1}]$ has dimension $d - 1 - \sum_{j=1}^{r+1} \lfloor (e_j - 1)/p \rfloor$.*

In fact, the bijection in part 1 can be given explicitly. Since k is algebraically closed, after some automorphism of $\mathbb{P}^1_k$, we can assume that f is not branched at infinity. Let $f : X \to \mathbb{P}^1_k$ be given by the equation $y^p - y = \sum_{i=1}^{n} f_i(\frac{1}{x - c_i})$, where f_i are polynomials over k of degrees not divisible by p (in particular $\deg(f_i) > 0$), and $c_i \in k$ are distinct. Then it is branched at n points, $\{c_1, \ldots, c_n\}$, and any Artin-Schreier cover branched at these points is of the above form. Let s be the p-rank of f. The Deuring-Shafarevich theorem [Sub] states that, for $X \xrightarrow{\mathbb{Z}/p} Y$ over k, $s_X - 1 = p(s_Y - 1) + n(p-1)$, where n is the number of branch points on Y. Here the p-rank of $Y = \mathbb{P}^1_k$ is 0. Therefore $s = (n-1)(p-1)$, and $n = r+1$, with r defined as above.

Let $e_i = \deg(f_i) + 1$. Then $e_i \not\equiv 1 \bmod p$. By the Riemann-Hurwitz formula,

$$2g - 2 = p(0-2) + \sum_{i=1}^{r+1} e_i(p-1).$$

Thus $g = (p-1)(\sum_{i=1}^{r+1} e_i - 2)/2 = d(p-1)/2$, so $\sum_{i=1}^{r+1} e_i = d+2$ and $[e_1, \ldots, e_{r+1}]$ is a partition of $d+2$.

4.2 Coalescing of Branch Points of a Lift

Proposition 4.2.1. *With the above notation, consider the component of $AS_{g,s}$ of an Artin-Schreier cover $f : X \to \mathbb{P}^1_k$ with p-rank s which corresponds to the partition $[e_1, ..., e_{r+1}]$ of $d+2$, with each $e_j \not\equiv 1 \bmod p$. Suppose f is branched at $\{c_1, \ldots, c_{r+1}\}$. As above, f is given by an equation of the form $y^p - y = \sum_{i=1}^{r+1} f_i(\frac{1}{x - c_i})$, where $e_i = \deg(f_i) + 1$. Then there exists a lift of f to R whose generic fiber is a degree p Kummer cover with $d+2$ branch points, e_i of which coalesce to c_i on $\mathbb{P}^1_k$ for $1 \leq i \leq r+1$.*

Conversely, any lift of f is a $\mathbb{Z}/p$-cover with $d+2$ branch points, e_i of which coalesce to c_i on $\mathbb{P}^1_k$ for $1 \leq i \leq r+1$.

Proof. Localizing at each branch point of f, we get $r+1$ local extensions, of $k[[x-c_i]], 1 \leq i \leq r+1$. Since $x - c_j$ is a unit in $k[[x-c_i]]$ for all $j \neq i$, $\frac{1}{x-c_j} \in k[[x-c_i]]$ and thus $f_j(\frac{1}{x-c_j}) \in k[[x-c_i]]$. Then there exists an element $z = -(f_j + f_j^p + f_j^{p^2} + \cdots)$ in $k[[x-c_i]]$ such that $z^p - z = f_j$. Therefore, after a change of variables, the local extension of $k[[x-c_i]]$ is given generically by $y^p - y = f_i(\frac{1}{x-c_i})$.

By the Oort conjecture, after possibly extending R, we can lift these local covers, which give us branched covers of Spec$R[[x-c_i]]$, branched at b_i points on the generic fiber, for some $b_i > 0$, all coalescing at c_i. By the different criterion 2.5.1, the generic different, $b_i(p-1)$, equals to the special different, $(\deg(f_i)+1)(p-1)$, so $b_i = \deg(f_i)+1 = e_i$. By the proof of Theorem 2.2 in [CGH08], we can patch these local lifts together to get a smooth $\mathbb{Z}/p$-cover $X_R \to \mathbb{P}^1_R$, with e_i branch points coalescing to the point c_i on $\mathbb{P}^1_k$.

Let $X_K \to \mathbb{P}^1_K$, branched at m points, be the generic fiber of the lift $X_R \to \mathbb{P}^1_R$. Then by the Riemann-Hurwitz formula and flatness of $X_R \to \mathbb{P}^1_R$, $(m-2)(p-1)/2 = g_{X_K} = g_X = d(p-1)/2$, so $m = \sum_{i=1}^{r+1} e_i = d+2$.

Now we prove the converse. Suppose $F : X_R \to \mathbb{P}^1_R$ is a lift of f. Localizing $\mathbb{P}^1_R$ at the closed point $c_i \in \mathbb{P}^1_k$, for $1 \le i \le r+1$, we get the inclusion Spec$\hat{\mathcal{O}}_{\mathbb{P}^1_R, c_i} \to \mathbb{P}^1_R$. Now taking its fiber product with F, we get an extension $R[[z]]$ of $R[[x-c_i]]$ branched at only those branch points of F coalescing at c_i. Suppose there are n_i of them.

The reduction of $R[[z]]/R[[x-c_i]]$ is an extension of $k[[x-c_i]]$ given generically by $y^p - y = f_i(\frac{1}{x-c_i})$, as shown above. Again, by the different criterion, $R[[z]]/R[[x-c_i]]$ has to be branched at $\deg(f_i)+1 = e_i$ points. Therefore, $n_i = e_i \not\equiv 1 \bmod p$. □

Remark 4.2.2. We can therefore interpret Theorem 4.1.1 as a description of K-covers $f : X \to \mathbb{P}^1_K$ with good reduction, in terms of how their branch points coalesce on the special fiber. Namely, if $X_R \to \mathbb{P}^1_R$ is the smooth model of f, then e_i points on $\mathbb{P}^1_R$ coalesce to the i-th branch point on the special fiber. Moreover, let $\mathcal{H}_{m,p}$ be the space of p-covers of $\mathbb{P}^1_K$ branched at m points, and let $\mathcal{H}^{good}_{m,p}$ be the subspace of $\mathcal{H}_{m,p}$ consisting of those covers having good reduction. Then we get a stratification of $\mathcal{H}^{good}_{m,p}$ into strata $\mathcal{H}^{good}_{m,p,n}$ of covers whose reduction have n branch points.

Remark 4.2.3. Part 2 of Theorem 4.1.1 can be used to describe the strata $\mathcal{H}^{good}_{m,p,n}$ in characteristic 0. Since we construct lifts to R by lifting the coefficients of the defining polynomials for the covers over k, the component of $\mathcal{H}^{good}_{m,p,n}$ consisting of covers with branch locus partition $[e_1, \ldots, e_n]$, where e_i points coalesce to one point for each i, is a p-adic neighborhood of a subvariety of $\mathcal{H}^{good}_{m,p}$ of dimension $m - 3 - \sum_{i=1}^{n} \lfloor (e_i - 1)/p \rfloor$.

Corollary 4.2.4. *Let $f : X \to \mathbb{P}^1_R$ be a lift of a $\mathbb{Z}/2$-cover of $\mathbb{P}^1_k$. Then the number of branch points of f coalescing to one point over k is even.*

CHAPTER 5

LIFTINGS OF $(\mathbb{Z}/2)^3$-COVERS

In this chapter, we apply results in the previous two chapters to construct explicit lifts for $(\mathbb{Z}/2)^3$-covers of various conductor types. We first use Mitchell's classification [Mit] to show that covers of type $(4,4,4)$ can only be lifted with equidistant geometry. Then we construct lifts for all covers of type $(4,4,2r)$, $r \geq 3$, with certain branch locus geometry.

5.1 Hurwitz Trees for $(\mathbb{Z}/2)^3$-Covers of Type $(4,4,4)$

In order to simplify the notations, we will only consider the subtrees of the Hurwitz tree rooted at a vertex connected to v_1 (see Section 2.4.2). Call them *branches* of the Hurwitz tree, and the size of a branch denotes the number of leaf nodes in that subtree. In each particular case, we will specify whether the leaves in a branch are equidistant, or there are further branching.

Definition 5.1.1. We say that a Hurwitz tree has *branch partition* $(b_1, \ldots, b_k)$, if there are k edges coming from the vertex connected to the root of the tree, each with b_i leaves. We say that a characteristic 0 cover has *branch locus geometry* $(b_1, \ldots, b_k)$ if its Hurwitz tree has branch partition $(b_1, \ldots, b_k)$, whose leaves correspond to branch points of the cover.

Proposition 5.1.2 (Mitchell). *The only possible Hurwitz trees for a* $(\mathbb{Z}/2)^2$*-cover over* R *of type* $(4,4)$ *have branch partition* $(1,1,1,1,1,1), (3,3)$ *and* $(2,2,2)$ *(Figure 5.1).*

Lemma 5.1.3. *Let* C *be a* $\mathbb{Z}/2$*-cover over* k*, and* $\hat{C}$ *a lift of* C *to* R *with non-equidistant geometry. Then the Hurwitz tree of* $\hat{C}$ *must only have branches with an even number of branch points.*

Proof. Suppose the Hurwitz tree of $\hat{C}$ has a branch with an odd number of branch points, i.e. there are an odd number of branch points closer to each other than any other branch point.

Since C is a local cover with Galois group $\mathbb{Z}/2$, there exists a global cover C' of $\mathbb{P}^1_k$, branched only

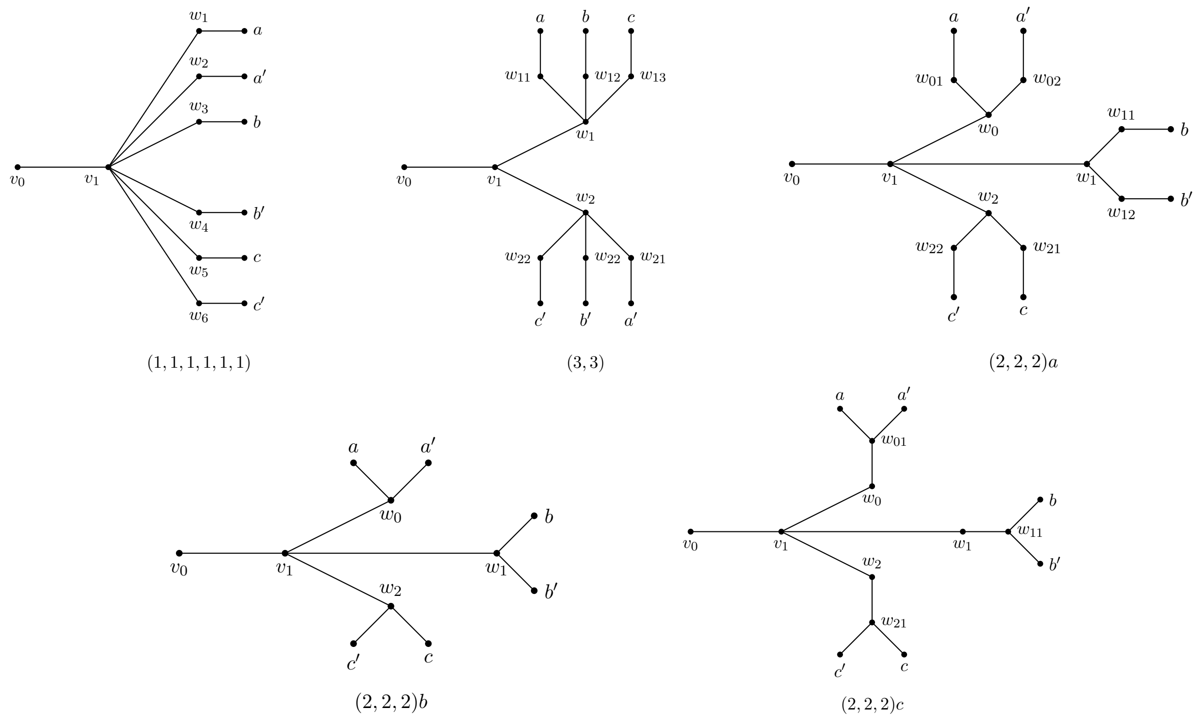

Figure 5.1: Hurwitz trees for Klein-four covers of type (4,4)

at ∞, such that the localization of C' at ∞ is C[Har80]. By the proof of Theorem 2.2 in [CGH08], C' lifts to a characteristic 0 cover $\hat{C}'$, with localization at ∞ isomorphic to $\hat{C}$, thus having the same geometry of branch locus. After a suitable change of variables, we then have a $\mathbb{Z}/2$-cover of $\mathbb{P}^1_R$ having good reduction, with an odd number of branch points coalescing to one point on $\mathbb{P}^1_k$, which is impossible by 4.2.4. □

Proposition 5.1.4. *The only possible Hurwitz tree for a lift of a* $(\mathbb{Z}/2)^3$*-cover over* k *of type* $(4, 4, 4)$ *is* $(1, 1, 1, 1, 1, 1, 1)$*, i.e. with equidistant geometry.*

Proof. We study possible Hurwitz trees for the lift C, a $(\mathbb{Z}/2)^3$-cover over R, by looking at subtrees corresponding to its $(\mathbb{Z}/2)^2$-subcovers. Let C_1, C_2, C_3 be three generating $\mathbb{Z}/2$-subcovers of C. Below, I will use the same letter to indicate that several branch points belong to the same branch in the Hurwitz tree. For example, a_i and a_j are closer to each other than a_i is to b_k.

Case 1: Suppose $C_1 \times C_2$ has Hurwitz tree $(2, 2, 2)$, with branch points $a_1, a_2, b_1, b_2, c_1, c_2$, where

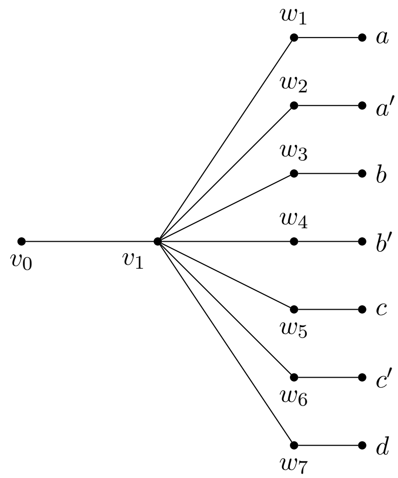

Figure 5.2: Equidistant Hurwitz tree for $(\mathbb{Z}/2)^3$-cover of type $(4,4,4)$

C_1, C_2 have branch loci $\{a_1, a_2, b_1, b_2\}$ and $\{b_1, b_2, c_1, c_2\}$ respectively. Then by the branch locus criterion, without loss of generality, we can assume that C_3 has branch points a_1, b_1, c_1 and a new branch point d.

By 5.1.3, since the Hurwitz tree of C_3 has at least two branches with only one branch point, it has to have equidistant geometry. Therefore d is not close to any branch point of $C_1 \times C_2$, i.e. C has Hurwitz tree $(2,2,2,1)$. Then the subtree corresponding to $C_1 \times C_3$ has branch locus $\{a_1, a_2, b_1, b_2, c_1, d\}$, thus is of the shape $(2,2,1,1)$, not an allowed Hurwitz tree for Klein-four covers.

Case 2: Suppose $C_1 \times C_2$ has Hurwitz tree $(3,3)$, with branch points $a_1, a_2, a_3, b_1, b_2, b_3$, where C_1, C_2 have branch loci $\{a_1, a_2, b_1, b_2\}$ and $\{a_1, a_3, b_1, b_3\}$ respectively. Then we can assume that C_3 has branch points a_1, b_2, b_3 and a new branch point d.

Applying 5.1.3 to C_3, d must be closer to the a's than the b's, i.e. C has Hurwitz tree $(4,3)$. Then the third $\mathbb{Z}/2$-subcover C_{12} of $C_1 \times C_2$ has branch points a_2, a_3, b_2, b_3, and $C_{12} \times C_3$ has branch locus $\{a_1, a_2, a_3, d, b_2, b_3\}$. Thus the subtree corresponding to $C_{12} \times C_3$ is of shape $(4,2)$, not an allowed Hurwitz tree for Klein-four covers.

Therefore, a $(\mathbb{Z}/2)^3$-cover over k of type $(4,4,4)$ can only be lifted with equidistant geometry. $\square$

Remark 5.1.5. In this special case of $(\mathbb{Z}/2)^2$-cover of type $(4,4,4)$, the lift only has 7 branch points. Since $(\mathbb{Z}/2)^3$ has 7 Klein-four subgroups, there are 7 Klein-four subcovers, all with distinct branch loci. Therefore we can look at a candidate Hurwitz tree for the $(\mathbb{Z}/2)^3$-cover, take away one branch point at a time, and check if the remaining subtree is one of the allowed Klein-four Hurwitz trees. This method will allow us to reach the same conclusion. However, the above proof can be generalized to more general $(\mathbb{Z}/2)^3$-covers, if we know a classification of Klein-four Hurwitz trees with higher conductors.

Proposition 5.1.6. *For almost all $b \in k$ and $u_2, u_3 \in k^*$, the characteristic 2 $(\mathbb{Z}/2)^3$-cover C of type $(4,4,4)$ given by*

$$C_1 : y_1^2 - y_1 = \frac{1+bt^2}{t^3}$$
$$C_2 : y_2^2 - y_2 = \frac{u_2}{t^3}$$
$$C_3 : y_3^2 - y_3 = \frac{u_3}{t^3}$$

can be lifted to characteristic 0.

Proof. After enlarging R, we can assume that $2^{1/3} \in R$. First we lift C_1 to $\tilde{C}_1 : Y_1^2 - (1+2^{4/3}C_1T^{-2} + 2^{2/3}C_2T^{-1})Y_1 = T^{-3} + BT^{-1} + o(1)$, where $C_1, C_2 \in R$, $\bar{B} = b$ and $o(1)$ denotes a polynomial in $R[T^{-1}]$ with Gaussian valuation strictly greater than 0. After the change of variables $X = 2^{2/3}T^{-1}$, $\tilde{C}_1$ is given by

$$Y_1^2 - (1 + C_1X^2 + C_2X)Y_1 = 2^{-2}X^3 + 2^{-2/3}BX + o(1).$$

Multiplying both sides by 4, and defining $Y_1' = 2Y_1 - (1 + C_1X^2 + C_2X)$, we have

$$Y_1'^2 = X^3 + 2^{4/3}BX + (1 + C_1X^2 + C_2X)^2 + o(4) =: F(X).$$

Then factoring $F(X)$, we can write

$$(Y_1')^2 = (1 - X_1X)(1 - X_2X)(1 - X_3X)(1 - X_4X),$$

with $X_i \in R$ for $1 \leq i \leq 4$, such that $X_i \neq X_j \mod \pi$ for $i \neq j$, $X_i \neq 0 \mod \pi$, and $X_3 \neq X_1 + X_2 \mod \pi$. Note that these are open conditions, so they do not decrease the dimension of possible lifts, which is 2 with parameters C_1, C_2.

Let $a_1, a_2, a_3 \in R$, after possibly enlarging R, be such that $X_1 = (a_1 + a_3)^2, X_2 = (a_2 + a_3)^2$ and $X_3 = (a_1 + a_2 + a_3)^2$. Then

$$\tilde{C}_1 : Y_1^2 = (1 - (a_1 + a_2)^2 X)(1 - (a_2 + a_3)^2 X)(1 - (a_1 + a_2 + a_3)^2 X)(1 - X_4 X),$$

and we construct covers

$$\tilde{C}_2 : Y_2^2 = (1 - a_1^2 X)(1 - (a_1 + a_2)^2 X)(1 - (a_1 + a_3)^2 X)(1 - (a_1 + a_2 + a_3)^2 X)$$
$$\tilde{C}_3 : Y_3^2 = (1 - a_2^2 X)(1 - (a_1 + a_2)^2 X)(1 - (a_2 + a_3)^2 X)(1 - (a_1 + a_2 + a_3)^2 X).$$

We can check that a_1, a_2, a_3 satisfy condition (*) in [Mat99] for most $a \in k$, since (*) is an open condition. Then by Proposition 2 in section 2.1 of [Mat99], $\tilde{C}_2, \tilde{C}_3$ have good reduction with respect to $T = 2^{-2/3} X^{-1}$, and the reductions are given by

$$C_2 : y_2^2 - y_2 = \frac{u_2}{t^3}$$
$$C_3 : y_3^2 - y_3 = \frac{u_3}{t^3},$$

where $u_2 = \overline{a_2 a_3 (a_2 + a_3)}^{1/2}, u_3 = \overline{a_1 a_3 (a_1 + a_3)}^{1/2}$. Now since the space of possible lifts is dimension 2, we can find a triple (a_1, a_2, a_3) satisfying this relation for most $u_2, u_3 \in k^*$.

Therefore, $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3$ have simultaneously good reduction. Moreover, they have one common branch point $1/(a_1 + a_2)^2$, and pairwise share one other branch point, satisfying the branch locus criterion. We conclude that the normalization of $\tilde{C}_1 \times \tilde{C}_2 \times \tilde{C}_3$ is a lift of C. □

5.2 Lifting $(\mathbb{Z}/2)^3$-Covers of Type $(4, 4, 2r), r \geq 3$

In this section, I will construct lifts of any $(\mathbb{Z}/2)^3$-cover of type $(4, 4, 2r)$ for $r \geq 3$, using methods in Mitchell's thesis [Mit] and results in Pagot's thesis [Pag]. The lifts have Hurwitz tree $(3, 3, 3, 2, \ldots, 2)$, with $r - 3$ branches of size 2. Define $\rho = 2^{\frac{1}{2r-1}} \in \pi R$.

Lemma 5.2.1. *Let $\alpha \in k^*, \beta \in k, A \in R^*$, and suppose that $U \in R^*$ is any element such that $-AU^2 \equiv \alpha$ mod π and $U - A \in R^*$. Then there exists $V \in R^*$ such that the following property holds: Let*

$$T_1 = 0, T_2 = \rho^{4r-4}A, T_3 = \rho U, T_4 = \rho U + \rho^{4r-4}V.$$

Then the cover $Y^2 = F(T^{-1}) = \prod_{i=1}^{4}(1 - T_iT^{-1})$ of $\mathbb{P}^1_R$ has good reduction, namely with reduction $z^2 - z = \frac{\alpha}{t^3} + \frac{\beta}{t}$.

Proof. This proof is similar to the proof of Lemma 4.2.2 in [Mit], but with different and more general distances between branch points.

Let $V = -\rho^2 B - A + (-\rho^3(\rho^2 B + A)U)^{1/2}$, for some $B \in R$ with $B \equiv \beta$ mod π. Then V is a solution to the polynomial equation

$$V^2 + 2(\rho^2 B + A)V + \rho^3(\rho^2 B + A)U + (\rho^2 B + A)^2 = 0;$$

$$\text{or equivalently,} \rho^{4r-5}UV + U^2 - (\rho^{2r-2}B + \rho^{2r-4}A + U + \rho^{2r-4}V)^2 = 0.$$

Thus

$$(\rho^{4r-5}UV + U^2)^{1/2} = -\rho^{2r-2}B - \rho^{2r-4}A - U - \rho^{2r-4}V, \tag{5.1}$$

where $(\rho^{4r-5}UV + U^2)^{1/2}$ denotes the appropriate square root of $\rho^{4r-5}UV + U^2$. After possibly enlarging R, we can assume this element is in R, along with V.

Substituting the values of T_i into F and using the definition of ρ, we have that

$$\begin{aligned} F(T^{-1}) &= (1-\rho^{4r-4}AT^{-1})(1-\rho UT^{-1})(1-(\rho U+\rho^{4r-4}V)T^{-1}) \\ &= 1-(\rho^{4r-4}A+2\rho U+\rho^{4r-4}V)T^{-1}+(\rho^{4r-3}UV+\rho^2U^2)T^{-2}-4AU^2T^{-3}+o(4), \end{aligned}$$

where $o(4)$ denotes a polynomial with Gauss valuation strictly greater than $v(4)$.

Again after enlarging R, let

$$q=(\rho^{4r-3}UV+\rho^2U^2)^{1/2}=\rho(\rho^{4r-5}UV+U^2)^{1/2}\pi R,$$

and define $Q(T^{-1})=1+qT^{-1}\in R[T^{-1}]$.

Then by equation (5.1) and the definitions of ρ and q,

$$\begin{aligned} &Q(T^{-1})^2+4BT^{-1}-4AU^2T^{-3} \\ =&Q(T^{-1})^2-2\rho((\rho^{4r-5}UV+U^2)^{1/2}+\rho^{2r-4}A+U+\rho^{2r-4}V)T^{-1}-4AU^2T^{-3} \\ =&1+2qT^{-1}+q^2T^{-2}-2qT^{-1}-\rho^{2r}(\rho^{2r-4}A+U+\rho^{2r-4}V)T^{-1}-4AU^2T^{-3} \\ =&1-(\rho^{4r-4}A+2\rho U+\rho^{4r-4}V)T^{-1}+(\rho^{4r-3}UV+\rho^2U^2)T^{-2}-4AU^2T^{-3} \\ =&F(T^{-1})+o(4). \end{aligned}$$

After the change of variables $Y=-2Z+Q(T^{-1})$, and plugging in values of U and V, the equation $Y^2=F(T^{-1})$ gives

$$4Z^2-4ZQ(T^{-1})+Q(T^{-1})^2=Q(T^{-1})^2+4BT^{-1}-4AU^2T^{-3}+o(4).$$

$$\text{Equivalently,} Z^2-ZQ(T^{-1})=BT^{-1}-AU^2T^{-3}+o(1).$$

Finally, since $Q(T^{-1})\equiv 1 \bmod \pi$, this reduces to $z^2-z=\frac{\alpha}{t^3}+\frac{\beta}{t}$. □

Proposition 5.2.2. *For all $(\mathbb{Z}/2)^3$-covers defined by a ring extension $k[[z]]/k[[t]]$ of type $(4,4,2r), r\geq$*

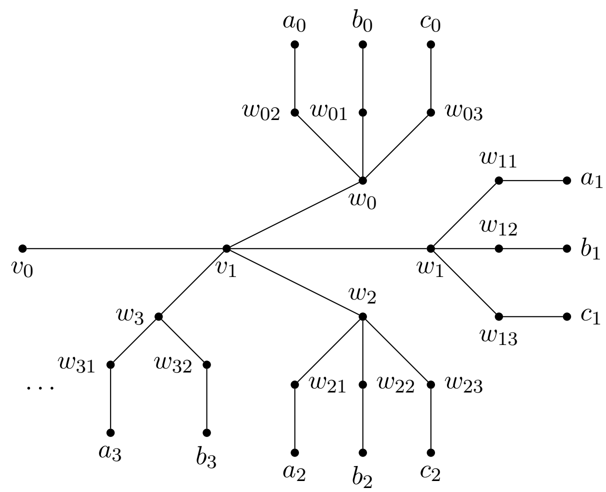

Figure 5.3: Hurwitz tree with branch partition $(3, 3, 3, 2, \dots, 2)$

3, *there exists a lift to characteristic* 0 *with branch locus geometry* $(3, 3, 3, \underbrace{2, \dots, 2}_{r-3})$. *I.e. its Hurwitz tree has* 3 *branches of size* 3 *and* $r - 3$ *branches of size* 2 *(see Figure 5.3). In particular, the branch points of a lift here can never be equidistant.*

Proof. We can assume that $k[[z]]/k[[t]]$ is defined as the composition of subcovers of the form

$$C_1 : y_1^2 - y_1 = \frac{a_1}{t^3} + \frac{b_1}{t},$$
$$C_2 : y_2^2 - y_2 = \frac{a_2}{t^3} + \frac{b_2}{t},$$
$$C_3 : y_3^2 - y_3 = \frac{1}{t^{2r-1}},$$

where $a_1, a_2 \neq 0$ are distinct. Fix $A \in R^*$ and $U_1, U_2 \in R^*$ such that $-AU_i^2 \equiv a_i \bmod \pi$ and $U_i - A \in R^*$. Then by 5.2.1, there exist $V_1, V_2 \in R^*$, such that

$$\mathcal{C}_i : Y_i^2 = (1 - \rho^{4r-4} A T^{-1})(1 - \rho U_i T^{-1})(1 - (\rho U_i + \rho^{4r-4} V_i) T^{-1})$$

is a lift of C_i for $i = 1, 2$. Note that since $a_1 \neq a_2$ and A is a unit, $v(U_1 - U_2) = 0$.

Now let $T_1 = 0, T_2 = U_1, T_3 = U_2$, and choose $T_i, 4 \leq i \leq r$, such that $v(T_i - T_j) = 0$ for all $i \neq j$. Then by Lemma 5.1.2 of [Pag](see also Proposition 3.3 of [MatNotes]), we can define some

$F(X) = \prod_{i=1}^{r}(X - T_i)(X - \tilde{T}_i)$ such that $v(T_i - \tilde{T}_i) = v(2)$, and $Y^2 = F(X)$ has good reduction relative to the coordinate $T = \rho X$, with reduction C_3. Then $\tilde{T}_i = T_i + 2W_i$ for some $W_i \in R^*$, and this lift $\mathcal{C}_3$ is defined by

$$Y_3^2 := ((\rho/T)^r Y)^2 = \prod_{i=1}^{r}(1 - \rho T_i T^{-1})(1 - (\rho T_i + \rho^{2r} W_i)T^{-1}),$$

Observe that 0 is the common branch point for all three lifts, while $\rho^{4r-4}A$ is a branch point that is shared by $\mathcal{C}_1, \mathcal{C}_2$; ρU_1 is shared by $\mathcal{C}_1, \mathcal{C}_3$; and ρU_2 is shared by $\mathcal{C}_2, \mathcal{C}_3$. Thus the lifts satisfy the branch cycle criterion (Theorem 3.4.1), and the normalization of the product of $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ is a lift of $k[[z]]/k[[t]]$. It is straightforward to check that this configuration of branch points is as indicated in the picture above. $\square$

BIBLIOGRAPHY

[BW09] Brewis, L.H., Wewers, S. Artin characters, Hurwitz trees and the lifting problem. Math. Ann. 345, 711–730 (2009).

[BW06] Irene I. Bouw and Stefan Wewers. The local lifting problem for dihedral groups. Duke Math. J., 134(3):421–452, 2006.

[B1] Bourbaki, N.: Algèbre Commutative, Hermann, Paris, 1961.

[CGH08] Ted Chinburg, Robert Guralnick, and David Harbater. Oort groups and lifting problems. Compositio Math., 144(4):849–866, 2008.

[CGH17] Ted Chinburg, Robert Guralnick, and David Harbater. Global Oort groups. J. Algebra, 473:374–396, 2017.

[Dan20] Dang, Huy. Connectedness of the moduli space of Artin-Schreier curves of fixed genus. J. Algebra 547 (2020), 398–429.

[GM98] Barry Green and Michel Matignon. Liftings of Galois covers of smooth curves. Compos. Math., 113(3):237–272, 1998.

[GM99] Barry Green and Michel Matignon. Order p automorphisms of the open disc of a p-adic field. J. Amer. Math. Soc., 12(1):269–303, 1999.

[Gar96] Marco A. Garuti. Prolongement de revêtements galoisiens en géométrie rigide. Compositio Math., 104(3):305–331, 1996.

[Har80] David Harbater. Moduli of p-covers of curves. Comm. Algebra, 8(12):1095–1122, 1980.

[Har77] Robin Hartshorne. Algebraic geometry. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[Hen00] Yannick Henrio. Arbres de hurwitz et automorphismes d'ordre p des disques et des couronnes p-adiques formels.

[Lehr] Lehr, C. G. (2001). Reduction of wildly ramified covers of curves. Available from Dissertations & Theses @ University of Pennsylvania; ProQuest Dissertations & Theses Global.

[Liu] Qing Liu, Algebraic Geometry and Arithmetic Curves. Oxford Graduate Texts in Mathematics, 6 (2002).

[Mat99] Michel Matignon. p-Groupes abéliens de type $(p, \ldots, p)$ et disques ouverts p-adiques. Manuscripta Math. 99, 93–109 (1999).

[MatNotes] Michel Matignon, Lifting $(\mathbb{Z}/2\mathbb{Z})^2$-actions according to G. Pagot, (3-10-2003). https://www.math.u-bordeaux.fr/∼mmatigno/preprints.html

[Mit] Mitchell, George, The Local Lifting Problem for Curves with Quaternion Actions (2022). CUNY Academic Works.

[Obu15] Andrew Obus. A generalization of the Oort Conjecture. ArXiv e-prints, February 2015. arxiv:1502.07623.

[Obu16] Andrew Obus. The local lifting problem for A4. Algebra Number Theory, 10(8):1683–1693, 2016.

[Obu17] Andrew Obus. Lifting of curves with automorphisms. arXiv: Algebraic Geometry, 2017.

[OW14] Andrew Obus and Stefan Wewers. Cyclic extensions and the local lifting problem. Annals of Mathematics, 180(1):233–284, 2014.

[Pop14] Florian Pop. The Oort conjecture on lifting covers of curves. Annals of Mathematics, 180(1):285–322, 2014.

[OSS] T. Sekiguchi, F. Oort, and N. Suwa. On the deformation of Artin-Schreier to Kummer. Ann. Sci. Ecole Norm. Sup. (4), 22(3):345–375, 1989.

[Pag] Guillaume Pagot, $\mathbb{F}_p$-espaces vectoriels de formes différentielles logarithmiques sur la droite projective, Journal of Number Theory, Volume 97, Issue 1, 2002, Pages 58-94.

[PZ] Pries, Rachel; Zhu, Hui June. The p-rank stratification of Artin-Schreier curves. Annales de l'Institut Fourier, Volume 62 (2012) no. 2, pp. 707-726.

[Ser] Serre (1979). *Local Fields*, Springer Graduate Texts in Mathematics

[SGA03] Revêtements étales et groupe fondamental (SGA 1). Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 3. Société Mathématique de France, Paris, 2003.

[Sub] Subrao, Doré, The p-rank of Artin-Schreier curves. Manuscripta Math. 16 (1975), no. 2, 169–193.

[Wea17] Bradley Weaver. Non-Abelian Groups of Order Eight and the Local Lifting Problem. PhD thesis, University of Virginia, 2018.