

A Monadic Interpretation of Categorical Mackey Functors

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ABSTRACT
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We study a monadic version of categorical Mackey functors proposed by Bonventre which we call $\hat{\Sigma}_G \wr (-)$ -algebras or *SGAs*. These are algebras over the monad $\hat{\Sigma}_G \wr (-)$ in categories fibered over \mathbf{Fin}^G satisfying an additivity condition. The monad operation encode genuine commutative operations, which we can also interpret as transfers. These have several conditions we can strengthen or weaken, offering substantial flexibility. By varying the conditions on these algebras, we show we can recover the permutative Mackey functors of Bohmann-Osorno and the symmetric monoidal Mackey functors of Hill-Hopkins. In the process we construct a convenient strict $(2,1)$ -category of spans. We also define *commutative monoids* in an SGA.

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Chapter 1

Introduction

One of the most fundamental concepts in category theory and algebra is that of a symmetric monoidal category - a category with a monoidal product, unital, associative, and commutative up to natural isomorphism; in the case of a *permutative category* associativity and unitality are strict. For instance we have the symmetric monoidal category of *Set* with the cartesian product, or $R - Mod$ with \otimes_R . With this structure we can define rings and algebras as examples of a general concept of monoids. Additionally, by taking algebraic K -theory or the classifying space of a symmetric monoidal category, we can connect the structure of the category to topology.

Group actions are similarly omnipresent across mathematics, appearing frequently in topology, geometry, and number theory. In particular we consider a fixed finite group G . Recently, spurred on by the proof of the Kervaire Invariant 1 Problem by Hill-Hopkins-Ravenel [HHR21], and connections to trace methods in algebraic K -theory, equivariant homotopy theory and equivariant algebra have come to the spotlight as techniques from this field have found more widespread use.

Central to these are *Mackey functors* best thought of as the equivariant analogue of abelian groups. These can be defined as additive functors from Fin^G , the category of finite G -sets and equivariant maps, to Ab with both covariant *transfer* and contravariant *restriction* maps, related by the *double coset formula*.

Remark 1.0.1. Throughout equivariant homotopy theory and algebra, we often see a distinction between *naive* and *genuine commutative* objects. The former merely has both a commutative monoidal structure and a G -action, whereas in the latter the two have a more subtle interplay. For instance we have abelian groups with a G -action compared to Mackey functors or Borel G -spectra versus genuine G -spectra.

With this in mind, we might expect many of the symmetric monoidal categories encountered in equivariant homotopy theory and algebra to have some additional structure. Here lays a gap which is only natural to fill:

How we can define a genuine equivariant symmetric monoidal category? That is to say a suitable equivariant analogue of a symmetric monoidal category.

Multiple authors have presented versions, often with similar names, so we use the term *genuine equivariant symmetric monoidal category* to refer to the general idea of a construction aiming to answer this question, and refer to their constructions as different models of it. Similar to how one can refer to ∞ -categories in the abstract and to specific models such as quasicategories.

We start by outlining several desiderata of a theory of genuine equivariant symmetric monoidal categories.

- Mackey functors and semi-Mackey functors give discrete ones; taking isomorphism classes

gives us semi-Mackey functors.

- There is an equivariant K -theory functor to genuine G -spectra.
- Equivariant K -theory is surjective up to homotopy equivalence. Ideally K -theory has a homotopy-inverse analogous to Thomason's theorem.
- They include naive equivariant symmetric monoidal categories, but are not limited to them.
- They include the examples of finite G -sets, G -Mackey functors, and genuine G -spectra. These each should also include or recover the data of $Fin^H, Mack_H, Sp_H$, respectively, as well as restriction and norm functors N_H^G for all $H \leq G$.

The Eilenberg-MacLane spectrum $H(-)$ and $\underline{\pi}_0$ functor are monoidal.

- There is a notion of a genuine commutative monoid, and in the latter two examples they are Tambara functors and G -ring spectra.
- For $G = \{e\}$ the theory reduces to that of non-equivariant symmetric monoidal categories.
- Genuine equivariant colored operads specialize to them, in the same way that symmetric colored operads (multicategories) specialize to permutative categories.

We now introduce several of the models. For completeness we start with two naive versions.

- Symmetric monoidal categories with a G -action by symmetric monoidal functors.

- A *symmetric monoidal object in G -categories* is a G -object in categories with an equivariant biproduct \otimes and fixed unit object, so that the unitor, associator, and symmetry natural isomorphisms are G -isomorphisms [Rub20].

It is straightforward to check that this is equivalent to a symmetric monoidal category with G -action by strict symmetric monoidal functors.

Remark 1.0.2. We note that in a symmetric monoidal category with G -action by lax or strong monoidal functors, the fixed point category \mathcal{C}^H is generally not symmetric monoidal as $g(1_{\mathcal{C}}) \neq 1_{\mathcal{C}}$, so $1_{\mathcal{C}} \notin \mathcal{C}^H$ and $g(x) \otimes g(y) \neq g(x \otimes y)$, so $x, y \in \mathcal{C}^H \not\Rightarrow x \otimes y \in \mathcal{C}^H$.

We observe that a symmetric monoidal category with strong monoidal strictly unital G -action, is equivalent to a G -category \mathcal{C} with a pseudo-equivariant functor $\mathcal{C} \otimes \mathcal{C} \rightarrow \mathcal{C}$, with a fixed unit object, so that the unitor, associator, and symmetry natural isomorphisms are G -isomorphisms.

By [Mer17] Cor 3.4, this makes $\mathcal{C}^{hH} := Fun(EG, \mathcal{C})^H = Fun(EH, \mathcal{C})^H$ a symmetric monoidal category as we have the induced functor $\otimes : \mathcal{C}^{hH} \times \mathcal{C}^{hH} \rightarrow \mathcal{C}^{hH}$. Additionally, by [Mer17] Prop 3.3, given a pseudoequivariant functor $F : \mathcal{C} \otimes \mathcal{C} \rightarrow \mathcal{C}$, by applying $Fun(EG, -)$ we have an equivariant functor. Consequently, given a symmetric monoidal category with G -action by strong symmetric monoidal strictly unital functors, by applying $Fun(EG, -)$ we have a symmetric monoidal object in G -categories.

The first are those modelled on Mackey functors, we can think of these as being indexed on Fin^G or \mathcal{O}_G , the full subcategory of transitive G -sets.

- First are the *permutative Mackey functors* or *categorical Mackey functors* of Bohmann-Osorno [BIO15]. These were originally defined as enriched functors from $(\mathbf{GE}')^{op} \rightarrow$

$Perm$, where \mathbf{GE}' is a version of the span category on Fin^G enriched in permutative categories, and $Perm$ is the category of permutative categories which is enriched over itself. They define a version of K -theory and use it to construct Eilenberg-MacLane genuine G -spectra which is one of the key desiderata. We construct an equivalent enriched span category \mathbf{GE}_{ord} which we use instead as it is more convenient.

- Next are the *symmetric monoidal Mackey functors* of Hill-Hopkins as well as a generalization called *genuine G -symmetric monoidal structure* [HH16]. These are defined much less rigidly but also less precisely. As a result they are comparatively unwieldy. They also define *genuine G -commutative monoids* which are known to include some of the desired examples.
- Third we have the G -symmetric ∞ -categories of Nardin-Shah, part of the series of parametrized higher category theory and higher algebra [BDG⁺16, NS22]. These can be viewed as a higher categorical version of Mackey functors taking values in ∞ -categories. This theory connects to parametrized ∞ -operads and includes some of the desired examples. These are both the most general and in a sense the morally correct approach. However they are abstract and unwieldy and for many purposes working with 1- and 2-categories is simpler and more elucidating.

The second group consist of a single category symmetric monoidal or permutative category with extra operations on it. The first two of which are closely related to $\mathbb{E}_\infty - G$ -spaces.

- A $\Gamma - G$ -category as defined by Shimikawa is a functor $X : Fin_* \rightarrow Cat_*^G$ [Shi89, Shi91]. It is *special* if the map $X(n)_\rho \rightarrow X(1)_\rho^n$ is an H -equivalence, for all $H \leq G$ and all $\rho : H \rightarrow \Sigma_n$, where H acts on them through $\rho : H \rightarrow G \times \Sigma_n$. (These are equivalent to *special*

Γ_G -categories.) These give an equivariant generalization of the Segalic construction of K -theory.

- A *permutative* (resp. *symmetric monoidal*) G -category of Guillou-May-Merling-Osorno is a pseudoalgebra (resp. algebra) over the categorical equivariant Barratt-Eccles operad with n^{th} level is the G -category $\text{Fun}(EG, E\Sigma_n)$ [GM12, GMMO19, GMMO23]. By work of [BBK⁺19] these cannot be described as permutative category with finitely many extra operations. But the operad is equivalent to a finitely generated one. These have a well-studied version of K -theory which is multiplicative.
- A *normed symmetric monoidal category* of Rubin is a symmetric monoidal object in G -categories with the additional structure of compatible H -equivariant *external norm functors* $\otimes_T : \mathcal{C}^T \rightarrow \mathcal{C}$, where T is a finite H -set and H acts on \mathcal{C}^T by simultaneously permuting the components and action on them, and natural *untwistor* isomorphisms relating the external norms to multiplication $\mathcal{C}^{|T|} \rightarrow \mathcal{C}$ [Rub20].
- A *G -parsummable category* by Lenz is a G -object in the category of parsummable categories, a variation of symmetric monoidal categories [Len22].

The main thrust of this paper is introducing the monad $\hat{\Sigma}_G \wr (-)$, whose (pseudo)algebras can be viewed as a new type of categorical Mackey functor. There are several parameters we can vary separately which together determine the strength of additivity an SGA has, whether restrictions are functorial or pseudofunctorial, whether they are strong or strict monoidal, and similarly for transfers. This also provides a precise way to describe the naturality of a double-coset isomorphism. SGAs were created by Peter Bonvetre and Luis Pereira and further

developed in joint work with Bonventre. Most of the technical results on the structure of SGAs and some in their proofs were devised by Bonventre

With this we reach a main success of this paper - carefully interpolating between the [BIO15, HH16] versions of symmetric monoidal and permutative Mackey functors by varying the different parameters.

A significant challenge when constructing categorical Mackey functors is that pullbacks are only defined up to isomorphism, so there is no preferred way to define composition of spans. In the 1-categorical case this is not an issue as we are only concerned with isomorphism classes of spans. Guillou-May [GM11] and Bohmann-Osorno [BIO15] deal with this problem by working in \mathbf{Fin}^G , the category of ordered G -sets including all (unordered) equivariant maps, with only a single object of each ordered isomorphism-type. In this category we can define the *lexicographical pullback*, giving us a preferred choice of pullbacks. However if we use this to define a 2-category of spans, it results in the bicategory \mathbf{GE} as opposed to a strict 2-category. Guillou-May sidestep this issue by cleverly defining \mathbf{GE}' , a slight modification which is a strict 2-category, and Bohmann-Orsono contains a minor error as their construction of $G\mathcal{E}$ is not in fact a strict 2-category, as noted in [JY22].

One main result of this paper is proving that there is not consistent choice of pullbacks in \mathbf{Fin}^G which would make \mathbf{GE} a strict 2-category. Consequently something else is necessary. We define \mathbf{GE}_{ord} , the (2,1)-category of spans in \mathbf{Fin}^G whose left leg is order preserving. This is equivalent to \mathbf{GE} and \mathbf{GE}' as a 2-category and biequivalent to \mathbf{GE}' as a category enriched in permutative categories. \mathbf{GE}_{ord} is more closely connected to the category \mathbf{Fin}^G and thus simpler to work with. As a result, later in the paper we use \mathbf{GE}_{ord} as the domain category for permutative Mackey functors.

On the other hand we have the work of Guillou-May-Merling-Osorno [GMMO19, GMMO23]. They define genuine symmetric monoidal G -categories as algebras over an operad. At the moment however the connection between their version of genuine equivariant symmetric monoidal categories and the various Mackey functor versions is poorly understood. We believe this to be a fundamentally different approach. A major question is to what extent these all arise from naive symmetric monoidal G -categories. Lenz has proved that all do up to a notion of equivalence [Len22].

We believe genuine G -symmetric monoidal categories could be viewed as a type of a genuine G (colored) operad in the same way that symmetric multicategories extend permutative categories. Indeed this is the approach taken in [NS22]. We do not know of similar work in a lower-category theoretic context. However it is not apparent how such a connection would be formed.

We believe that many of the ideas in this thesis have a natural application in the six-functor formalism and Beck-Chevalley transformation of algebraic geometry. In many ways this is formally similar to that of Mackey functors.

Much of the first section of this work deals with \mathbf{Fin}^G , the skeleton category of ordered finite G -sets. This is so that we can define products and coproducts of finite G -sets so that they form a permutative category, and so that we have a well-defined choice of pullbacks. This is needed in order to construct permutative Mackey functors in a strict 2-functorial way. We emphasize that although large sections of this paper rely heavily on the mechanics of ordered sets and maps, they are not philosophically meaningful to us, rather they are useful merely for the formal properties they satisfy.

There are other approaches. The traditional 1-categorical approach is to work with isomor-

phism classes of G -sets or spans of them, making pullbacks well-defined. Another approach is to work entirely with weak 2-categories and pseudofunctors. This is the approach taken by [HH16]. In many ways this is a more natural and “morally correct” approach. A disadvantage is that these are generally unwieldy and unpleasant to deal with. Additionally some work such as [BIO15, EM06] uses strict 2-categories, and they are needed for multiplicative K -theory constructions. Even when working with weak 2-categories, one will often want to strictify to strict 2-categories and permutative categories as in [Gui10]; so we ultimately lose the strictness at one point or another. A final way is to work with ∞ -categories, which could be thought of as even more natural. However in this case the higher level of abstraction can obscure more than it reveals, particularly as the work in this direction is somewhat less accessible.

With SGAs we can carefully tweak the different ways in which things can be weak, strong, or strict.

We create the Fin^G categories - essentially these are functors $(\text{Fin}^G)^{op} \rightarrow \text{Cat}$ which are weakly additive, and the equivalences and invertible 2-cells witnessing them are suitably natural. In a sense this is a rather messy and ad-hoc definition, but we believe it is the weakest set of conditions needed for SGAs to have the desired properties. Fortunately in the strongly additive case, the extra naturality conditions are satisfied a fortiori. These were first defined by Bonventre as a setting for SGAs and refined in this work.

1.1 Outline

After reviewing some basic definitions and notation, we discuss the background topics of equivariant mathematics and various ways of describing monoids. These motivate the constructions of the different types of genuine equivariant symmetric monoidal categories. In particular, the idea of what genuine means in such a context.

In the next three chapters, we review definitions and results from 1-category theory, 2-category theory, as well a brief overview of ∞ -categories. We will need some of these later in the paper.

We then review symmetric monoidal categories, multicategories and enriched categories, building up to defining $Perm$, the category of permutative categories enriched in the multicategory of permutative categories. We touch on results of [BIO15] on change of enrichment and multiplicative K -theory of [EM06], which are used to construct genuine G -spectra from functors enriched in permutative categories.

Next we discuss \mathbf{Fin}^G , the skeleton category of finite ordered G -sets and equivariant maps. We prove several technical results on lexicographical-pullbacks in \mathbf{Fin}^G which we use throughout the rest of the paper.

We then introduce *chosen pullback systems*, a selection of pullbacks in a category \mathcal{C} , and *draconian chosen pullback systems*, a choice which is suitably unital and associative, allowing us to define a strict (2,1)-category of spans in \mathcal{C} . We prove that no such structure can exist on \mathbf{Fin} or \mathbf{Fin}^G , essentially telling us there is no convenient way to define a strict Burnside 2-category on \mathbf{Fin}^G .

Instead we are forced to construct a slight variation originally proposed by Bonventre. We discuss the strict (2,1)-categories \mathbf{GE}' of [GM11] and \mathbf{GE}_{ord} which we define as the strict (2,1)-category of spans in \mathbf{Fin}^G with left leg order preserving. This idea was presented by Bonventre. These are biequivalent as categories enriched in permutative categories, and we choose to use \mathbf{GE}_{ord} as it is more cleanly defined than \mathbf{GE}' .

We start the second section by defining \mathbf{Fin}^G -categories which are categories fibered over $(\mathbf{Fin}^G)^{op}$ satisfying a particular weak additivity property. This was also first devised by Bonventre and then refined in this thesis. We define $\hat{\Sigma}_G$ then $\hat{\Sigma}_G \wr (-)$, the latter a monad in the category of \mathbf{Fin}^G -categories. For a \mathbf{Fin}^G -category $\underline{\mathcal{C}}$, intuitively $\hat{\Sigma}_G \wr \underline{\mathcal{C}}$ encodes the fibers of $\underline{\mathcal{C}}$ as well as genuine equivariant operations.

We then introduce a $\hat{\Sigma}_G \wr (-)$ -algebra or SGA as a pseudoalgebra over this monad which was proposed by Bonventre. We discuss several ways to strengthen and weaken this definition and then present several important examples. In the next chapter we prove technical results on the structure of SGAs, the key takeaway being that SGAs share much of the key information as a Mackey functor but in categories, with various invertible 2-cells being highly coherent.

In the next section we compare SGAs to two other models of genuine equivariant symmetric monoidal categories. We first show that given a *permutative* SGA, one satisfying some strictness assumptions, we can construct a permutative Mackey functor. And given a permutative Mackey functor satisfying a mild technical assumption we can construct an SGA. We observe that on the categories and 1-morphism present, these constructions are inverses, but we do not have a proof of a stronger statement.

We then observe that pseudo-SGAs, a weaker version, capture the idea described in [HH16] as a symmetric monoidal Mackey functor. We then define commutative monoids in an SGA

and compare them to the G -commutative monoids of [HH16].

We finish by discussing some different choices we could have made in this paper as well as directions of further research.

Part I

Conventions and Preliminaries

Chapter 2

Basics

Notation 2.0.1. Through this entire paper, G will denote a finite group, H, K will denote subgroups, and e will denote its identity element.

We discuss the importance of finiteness in Remark 3.2.12.

Notation 2.0.2. \mathbf{Fin} denotes a skeleton category of finite sets and all maps. For concreteness its objects are $\underline{n} := \{1, \dots, n\}$, with $\underline{0} = \emptyset$.

We let Σ denote its maximal subgroupoid and use cycle notation to denote its elements.

\mathbf{Fin}_* denotes finite pointed sets with elements $n_+ = \{*, 1, \dots, n\}$.

Notation 2.0.3. Σ_n denotes the automorphism group of \underline{n} in \mathbf{Fin} .

Notation 2.0.4. We use the term (*genuine*) *equivariant symmetric monoidal category* to mean a genuine equivariant version of a symmetric monoidal category, though there are several models for this.

Many terms similar to this are used by different authors to refer to very different constructions so may we refer to those by their authors' names to minimize ambiguity.

Notation 2.0.5. Throughout, “=” will only be used to denote strict equality; in diagrams we use “=” and “ Id ” interchangeably. We use “ \cong ” to denote an isomorphism, “ \simeq ” will denote an equivalence of categories, an equivalence of 2-categories, or a homotopy equivalence; and “ \sim ” will denote weak equivalences or ad hoc equivalence relations; “ \simeq_Q ” will denote a (zig-zag of) Quillen equivalence(s) between model categories.

Notation 2.0.6. For a category \mathcal{C} we will write $x \in \mathcal{C}$ to denote that x is an object of \mathcal{C} .

We will write “ $f: x \rightarrow y$ in \mathcal{C} ” to mean $f \in Hom_{\mathcal{C}}(x, y)$.

Notation 2.0.7. We will denote a commutative square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

as A, B, C, D ,

Notation 2.0.8. For a group G , BG denotes the associated groupoid.

Definition 2.0.9. A G -object in a category \mathcal{C} is a functor $BG \rightarrow \mathcal{C}$, equivalently an object in \mathcal{C} upon which G acts by automorphisms. These form the category $\mathcal{C}^G := Fun(BG, \mathcal{C})$ where morphisms are natural transformations, equivalently equivariant maps in \mathcal{C} .

Definition 2.0.10 ([Mer17]). For two G -categories \mathcal{C}, \mathcal{D} , a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is *pseudo-equivariant* if for $g \in G$, we have a natural isomorphism $gF \cong Fg$, which is suitably preserved along multiplication in G .

Definition 2.0.11. For a G -set A , $A//G$ is the *action groupoid* with objects elements of A and morphisms are pairs $a \mapsto ga$ for each $(a, g) \in A \times G$.

Definition 2.0.12. For a group G , its *chaotic category* or *indiscrete category* EG is the action

groupoid of G acting on itself by translation. This has an object for each element of G and a unique morphism between every two elements. We observe that G acts on this by translation.

Definition 2.0.13. Given two G -categories \mathcal{C}, \mathcal{D} , $\text{Fun}(\mathcal{C}, \mathcal{D})$ is the category of all functors $\mathcal{C} \rightarrow \mathcal{D}$, it has a G -action via conjugation and $\text{Fun}(\mathcal{C}, \mathcal{D})^G$ is the full subcategory of equivariant functors.

We note that $\text{Fun}(EG, \mathcal{C}) \simeq \mathcal{C}$, but they are not equivalent as G -categories. For $H \leq G$, we call $\mathcal{C}^{hH} := \text{Fun}(EG, \mathcal{C})^H$ the *homotopy fixed points* of \mathcal{C} .

Proposition 2.0.14 ([Mer17]). *Given a pseudo-equivariant functor $F : \mathcal{C} \rightarrow \mathcal{D}$, it induces an equivariant functor we also denote $F : \text{Fun}(EG, \mathcal{C}) \rightarrow \text{Fun}(EG, \mathcal{D})$.*

Chapter 3

Equivariant Background

In this chapter we introduce several concepts in the field of equivariant algebra which motivates later constructions. We present several important definitions and theorems. We also explain many key bits of intuition in equivariant algebra. Some topics we present only briefly here and only present more formally in other chapters. We hope this chapter could be of use to novices of equivariant homotopy theory.

In Chapter 8 we introduce a specific model of \mathbf{Fin}^G , the category of finite G -sets and equivariant maps, which has a preferred way of taking finite co/products and pullbacks. In this chapter any equivalent category suffices.

Notation 3.0.1. When discussing group actions, we always mean left actions. G/H denotes the set of left cosets, those of the form gH . We let $gHg^{-1} := \{ghg^{-1} \mid g \in G\}$ denote the conjugate subgroup of H .

Definition 3.0.2. The *orbit category* \mathcal{O}_G is the full subcategory of \mathbf{Fin}^G on transitive G -sets of the form G/H .

Lemma 3.0.3. *All transitive G -sets are isomorphic to those of the form G/H .*

Projection maps $\pi : G/K \rightarrow G/H$ for $K \leq H$ and conjugacy maps $c : G/H \rightarrow G/(gHg^{-1})$ generate the morphisms in this category.

For a G -object X , X^H denotes the H -fixed points of X , which in a general category can be defined as $\lim(BH \xrightarrow{X} \mathcal{C})$ when it exists.

We note that in Set , $X^H = Hom(G/H, X) \cong X^H$.

Lemma 3.0.4. *Any finite G -set is a finite coproduct of G -sets of the form G/H , which are exactly the transitive G -sets.*

We note that we mean this as a categorical product as opposed to an ordered one. Alternatively every element of \mathbf{Fin}^G is isomorphic to one of the form $\coprod_i G/H_i$.

Lemma 3.0.5. $\mathbf{Fin}^H \simeq \mathbf{Fin}_{(G/H)}^G$.

Proof. We first pick a set of coset representatives $\{g_i\}_i$ of G/H , where $g_0 = e$. Let $X \in \mathbf{Fin}^H$, we have the G -set $X \times G/H$, where $g_i h(x, g_j H) = (hx, g_i g_j H)$. Thus projection $X \times G/H \rightarrow G/H$ is G -equivariant. In the other direction, given $p : Y \rightarrow G/H$, $p^{-1}(eH)$ is an H -set.

□

Given $H \circlearrowleft X$ and $K \leq H$ then $K \circlearrowleft X$ and $X^H \subseteq X^K$. We call these both *restriction*. And gHg^{-1} acts on X by $ghg^{-1}(x) := h(x)$ which we call *conjugation*.

These are equivalent to the functors: $\mathbf{Fin}_{(G/H)}^G \rightarrow \mathbf{Fin}_{G/K}^G$ and $\mathbf{Fin}_{(G/H)}^G \rightarrow \mathbf{Fin}_{G/(gHg^{-1})}^G$ given by pullback along π, c .

Theorem 3.0.6. (*Elmendorf*) *We have a Quillen equivalence*

$$\text{Top}^G \simeq_Q \text{Fun}(\mathcal{O}_G^{\text{op}}, \text{Top})$$

Given by

$$X \longmapsto (G/H \mapsto X^H)$$

On the LHS weak equivalences are maps which induce weak homotopy equivalences on all fixed points, on the RHS weak equivalences are defined objectwise.

We note that there are many analogues of Elmendorf’s theorem across equivariant homotopy theory. This motivates our slogan:

“When doing equivariant homotopy theory, it is just as good to study fixed points as it is to study objects with a G -action.”

From the lemma we know that defining a (contravariant) functor from Fin^G which sends coproducts to (products) coproducts is equivalent to defining a (contravariant) functor on \mathcal{O}_G . As a result, throughout equivariant mathematics we often study *coefficient systems*, functors out of $\mathcal{O}_G^{\text{op}}$.

3.1 Commutative Monoids

Definition 3.1.1. A *monoid* is a set M with a *multiplication map* $\mu : M \times M \rightarrow M$ and an element $1_M \in M$ called the unit, such that

$$\mu(a, \mu(b, c)) = \mu(\mu(a, b), c), \mu(1_M, a) = a = \mu(a, 1_M)$$

and it is *commutative* if $\mu(a, b) = \mu(b, a)$.

We will generally assume monoids are commutative unless otherwise stated. This definition can be generalized from *Set* to a general symmetric monoidal category in Chapter 7.

We present three other equivalent ways to view commutative monoids.

The first is a Segalic construction. A *special Γ set* is a functor $F : \mathbf{Fin}_* \rightarrow \mathbf{Set}_*$ such that $F(*) = *$ and the natural map $F((n+m)_+) \rightarrow F(n_+) \times F(m_+)$ induced by the order-preserving maps sending $n+1, \dots, n+m$ to $*$ and $1, \dots, n$ to $*$ respectively, is an isomorphism. This we call this the *Segal condition*.

The Segal condition implies $F(n_+) \cong M^n$ and we view this as an identification. The intuition is that $F(1_+)$ is M , $F(* \rightarrow 1_+)$ picks out 1_M .

The fold map $F(n_+ \rightarrow 1_+)$ sending only $*$ to $*$ induces the monoidal product. The compatibility of these folds across different n and the action of Σ_n imply associativity and commutativity.

More generally for $f: n_+ \rightarrow m_+$, we view $F(f): M^n \rightarrow M^m$ as sending the n -tuple (a_1, \dots, a_n) to the m -tuple whose i^{th} entry is $\prod_{j \in f^{-1}(i)} a_j$, which means the product via μ over that set, and interpret as 1_M in the case of the empty set. The structure of \mathbf{Fin}_* then implies the operation is also unital. Numbers sent to $*$ are essentially discarded.

These can be in terms of spans.

We let $\mathit{Span}(\mathbf{Fin})$ be the span category of \mathbf{Fin} with objects the same as those of \mathbf{Fin} and morphisms A to B are isomorphism classes of spans $(A \leftarrow X \rightarrow B)$. Composition is by pullback which is well defined up to isomorphism. It is routine to check that the coproduct of sets gives the product of objects in $\mathit{Span}(\mathbf{Fin})$.

A *span-monoid*¹ is a product preserving functor $G : Span(\mathbf{Fin}) \rightarrow Set$. We let $M = G(\underline{1})$ and we identify $G(\underline{n})$ with M^n . $G(\emptyset = \emptyset \rightarrow \underline{1})$ gives a map $*$ $\rightarrow M$ whose image will be the unit. $G(\underline{2} = \underline{2} \rightarrow \underline{1})$ gives the multiplication. Via similar arguments as in the case of special Γ sets, the structure of $Span(\mathbf{Fin})$ is enough to show that multiplication in M is associative, unital, and commutative.

The other direction is much more intuitive as discussed in [Fre13]

Given a commutative monoid M , we let $G(\underline{n}) = M^n$.

For $f : \underline{n} \rightarrow \underline{r}$ we get a function $M^r \rightarrow M^n$, by $(a_1, \dots, a_r) \mapsto (b_i = a_{f(i)})_i$.

These contravariant maps we call *restrictions* and think of as rearranging and picking out the relevant entries of M^n .

For $g : \underline{r} \rightarrow \underline{m}$ we get a function $M^r \rightarrow M^m$ sending $(b_1, \dots, b_r) \mapsto (c_i = \sum_{k \in g^{-1}(i)} b_k)_i$, and 1_M in the case of the empty set. This covariant map we call a *transfer*, and this encodes the multiplication. This is well defined up to isomorphism of spans.

Given a span $h = (\underline{n} \xleftarrow{f} \underline{r} \xrightarrow{g} \underline{m})$ $G(h) : M^n \rightarrow M^m$ is defined as the composite of the restriction then transfer.

This is philosophically quite similar to special Γ -categories. Both assign sets to natural numbers and the Segal condition corresponds exactly to the span monoids being product-preserving.

Given a map $f : n_+ \rightarrow m_+$ in \mathbf{Fin}_* it creates the span in \mathbf{Fin} given by $\underline{n} \xleftarrow{f^{-1}} \{1, \dots, m\} \xrightarrow{f} \underline{m}$. In the case restriction is injective this gives an exact correspondence between maps of finite pointed sets and isomorphism classes of spans.

But in the case restriction is not injective we no longer have the exact correspondence.

¹We do not know of another term for this construction so use this ad-hoc one for clarity.

Intuitively this encodes multiple copies of an element appearing. For instance $(\underline{1} \leftarrow \underline{2} = \underline{2})$ gives $(a) \mapsto (a, a)$ and $(\underline{1} \leftarrow \underline{2} \rightarrow \underline{2})$ gives $(a) \mapsto (a + a)$.

In special Segal sets, we must use the diagonal map of sets to formulate this. For instance by $M \xrightarrow{\Delta} M \times M$ and $M \xrightarrow{\Delta} M \times M \xrightarrow{\mu} M$ respectively.

Lastly, we note that via the disjoint union of spans, $Span(\mathbf{Fin})$ is enriched in commutative monoids. We could just as well define span monoids as functors enriched in monoids $F: Span(\mathbf{Fin}) \rightarrow CMon$, and via a straightforward calculation the enrichment implies the product-preserving condition we had. Similarly this gives a monoidal structure on $F(\underline{1})$, which turns out to be the original one of $F(\underline{1}) \in CMon$ via an Eckman-Hilton argument.

An operadic description of monoids is as algebras over the operad $Comm$ enriched in Set , where $Comm(n) = *$.

3.2 Equivariant Monoids

We now consider suitable ways to define commutative monoids in an equivariant context. At the most basic level we can consider a G -object in commutative monoids, equivalently, a commutative monoid in the category of G -sets. These we call *naive* as they are the most basic.

However in many situations naive equivariance does not capture the full nature of the interplay between an algebraic structure and the group G .

The key intuition is in a genuine equivariant situation, G not only acts on an object, but also permutes the inputs of the monoidal operation simultaneously. For instance, loosely

speaking given an K -spectrum X , its Hill-Hopkins-Ravenel (HHR) norm is the H -spectrum $\bigwedge_{[k_i] \in H/K} X_i$ where $gh = h_i k$ acts by k acting on the separate copies of X , and h_i permuting them. Then $\bigwedge_{[h_i] \in G/J} X_i$ is converted back to a genuine spectrum [HHR21].

In a sense operations can be parametrized by maps of finite G -sets. We decompose these as the composition of a quotient or conjugation $G/K \rightarrow G/H$ to pass from one subgroup to another. And then a fold map - encoding the basic (nonequivariant) operation as we had used folds to encode operations in a monoid, and then quotients. These can also be parametrized by the G -corollas of [BP21].

Thus the naive vs. genuine distinction only appears in situations which are algebraic in some way with a form of commutativity.

We now explain this intuition as it pertains to spectra. Intuitively a (non-equivariant) connective spectrum is the homotopical version of an abelian group. Indeed they are equivalent to grouplike \mathbb{E}_∞ spaces. For each abelian group A we have the Eilenberg Mac Lane spectrum HA which we can view as the “discrete” spectrum on A . In the other direction, taking π_0 gives us abelian groups. The idea is that the delooping gives a direction in which we can “add”.

Borel spectra or naive G -spectra are just spectra with a G -action, we can think of these as the naive equivariant homotopical version of abelian groups. Indeed given an abelian group A with G -action, HA is a naive G -spectrum, and taking π_0 of a naive G -spectrum gives an abelian group A with G -action. By analogy this means that G can permute the “inputs” we add and that the commutativity respects this.

In a genuine G -spectrum, we have deloopings with respect to all finite G -representations, Given something monoidal with strict commutativity, naive implies genuine, as the action

of G by permuting inputs has no effect.

As an example, given an abelian group, ring, or strict symmetric monoidal category with a G -action, we can create their genuine versions, respectively a Mackey functor, Tambara functor, or genuine equivariant symmetric monoidal category. So we only are concerned with genuine versus naive when we have commutativity up to homotopy or isomorphism.

There are several different equivalent definitions for a Mackey functor and we will review a few of them.

Definition 3.2.1. A *Mackey functor* is a function $M : \{\text{subgroups of } G\} \rightarrow Ab$. With morphisms $I_K^H : M(K) \rightarrow M(H)$, $R_K^H : M(H) \rightarrow M(K)$, $c_g : M(H) \rightarrow M(gHg^{-1})$ for all $K \leq H \leq G$ and $g \in G$ satisfying the following identities:

- $I_H^H, R_H^H, c_H : M(H) \rightarrow M(H)$ are all identities for H and $h \in H$.
- $R_J^K R_K^H = R_J^H, I_K^H I_J^K = I_J^H$ for $J \leq K \leq H$.
- $c_g c_h = c_{gh}$
- $R_{gKg^{-1}}^{gHg^{-1}} c_g = c_g R_K^H, I_{gKg^{-1}}^{gHg^{-1}} c_g = I_K^H$
- $R_J^H I_K^H = \sum_{x \in JH/K} I_{J \cap xKx^{-1}c_x R_{x^{-1}Jx \cap K}^K}^J$ for $J, K \leq H$. This we refer to as the *double coset formula*.

Definition 3.2.2. A *Mackey functor* $\underline{M} = (M_*, M^*)$ consists of a covariant functor $M_* : \text{Fin}^G \rightarrow Ab$ and a contravariant functor $M^* : (\text{Fin}^G)^{op} \rightarrow Ab$ which agree on objects. We call $M^*(f)$ a *restriction* and $M_*(f)$ a *transfer*.

\underline{M} must satisfy the *double coset formula*: given a pullback square in Fin^G

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
p \downarrow & & \downarrow q \\
C & \xrightarrow{g} & D
\end{array}$$

$M_*(p)M^*(f) = M^*(g)M_*(q)$ as functions $\underline{M}(B) \rightarrow \underline{M}(C)$.

Lastly \underline{M} satisfies an additivity condition in that for $\iota_A : A \rightarrow A \amalg B$, $\iota_B : B \rightarrow A \amalg B$ the resulting map

$$M_*(\iota_A) \oplus M_*(\iota_B) : \underline{M}(A) \oplus \underline{M}(B) \rightarrow \underline{M}(A \amalg B)$$

is an isomorphism.

Lemma 3.2.3. *The following is a pullback diagram in \mathbf{Fin}^G ,*

$$\begin{array}{ccc}
A & \longrightarrow & G/K \\
\downarrow & & \downarrow \\
G/J & \longrightarrow & G/H
\end{array}$$

where $J, K \leq H$ and

$$A = \sqcup_{x \in J \backslash H / K} G / (J \cup xKx^{-1}).$$

This is the origin of the name *double coset formula*.

Definition 3.2.4. A *Mackey functor* is an additive functor $\mathbf{Span}(\mathbf{Fin}^G) \rightarrow \mathbf{Ab}$.

Definition 3.2.5. We define a *semi-Mackey functor* as a Mackey functor which takes values in commutative monoids instead of abelian groups. As our definitions never used inverses, nothing else needs to be modified here.

Equivalently, a semi-Mackey functor is a product preserving functor $\mathbf{Span}(\mathbf{Fin}^G) \rightarrow \mathbf{Set}$.

Example 3.2.6. Given an abelian group A with G -action, we have a fixed point Mackey functor $G/H \mapsto A^H$, restriction are inclusion of fixed points $A^H \hookrightarrow A^K$ for $K \leq H$, transfers are by summing $a \in A^K \mapsto \sum_{[h_i] \in H/K} h_i(a) \in A^H$.

Remark 3.2.7. In equiariant algebra and related fields one often talks about both *transfers* and *norms*. Intuitively these are the same, only that transfers refer to operations we view as additive and norms refer to ones we view as multiplicative. As a result when there is only one sort of operation the two are essentially synonymous.

Mackey functors form a closed symmetric monoidal abelian category, with symmetric monoidal product given by the *box product* (\square), defined by Day convolution. The monoidal unit is the *Burnside Mackey functor* A_G .

Furthermore, $Mack_G$, the category of G -Mackey functors forms an abelian category. This leads to our slogan: “Mackey functors are the equivariant version of abelian groups.”

Definition 3.2.8. A *Green functor* is a monoid in the symmetric monoidal category $(Mack_G, \square, A_G)$.

This has an equivalent definition in terms of concrete formulae similar to our first definition of Mackey functors.

In a Green functor \underline{M} , $\underline{M}(A)$ has the structure of a commutative ring. Restrictions are ring homomorphism and transfers are abelian group homomorphisms.

Intuition - Green functors are genuine equivariant with their additive structure, but only naive monoids with their commutative structure.

Definition 3.2.9. A *Tambara functor* is a Green functor with the additional structure of covariant *norm* maps which are maps of multiplicative monoids. They must satisfy several additional conditions similar to those in 3.2.1.

Tambara functors can also be defined as product preserving functors from the category of *bispans* in \mathbf{Fin}^G , just as we can define Mackey functors as product preserving functors from the category of *bispans* in \mathbf{Fin}^G .

Just as in a Mackey functor the left side of a span encodes restriction and the right side the transfer. A bispan has three legs $\bullet \leftarrow \bullet \rightarrow \bullet \rightarrow \bullet$. The left is restriction, the middle is the norm, the right is transfer. Composition of bispans is complicated and unenlightening so we omit it.

The intuition is that a Tambara functor is a monoid in the “genuine equivariant symmetric category” of Mackey functors. This has been formalized by [?].

Currently there is little work in the way of homological algebra

Example 3.2.10. Given a ring R with a G -action, the fixed point Mackey functor is also a Tambara functor with norms given by multiplying: $A \mapsto \prod_{[k_i] \in K/H} k_i(a)$

Example 3.2.11. Given a G -ring spectrum, its $\underline{\pi}_0$ Mackey functor is a Tambara functor. [Bru06]

The intuition here is that $\underline{\pi}_0$ is a genuine monoidal functor so it sends monoids to monoids.

Remark 3.2.12. A common question is why we restrict to G a finite group. This is only the case in algebraic settings.

First, transfers often arise from summing over the set of cosets in a group, which roughly speaking requires a group we can integrate over like a (pro)finite group or compact Lie group. Indeed genuine equivariant spectra work well in these contexts.

Second, However for genuine equivariant spectra to give us Mackey functor via $\underline{\pi}_0$ we need transfers arising from $X^K \rightarrow X^H$ for $K \leq H$ via

$$X^H = [\Sigma_G^\infty(G/H_+), X] \cong [\mathbb{S}_G, \Sigma_G^\infty(G/H_+) \wedge X] \rightarrow [\mathbb{S}_G, \Sigma_G^\infty(G/K_+) \wedge X] = [\Sigma_G^\infty(G/K_+), X] = X^K$$

This comes from the fact that $\Sigma_G^\infty(G/H_+)$ is self dual, which does not hold for G compact Lie.

Finally, following the intuition that in a genuine equivariant monoid, G permutes the inputs of an operation, this is only meaningful if G acts on finite sets.

Chapter 4

1-Categorical Preliminaries

Remark 4.0.1. Let Cat be the category of categories and functors. This has the potential to lead to size issues, but we assume that we can pass to a larger universe as needed.

Definition 4.0.2. A category \mathcal{C} is a *skeleton category* if it has no isomorphisms between different objects. Equivalently it has only one object of each isomorphism class.

Remark 4.0.3. Skeleton categories are often more convenient to work with and in many sections of this paper we will replace categories with equivalent skeleton categories.

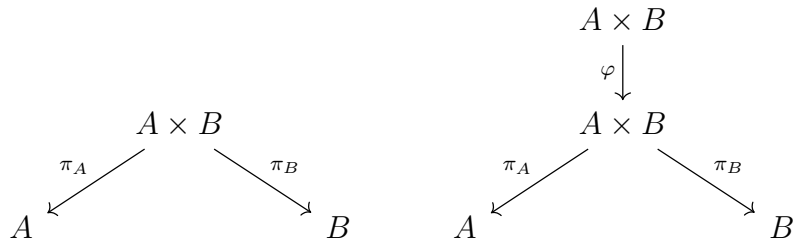
Using the axiom of global choice¹, every category is equivalent to a skeletal subcategory, by choosing one object from each isomorphism class.

However this does not generally make (co)limits strictly well-defined as opposed to defined up to unique isomorphism as (co)cones are part of the data of a (co)limit and objects in skeleton categories can still have non-trivial automorphism groups.

Example 4.0.4. Let \mathcal{C} be a skeleton category with object A, B . Further suppose that $(A \times$

¹This is essentially the axiom of choice for classes.

$(A \times B, \pi_A, \pi_B)$ is the product of A and B in \mathcal{C} , and that $\varphi : A \times B \xrightarrow{\cong} A \times B$ is a non-trivial isomorphism. Then $(A \times B, \pi_A \varphi, \pi_B \varphi)$ is also a product of A and B in \mathcal{C} .



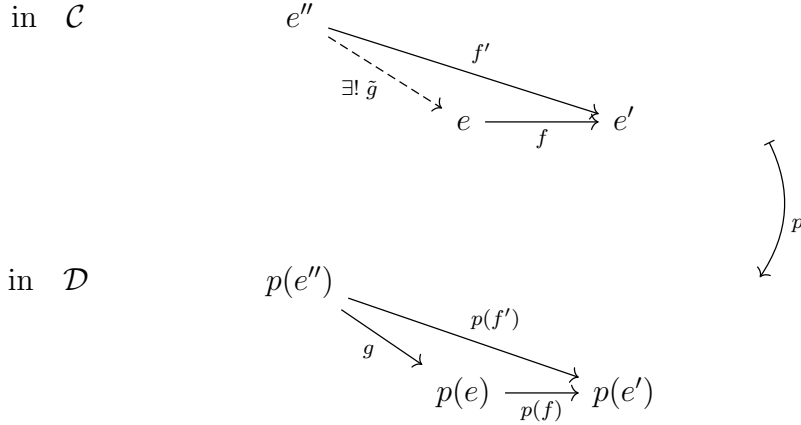
We recall another result which will prove useful.

Lemma 4.0.5. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be an equivalence of categories, given $G : \mathcal{D} \rightarrow \mathcal{C}$ and a natural isomorphism $\eta : Id_{\mathcal{C}} \Rightarrow GF$ there is a unique natural isomorphism $\epsilon : FG \Rightarrow Id_{\mathcal{D}}$ so that $F \dashv G, \eta, \epsilon$ is an adjoint equivalence.*

4.1 Cartesian Morphisms and Fibrations

All the results in this section are standard in the literature so do not prove them here. NLab and [FK18] provide some of the best explanations of these.

Definition 4.1.1. Let $p : E \rightarrow B$ be a functor, we say $f : e \rightarrow e'$ in E is *cartesian* if for all $f' : e'' \rightarrow e'$ in \mathcal{C} and $g : p(e'') \rightarrow p(e)$ in \mathcal{D} such that $p(f') = p(f)g$, there exists a unique lift \tilde{g} of g so that $f' = f\tilde{g}$.



We note that *cartesian* is not an absolute term but is only meaningful relative to p .

Remark 4.1.2. Everything in this section has a dual version simply by applying the definitions to $p^{op} : \mathcal{C}^{op} \rightarrow \mathcal{D}^{op}$.

We note some direct implications of the definition.

Lemma 4.1.3. *Let $f : e_1 \rightarrow e'$, $f' : e_2 \rightarrow e'$ two cartesian morphisms over $b \rightarrow p(e')$. Then $p(e_1) = p(e_2)$ and there is a unique isomorphism $\varphi : e_1 \rightarrow e_2$ in E that p sends to $Id_{p(a)}$ and such that $f = f'\varphi$.*

Lemma 4.1.4. *Let $h : b \rightarrow b'$, $h' : b; \rightarrow b''$ be morphisms in B and $\tilde{h}, \tilde{h}', \tilde{h}'h$ be cartesian lifts. Then there is a unique isomorphism φ between the sources of \tilde{h} and $\tilde{h}'h$ so that $\tilde{h}'\tilde{h} = \varphi\tilde{h}'h$.*

Definition 4.1.5. A functor $p : E \rightarrow B$ is a *fibration* if for all $e \in E, b \in B$ and $h : b \rightarrow p(e)$ there is a lift \tilde{h} in E which is cartesian.

We point out that this is not required to be unique.

Some author call this a *categorical fibration*, *fibered category*, *cartesian fibration*, or *Grothendieck fibration*.

Definition 4.1.6. Given a fibration p , a *cleavage* is a choice of cartesian lift of every morphism in B with target in the image of E .

By the axiom of global choice we can always find one.

Definition 4.1.7. A cleavage is *normal* if $h = Id$ implies $\tilde{h} = Id$.

It is *split* if it preserves composition in the sense that $(\tilde{h}k) = \tilde{h}\tilde{k}$.

Lemma 4.1.8. Let $p : E \rightarrow B$ be a fibration with a cleavage. It defines a pseudofunctor $B^{op} \rightarrow Cat$ which sends $b \in B$ to the fiber $E_b \subset E$ consisting of objects sent to b and morphisms sent to Id_b . On morphisms $h : b \rightarrow b'$ we define the functor

$$h^* : E_{b'} \rightarrow E_b$$

which sends an object e' to the source of the chosen cartesian lift of $h : b \rightarrow p(e')$, which we denote $h^*(b')$.

In general this only defines a pseudofunctor, but in the case that the cleavage is split and normal, this is a functor.

Definition 4.1.9. Given a functor (or more generally pseudofunctor) $F : B^{op} \rightarrow Cat$, we construct an associated split fibration $\int F \rightarrow B$ called the *Grothendieck construction on F* . The objects of $\int F$ are pairs $b \in B, a \in F(b)$. Its morphisms are pairs $f : c \rightarrow c', \alpha : F(f)(a) \rightarrow a'$. The functor π is simply the forgetful functor.

This is functorial in the sense that a natural transformation is sent to a map of split fibrations.

Remark 4.1.10. (f, α) is a cartesian lift of f if α is an isomorphism in $F(b')$. $\int F \xrightarrow{\pi} B$ is a split fibration where the chosen cartesian lifts are those with $\alpha = Id_{a'}$.

Proposition 4.1.11. The process of taking fibers and the Grothendieck construction are inverse weak 2-functors which make an equivalence of weak 2-categories between pseudofunctors $B^{op} \rightarrow Cat$ and fibrations over B .

We briefly note that *pseudofunctors* are a type of weak functor between 2-categories which we discuss more in Chapter 5.

Definition 4.1.12. For two fibrations (resp. fibrations with cleavages) $p : E \rightarrow B, p' : E' \rightarrow B'$, a *morphism of fibrations* is a pair of functors $F : E \rightarrow E', G : B \rightarrow B'$ so that $Gp = p'F$ and F sends cartesian morphism (resp. also takes chosen cartesian lifts of p to chosen lifts of p').

Definition 4.1.13. Given two morphisms of fibrations (F, G) and (F', G') , a *natural transformation of fibrations* consists of natural transformations $\eta : F \Rightarrow F', \epsilon : G \Rightarrow G'$ so that η lies above ϵ . In the case $G = G' = Id_B$ and $\epsilon = Id$ as well, this mean the components of η all lie above identity morphisms in B .

Chapter 5

2-Categories

In this chapter we briefly review key concepts from the theory of 2-categories, sometimes omitting technicalities present in some definitions.

Remark 5.0.1. Throughout this paper we almost exclusively deal with strict 2-categories and strict 2-functors as opposed to weak 2-categories or pseudofunctors. Unless otherwise specified, we use *2-category* and *2-functor* to refer to the strict version.

Much of the content of this paper could be done just as well with bicategories (weak 2-categories) and pseudofunctors and in some cases this would be more natural. For instance the main theme of Chapter 9 is finding conditions when pullbacks can be taken in a strictly coherent way so as to create strict 2-categories.

We prefer to work with strict 2-categories for two main reasons. First, much of the existing literature we build on, such as [BIO15], uses strict 2-categories. Second, strict 2-categories are much simpler to deal with, and in the context of this paper we rarely lose generality or any of the key ideas.

For the same two reasons, we prefer to work with permutative categories as opposed to

symmetric monoidal ones when possible. (Though we generally have no choice but to use strong monoidal functors as opposed to strict monoidal ones).

In the opposite direction, one can work with ∞ -categories¹, which in many ways is even more natural. Indeed some work has been done in this direction in the papers on Spectral Mackey Functors and Parametrized Higher Algebra [Bar17, BGS20, BDG⁺16, NS22]. Throughout we mention connections to this work.

Definition 5.0.2. A (*strict*) *2-category* is simply a category enriched in categories. This has objects, a hom-category $Hom(A, B)$ for every pair of objects, and identity object in $Hom(A, A)$, and a composition functor $Hom(B, C) \times Hom(A, B) \rightarrow Hom(A, C)$. These satisfy the usual associativity and unitality conditions.

We may refer to the objects as *0-cells*, the objects in the hom-categories as *morphisms*, *1-morphisms*, or *1-cells*, and the morphisms in the hom-categories as *2-morphisms* or *2-cells*.

In this section, let \mathcal{C} be a strict 2-category unless otherwise specified.

Definition 5.0.3. Given a diagram of objects, morphisms, and 2-morphisms in \mathcal{C} , we say it *commutes* if all comparable pasting diagrams that can be formed strictly agree.

Example 5.0.4. Cat , the 2-category of categories, functors, and natural transformations is the prototypical 2-category.

Most of the 2-categories discussed in this paper are a variant on Cat , specifically a 2-category whose objects are a certain type of category, morphisms are a type of functor, and 2-cells are a type of natural transformation.

¹The namesake of Infty mentioned in the acknowledgements

Example 5.0.5. We observe that every category can be viewed as a 2-category with discrete hom-categories.

Definition 5.0.6. A (*strict*) 2-functor between 2-categories is simply a functor enriched in categories.

It sends objects to objects, and has a functor between the suitable hom-categories, satisfying strict unitality and compositionality identities.

There are numerous weakenings of the definition of a 2-category, but one which is most relevant here.

Definition 5.0.7. A *bicategory* is a type of weak 2-category. A bicategory has objects, a hom-category for every pair of objects, with an identity object in $Hom(A, A)$ and a composition bifunctor. These satisfy the typical unitality and associativity but only up to natural isomorphisms as opposed to the equality we have in a strict 2-category. The associator and unitor transformations must also satisfy coherence diagrams.

Definition 5.0.8. A *pseudofunctor* between 2-categories or bicategories $F : \mathcal{C} \rightarrow \mathcal{D}$ is a weaker version of a 2-functor, in which unitality and composition are only preserved up to natural isomorphism. We note that we can define this with \mathcal{C} a 1-category as well via Example 11.1.17.

Formally we have an assignment $F : Obj(\mathcal{C}) \rightarrow Obj(\mathcal{D})$, a functor $F_{x,y} : Hom_{\mathcal{C}}(x, y) \rightarrow Hom_{\mathcal{D}}(F(x), F(y))$, an isomorphism $\eta_x : Id_{F(x)} \Rightarrow F_{x,x}(Id_x)$ in $Hom_{\mathcal{D}}(F(x), F(x))$, for $f \in Hom_{\mathcal{C}}(x, y), g \in Hom_{\mathcal{C}}(y, z)$ an isomorphism $\mu_{f,g} : F_{y,z}(f) \circ F_{x,y}(f) \cong F_{x,z}(g \circ f)$, which is natural in f, g . We require these satisfy associativity and unitality diagrams, which also incorporate the associativity and unitality isomorphisms making \mathcal{D} a bicategory.

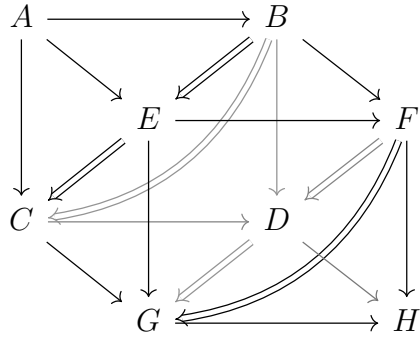
This is the natural type of morphism between bicategories, among which we cannot in general define strict 2-functors.

Definition 5.0.9. We say $F : \mathcal{C} \rightarrow \mathcal{D}$ is an *equivalence of 2-categories* if it is essentially surjective on objects (i.e. surjective up to equivalence in \mathcal{D}) and induces equivalences on the hom-categories.

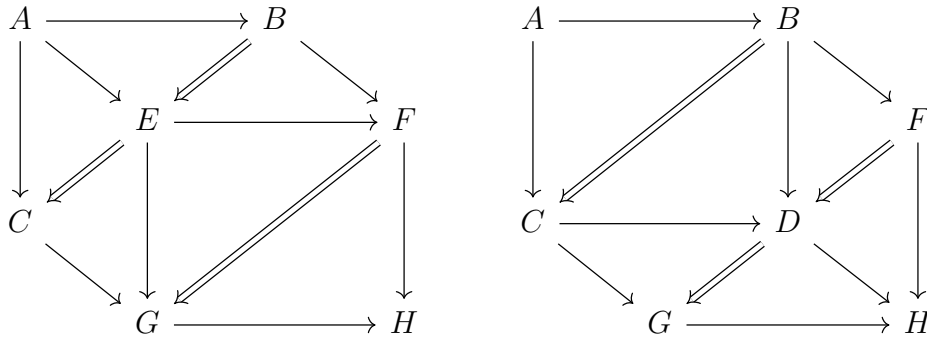
Equivalently it has a weak inverse pseudofunctor $G : \mathcal{D} \rightarrow \mathcal{C}$ and pseudonatural transformation equivalences: $GF \simeq Id_{\mathcal{C}}, FG \simeq Id_{\mathcal{D}}$.

We note that F can be an equivalence even if its weak inverse is only a pseudofunctor.

Example 5.0.10. A *commuting cube* in \mathcal{C} is a diagram of the form



where the following pasting diagrams agree:



We will often display diagrams in 2-categories as cubes in this way.

Remark 5.0.11. We note that the same basic rules of diagram-chasing in 1-categories apply to diagrams in 2-categories, replacing commuting squares with commuting cubes.

Example 5.0.12. If two commuting cubes share a face in common, they can be glued along a face and then the composite cube also commutes.

Chapter 6

∞ -categories

We only briefly discuss ∞ -categories in this paper and never in technical terms so we only give an intuitive definition.

Definition 6.0.1. An ∞ -category, or $(\infty, 1)$ -category is a weak higher category with n -morphisms for all $0 \leq n < \infty$. All n -morphisms are invertible (up to higher morphisms) for $n > 1$.

Composition is only well-defined up to higher morphisms; associativity and unitality are only satisfied up to higher morphisms.

Remark 6.0.2. In an ∞ -category essentially nothing is defined in a strictly unique way, rather things are unique up to a contractible space of choices.

We can also think of an ∞ -category as being like a category enriched in spaces.

Quasi-categories, a type of simplicial set satisfying a lifting property, are the most common model of ∞ -categories. This theory is fleshed out in [Lur09].

Definition 6.0.3. The *nerve* $N : Cat \rightarrow Cat_\infty$ is a functor taking in a 1-category and

outputting an ∞ -category.

Definition 6.0.4. The *Duskin nerve* $N^D : \text{Cat}_{(2,1)} \rightarrow \text{Cat}_\infty$ is a functor taking in a (2,1)-category and outputting an ∞ -category.

We will generally identify a 1- of (2, 1)-category with its nerve.

Remark 6.0.5. Most 1-categorical notions (e.g. (co)limits, cartesian morphisms, fibrations, etc.) have ∞ -categorical analogues.

Generally speaking if an object (resp. morphism, category, functor) has a 1-categorical property, its nerve satisfies the ∞ -categorical analogue.

Chapter 7

Symmetric Monoidal Categories, Multicategories, and Enrichment

Remark 7.0.1. This chapter reviews definitions and results found in the literature. Most of this is found in [EM06], the section on *Perm* as a PC-category is best explained in [BIO15], and the most comprehensive reference is [JY22].

7.1 Symmetric Monoidal and Permutative Categories

Definition 7.1.1. A *monoidal category* is a category \mathcal{C} with a *monoidal product* functor

$$\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$$

which is unital and associative up to coherent isomorphisms. It has an object $1_{\mathcal{C}}$, and *unitors* natural isomorphisms

$$\eta_r : a \otimes 1_{\mathcal{C}} \cong a \cong 1_{\mathcal{C}} \otimes a : \eta_l,$$

and an *associator* natural isomorphism

$$\alpha : (a \otimes b) \otimes c \cong a \otimes (b \otimes c).$$

Finally the unitality and associativity isomorphisms must be suitably coherent in the sense that every diagram made of associators and unitors commutes as in [Mac63].

Definition 7.1.2. A *symmetric monoidal category* is a monoidal category which also has a symmetry isomorphism

$$\tau : a \otimes b \cong b \otimes a$$

which is coherent with unitality and associativity.

We will almost exclusively deal with symmetric monoidal categories in this paper as opposed to monoidal ones.

Definition 7.1.3. A *permutative category* is a symmetric monoidal category in which the unitality and associativity isomorphisms are identities.

Definition 7.1.4. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is *lax monoidal* if there is a natural *distributivity* transformation

$$\delta : F(-) \otimes_{\mathcal{D}} F(-) \Rightarrow F(- \otimes_{\mathcal{C}} -)$$

and a morphism

$$\eta_F : 1_{\mathcal{D}} \rightarrow F(1_{\mathcal{C}})$$

which are compatible in that δ and η_F satisfy coherence conditions related to the unitors and associators of \mathcal{C} and \mathcal{D} .

If \mathcal{C}, \mathcal{D} are symmetric monoidal, we say F is *symmetric monoidal* if the following diagram commutes.

$$\begin{array}{ccc} F(x) \otimes F(y) & \xrightarrow{\delta} & F(x \otimes y) \\ \tau \downarrow & & \downarrow F(\tau) \\ F(y) \otimes F(x) & \xrightarrow{\delta} & F(y \otimes x) \end{array}$$

F is *strong monoidal* if δ, η_F are isomorphisms and *strict monoidal* if they are equalities. We say F is *strictly unital* if η_F is an equality and if on of the inputs is $1_{\mathcal{C}}$ then δ is as well.

We will primarily work with strictly unital strong symmetric monoidal functors in this paper between permutative categories. In this case the compatibility conditions of monoidal functors simplify to the following diagram commuting

$$\begin{array}{ccc} F(x) \otimes F(y) \otimes F(z) & \xrightarrow{Id_{F(x)} \otimes \delta} & F(x) \otimes F(y \otimes z) \\ \delta \otimes Id_{F(z)} \downarrow & & \downarrow \delta \\ F(x \otimes y) \otimes F(z) & \xrightarrow{\delta} & F(x \otimes y \otimes z) \end{array}$$

In nature, symmetric monoidal categories are common but it is comparatively uncommon to find permutative categories. Fortunately we can remedy this.

Theorem 7.1.5 (Isbell. [Isb69, JS93]). *Given a symmetric monoidal category \mathcal{C} there is a permutative category \mathcal{C}' with a strong monoidal functor $\mathcal{C} \rightarrow \mathcal{C}'$ which is an equivalence of categories.*

Essentially this tells us that we can replace symmetric monoidal categories with permutative ones for free. However the analagous result does not work with regards to permutative categories which are strictly symmetric.

Definition 7.1.6. Given monoidal functors $F, G : \mathcal{C} \rightarrow \mathcal{D}$, $\theta : F \Rightarrow G$ is a *monoidal natural transformation* if for $x, y \in \mathcal{C}$ the diagrams commute:

$$\begin{array}{ccc}
 F(x) \otimes_{\mathcal{D}} F(y) & \xrightarrow{\theta_x \otimes_{\mathcal{D}} \theta_y} & G(x) \otimes_{\mathcal{D}} G(y) \\
 \delta_F \downarrow & & \downarrow \delta_G \\
 F(x \otimes_{\mathcal{C}} y) & \xrightarrow{\theta_{x \otimes_{\mathcal{C}} y}} & G(x \otimes_{\mathcal{C}} y)
 \end{array}
 \qquad
 \begin{array}{ccc}
 & 1_{\mathcal{D}} & \\
 \eta_F \swarrow & & \searrow \eta_G \\
 F(1_{\mathcal{C}}) & \xrightarrow{\theta_{1_{\mathcal{C}}}} & G(1_{\mathcal{C}})
 \end{array}$$

There are several other ways to characterize symmetric monoidal and permutative categories which we briefly discuss.

- An *unbiased* version replaces the \otimes bifunctor with functors $\otimes^n : \mathcal{C}^n \rightarrow \mathcal{C}$ for each $n \geq 0$, which essentially multiply n objects. $n = 0$ is the inclusion of the unit object $I_{\mathcal{C}} : * \rightarrow \mathcal{C}$. Symmetry is realized by natural isomorphisms $\otimes^n \sigma \cong \otimes^n$ for a permutation isomorphism $\sigma : \mathcal{C}^n \rightarrow \mathcal{C}^n$. Associativity by natural isomorphisms relating the \otimes^n for multiple n at once.
- This has an equivalent *operadic description*. The *categorical Barratt-Eccles operad*, enriched in categories, is defined by $\mathcal{E}_n = E\Sigma_n$ which we recall has an object for each element of Σ_n and a unique isomorphism between each pair of objects.. Algebras over it are permutative categories and pseudoalgebras over it are symmetric monoidal categories. Symmetry is encoded by the operad action $\mathcal{E}_n \times \mathcal{C}^n \rightarrow \mathcal{C}$ and associativity by the compatibility of this action across multiple n .

(We recall that a *pseudoalgebra* is one in which the algebra coherence equalities are replaced by natural isomorphisms, as defined in [CG13].)

- A *special Γ category* is a functor $F : \mathbf{Fin}_* \rightarrow \mathbf{Cat}_*$ such that $F(*) \simeq *$ and satisfying the *Segal condition* that

$$F(\underline{n+m}) \rightarrow F(\underline{n}) \times F(\underline{m})$$

is an equivalence of pointed categories.

These correspond to permutative categories.

- Another version uses *spans* and considers (weakly) product-preserving functors

$$Span(\mathbf{Fin}) \rightarrow \mathbf{Cat}$$

Where $Span(\mathbf{Fin})$ has the same objects as \mathbf{Fin} and morphisms are isomorphism classes of spans. Its products are given by coproducts in \mathbf{Fin} . We give a complete description in 9.2.1.

- Finally we have the monadic version of [BP21], which heavily influenced this paper. We consider the monad $F \wr (-)$ in \mathbf{Cat} which sends \mathcal{C} to the Grothendieck construction of the functor $\mathbf{Fin}^{op} \rightarrow \mathbf{Cat}$ given by $I \mapsto \mathcal{C}^I$. This is a monad with $F \wr F \wr \mathcal{C} \rightarrow F \wr \mathcal{C}$ defined by

$$\left(((c_{ij})_{i \in I_j}, I_j)_{j \in J}, J \right) \mapsto \left((c_{ij})_{ij}, \coprod_{j \in J} I_j \right)$$

If \mathcal{C} is a symmetric monoidal category it gives an algebra over $F \wr (-)$ via $\mathcal{C}^I \rightarrow \mathcal{C}$ by multiplication. And in the other direction given an algebra over $F \wr (-)$, we can define a multiplicative structure on it in this way.

Many analogues of symmetric monoidal categories are based on these versions such as the symmetric monoidal ∞ -categories of [Lur17]. We see that most the various versions of

equivariant symmetric monoidal categories take one of these approaches and adds a G in, for instance replaces finite sets with finite G -sets.

Definition 7.1.7. A *monoid* in a symmetric monoidal category \mathcal{C} is an object M with a *multiplication* map $\mu : M \otimes M \rightarrow M$ and a *unit* map $\eta : 1_{\mathcal{C}} \rightarrow M$ so that the associativity and unitality diagrams commute:

$$\begin{array}{ccc}
 M \otimes M \otimes M & \xrightarrow{\mu \times Id} & M \otimes M \\
 Id \times \mu \downarrow & & \downarrow \mu \\
 M \otimes M & \xrightarrow{\mu} & M
 \end{array}
 \qquad
 \begin{array}{ccccc}
 1_{\mathcal{C}} \otimes M & \xrightarrow{\eta \times Id} & M \otimes M & \xleftarrow{Id \times \eta} & M \otimes 1_{\mathcal{C}} \\
 & \searrow \eta \ell & \downarrow \mu & \swarrow \eta r & \\
 & & M & &
 \end{array}$$

M is *commutative* if this diagram commutes as well:

$$\begin{array}{ccc}
 M \otimes M & \xrightarrow{\tau} & M \otimes M \\
 & \searrow \mu & \swarrow \mu \\
 & & M
 \end{array}$$

In a loose sense, symmetric monoidal categories can be thought of as weak monoids in the symmetric monoidal category of categories.

Proposition 7.1.8. Let $\mathcal{C}, \otimes, 1_{\mathcal{C}}$ be a symmetric monoidal category. The category of commutative monoids in \mathcal{C} is equivalent to the category of strong symmetric monoidal functors $(\mathbf{Fin}, \amalg, \emptyset) \rightarrow (\mathcal{C}, \otimes, 1_{\mathcal{C}})$.

Definition 7.1.9. A monoidal category \mathcal{C} is *closed monoidal* if for all objects $B \in \mathcal{C}$, the functor $(-) \otimes B$ has a right adjoint, which we denote by $\underline{Hom}(B, -)$ and that this is natural in B .

The intuition is that $\underline{Hom}(A, B)$ is an object in \mathcal{C} of morphisms $A \rightarrow B$. This is further motivated by the bijection (in a locally small category)

$$\text{Hom}_{\mathcal{C}}(1_{\mathcal{C}}, \underline{\text{Hom}}_{\mathcal{C}}(A, B)) \cong \text{Hom}_{\mathcal{C}}(A, B).$$

We also have an *evaluation* morphism

$$ev : A \otimes \underline{\text{Hom}}_{\mathcal{C}}(A, B) \rightarrow B.$$

7.2 Enrichment in Symmetric Monoidal Categories

Definition 7.2.1. Let \mathcal{V} be a symmetric monoidal category. A *category \mathcal{C} enriched over \mathcal{V}* has objects X , and for every $X, Y \in \mathcal{C}$, a hom-object $\mathcal{C}(X, Y) \in \mathcal{V}$ with units $Id_X : 1_{\mathcal{V}} \rightarrow \mathcal{C}(X, X)$ and composition maps

$$comp : \mathcal{C}(Y, Z) \otimes \mathcal{C}(X, Y) \rightarrow \mathcal{C}(X, Z).$$

Such that the diagrams commute in \mathcal{V} :

$$\begin{array}{ccc} \mathcal{C}(Y, Z) \otimes \mathcal{C}(X, Y) \otimes \mathcal{C}(W, X) & \xrightarrow{Id \otimes comp} & \mathcal{C}(Y, Z) \otimes \mathcal{C}(W, Y) \\ \downarrow comp \otimes Id & & \downarrow comp \\ \mathcal{C}(X, Z) \otimes \mathcal{C}(W, X) & \xrightarrow{comp} & \mathcal{C}(W, Z) \end{array}$$

$$\begin{array}{ccc} 1_{\mathcal{C}} \otimes \mathcal{C}(Y, X) & \xrightarrow{Id_X \otimes Id} & \mathcal{C}(X, X) \otimes \mathcal{C}(Y, X) \\ & \searrow \eta_l & \downarrow comp \\ & & \mathcal{C}(Y, X) \end{array} \qquad \begin{array}{ccc} \mathcal{C}(X, Y) \otimes 1_{\mathcal{C}} & \xrightarrow{Id \otimes Id_X} & \mathcal{C}(X, Y) \otimes \mathcal{C}(X, X) \\ & \searrow \eta_r & \downarrow comp \\ & & \mathcal{C}(X, Y) \end{array}$$

Notation 7.2.2. The convention when discussing enriched categories is to use \mathcal{V} to denote the enriching category. We call these \mathcal{V} -categories. Some authors switch the composition ordering convention and instead write $\mathcal{C}(X, Y) \otimes \mathcal{C}(Y, Z) \rightarrow \mathcal{C}(X, Z)$.

For $\mathcal{V} = Set$, this is exactly the definition of a locally small category.

Definition 7.2.3. Given a category \mathcal{C} enriched in \mathcal{V} , its *opposite category* \mathcal{C}^{op} is a category enriched in \mathcal{V} with the same objects as \mathcal{C} . And $\mathcal{C}^{op}(X, Y) := \mathcal{C}(Y, X)$.

$comp : \mathcal{C}^{op}(Y, Z) \otimes \mathcal{C}^{op}(X, Y) \rightarrow \mathcal{C}^{op}(X, Z)$ is defined as the composition:

$$\mathcal{C}^{op}(Y, Z) \otimes \mathcal{C}^{op}(X, Y) = \mathcal{C}(Z, Y) \otimes \mathcal{C}(Y, X) \xrightarrow{\tau} \mathcal{C}(Y, X) \otimes \mathcal{C}(Z, Y) \xrightarrow{comp_{\mathcal{C}}} \mathcal{C}(Z, X) = \mathcal{C}^{op}(X, Z).$$

Remark 7.2.4. We note the importance of the symmetry isomorphism τ in this.

Definition 7.2.5. A \mathcal{V} -enriched functor $F : \mathcal{C} \rightarrow \mathcal{D}$ sends object of \mathcal{C} to those of \mathcal{D} , and for $X, Y \in \mathcal{C}$ has a morphism in \mathcal{V} , $F_{X,Y} : \mathcal{C}(X, Y) \rightarrow \mathcal{D}(F(X), F(Y))$ so that

the diagrams in \mathcal{V} commute:

$$\begin{array}{ccc} \mathcal{C}(Y, Z) \otimes \mathcal{C}(X, Y) & \xrightarrow{comp_{\mathcal{C}}} & \mathcal{C}(X, Z) \\ \downarrow F_{Y,Z} \otimes F_{X,Y} & & \downarrow F_{X,Z} \\ D(F(Y), F(Z)) \otimes D(F(X), F(Y)) & \xrightarrow{comp_{\mathcal{D}}} & D(F(X), F(Z)) \end{array} \quad \begin{array}{ccc} 1_{\mathcal{V}} & & \\ \downarrow Id_X & \searrow Id_{F(X)} & \\ \mathcal{C}(X, X) & \xrightarrow{F_{X,X}} & D(F(X), F(X)) \end{array}$$

Remark 7.2.6. Given a \mathcal{V} -functor $F : \mathcal{C} \rightarrow \mathcal{D}$ it defines a \mathcal{V} -functor $F^{op} : \mathcal{C}^{op} \rightarrow \mathcal{D}^{op}$.

Definition 7.2.7. For enriched functors $F, G : \mathcal{C} \rightarrow \mathcal{D}$ a \mathcal{V} -enriched natural transformation $\alpha : F \Rightarrow G$ consists of morphisms $\alpha_X : 1_{\mathcal{V}} \rightarrow \mathcal{D}(F(X), G(X))$ for each $X \in \mathcal{C}$ so that the diagram commutes:

$$\begin{array}{ccc}
& \mathcal{C}(X, Y) & \\
\eta_\ell \swarrow & & \searrow \eta_r \\
1_{\mathcal{V}} \otimes \mathcal{C}(X, Y) & & \mathcal{C}(X, Y) \otimes 1_{\mathcal{V}} \\
\alpha_Y \otimes G_{X, Y} \downarrow & & \downarrow F_{X, Y} \otimes \alpha_X \\
\mathcal{E}(F(Y), G(Y)) \otimes \mathcal{E}(G(X), G(Y)) & & \mathcal{E}(F(X), F(Y)) \otimes \mathcal{E}(F(X), G(X)) \\
& \searrow \text{comp} & \swarrow \text{comp} \\
& \mathcal{E}(F(X), G(Y)) &
\end{array}$$

Definition 7.2.8. In this way \mathcal{V} -enriched categories form a strict 2-category which we denote by $\mathcal{V}\text{-Cat}$. We might also use the 1-category attained by ignoring the natural transformations.

7.3 Multicategories

Definition 7.3.1. A *multicategory* \mathcal{M} has objects $\{x\}$ and n -ary *multimorphisms*, a set (or proper class)

$$Hom_{\mathcal{M}}(x_1, \dots, x_n; y) \text{ for objects } x_1, \dots, x_n, y$$

an identity morphism $Id_x \in Hom_{\mathcal{M}}(x; x)$, and a composition map

$$\begin{aligned}
& Hom_{\mathcal{M}}(y_1, \dots, y_n; z) \times Hom_{\mathcal{M}}(x_{1,1}, \dots, x_{1,k_1}; y_1) \times \dots \times Hom_{\mathcal{M}}(x_{n,1}, \dots, x_{n,k_n}; y_n) \\
& \longrightarrow Hom_{\mathcal{M}}(x_{1,1}, \dots, x_{n,k_n}; z)
\end{aligned}$$

satisfying suitable associativity and unitality conditions [EM06]. We note that we allow *nullary* morphisms where $n = 0$. This is the same as a colored non-symmetric operad.

A *symmetric multicategory* \mathcal{M} is one with isomorphisms $\sigma : \text{Hom}_{\mathcal{M}}(x_1, \dots, x_n; y) \rightarrow \text{Hom}_{\mathcal{M}}(x_{\sigma(1)}, \dots, x_{\sigma(n)}; y)$ for each permutation $\sigma \in \Sigma_n$, that is suitably compatible with composition and unitality.

This is the same as a colored symmetric operad.

Definition 7.3.2. We say a multicategory \mathcal{M} is *unital* if it contains an object $1_{\mathcal{M}}$ and *unitor* binary morphisms $\eta_l \in \text{Hom}_{\mathcal{M}}(1_{\mathcal{M}}, x; x), \eta_r \in \text{Hom}_{\mathcal{M}}(x, 1_{\mathcal{M}}; x)$ which are natural in x such that they induce bijections natural in y : $\text{Hom}_{\mathcal{M}}(x, 1_{\mathcal{M}}; y) \cong \text{Hom}_{\mathcal{M}}(x; y) \cong \text{Hom}_{\mathcal{M}}(1_{\mathcal{M}}, x; y)$. We require the unitors to be compatible with associativity and symmetry if \mathcal{M} is symmetric; satisfying conditions analogous to those of symmetric monoidal categories.

Definition 7.3.3. A *multifunctor* $F : \mathcal{M} \rightarrow \mathcal{N}$ sends objects of \mathcal{M} to those of \mathcal{N} and functions

$$\text{Hom}_{\mathcal{M}}(x_1, \dots, x_n; y) \rightarrow \text{Hom}_{\mathcal{N}}(F(x_1), \dots, F(x_n); F(y))$$

subject to the obvious unitality and composition-preserving conditions. This is the same as a morphism of (non-symmetric) operads.

A *symmetric multifunctor* is a multifunctor between symmetric multicategories is one which is also Σ_n -equivariant on hom-sets. This is the same as a morphism of (symmetric) operads.

If \mathcal{M}, \mathcal{N} are unital, we require that F is strictly unital, so that $F(1_{\mathcal{M}}) = 1_{\mathcal{N}}$, and it preserves the unitors.

Example 7.3.4. Given a symmetric monoidal category \mathcal{V} we can form a multicategory $\hat{\mathcal{V}}$ with the same objects and

$$Hom_{\hat{\mathcal{V}}}(x_1, \dots, x_n; y) := Hom_{\mathcal{V}}(x_1 \otimes \dots \otimes x_n, y)$$

We use the conventions that the empty monoidal product is $1_{\mathcal{V}}$ so $Hom_{\hat{\mathcal{V}}}((); y) := Hom_{\mathcal{V}}(1_{\mathcal{V}}, y)$ and $x_1 \otimes \dots \otimes x_n = (x_1 \otimes \dots \otimes x_{n-1}) \otimes x_n$.

Remark 7.3.5. Given a permutative category \mathcal{V} , we can form a symmetric unital multicategories enriched in \mathcal{V} by having $\mathcal{M}(x_1, \dots, x_n; y)$ be an object of \mathcal{V} instead of a set or proper class. Composition is defined similarly as to enriched categories and analogous unitality and associativity identities hold.

Definition 7.3.6. A category \mathcal{C} enriched in a multicategory \mathcal{M} has objects X and an object $\mathcal{C}(X, Y) \in \mathcal{M}$ for every pair of objects in \mathcal{C} . We have a *composition multimorphism* in $Hom_{\mathcal{M}}(\mathcal{C}(Y, Z), \mathcal{C}(X, Y); \mathcal{C}(X, Z))$ and a unit in $Hom_{\mathcal{M}}(; \mathcal{C}(X, X))$ satisfying associativity in the sense that the two composite 3-morphisms agree in the left diagram and unital in that the three unary morphisms agree in the right diagram.

$$\begin{array}{ccc}
\mathcal{C}(C, D), \mathcal{C}(B, C), \mathcal{C}(A, B) & \xrightarrow{comp, Id} & \mathcal{C}(C, D), \mathcal{C}(A, C) \\
\downarrow Id, comp & & \downarrow comp \\
\mathcal{C}(B, D), \mathcal{C}(A, B) & \xrightarrow{comp} & \mathcal{C}(A, D)
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{C}(A, B) & \xrightarrow{Id_B, Id_{\mathcal{C}(A, B)}} & \mathcal{C}(B, B), \mathcal{C}(A, B) \\
\downarrow Id_{\mathcal{C}(A, B)}, Id_A & \searrow Id_{\mathcal{C}(A, B)} & \downarrow comp \\
\mathcal{C}(A, B), \mathcal{C}(A, A) & \xrightarrow{comp} & \mathcal{C}(A, B)
\end{array}$$

Most of constructions with categories enriched over a (symmetric) monoidal category can be generalized to categories enriched in (symmetric) multicategories.

We can define \mathcal{M} -enriched functors and natural transformations analogously to the case of enrichment in a symmetric monoidal category. This forms the category of \mathcal{M} -categories and \mathcal{M} -functor which we denote $\mathcal{M} - Cat$. Similarly we can define the opposite category of an \mathcal{M} -category in a functorial way, all in the case that \mathcal{M} is symmetric.

For a permutative category \mathcal{V} , a category enriched over \mathcal{V} is the same as one enriched over the multicategory $\hat{\mathcal{V}}$.

Given a multifunctor $F : \mathcal{M} \rightarrow \mathcal{N}$ we have a *change of enrichment 2-functor*

$$F_{\bullet} : \mathcal{M} - \text{Cat} \longrightarrow \mathcal{N} - \text{Cat}$$

For a \mathcal{M} -category \mathcal{C} , $F_{\bullet}(\mathcal{C})$ has the same objects as \mathcal{C} and $F_{\bullet}(\mathcal{C})(X, Y) := F(\mathcal{C}(X, Y))$.

Definition 7.3.7. We say a multicategory \mathcal{M} is *closed* if for object x_1, \dots, x_n, y there is an object $\underline{Hom}_{\mathcal{M}}(x_1, \dots, x_n; y) \in \mathcal{M}$ and an evaluation multimorphism

$$ev \in Hom_{\mathcal{M}}(\underline{Hom}_{\mathcal{M}}(x_1, \dots, x_n; y), x_1, \dots, x_n; y)$$

satisfying a bijection universal property similar to that of the adjunction seen in 7.1.9 (closed monoidal cat). [BLM12].

In this way we can view \mathcal{M} as enriched over itself.

We note that $\hat{\mathcal{C}}$ is closed if and only if \mathcal{C} is closed.

Proposition 7.3.8. *Let \mathcal{M}, \mathcal{N} be closed unital multicategories and $F : \mathcal{M} \rightarrow \mathcal{N}$ a unital multifunctor. We have an \mathcal{N} -functor*

$$\Phi_F : F_{\bullet}(\mathcal{M}) \rightarrow \mathcal{N},$$

on objects $\Phi_F(X) := F(X)$, and on hom-objects

$$\Phi_{F(X, Y)} : F(\underline{Hom}_{\mathcal{M}}(X, Y)) \rightarrow \underline{Hom}_{\mathcal{N}}(F(X), F(Y))$$

is defined as the adjoint to

$$F(ev_{\mathcal{M}}) : F(\underline{Hom}_{\mathcal{M}}(X, Y)), F(X) \rightarrow F(Y)$$

This is generalization of the Φ of [BIO15] § 6.

Definition 7.3.9. *Perm* is defined as the multicategory of permutative categories and strong monoidal strictly unital multilinear functors.

Its objects are permutative categories.

$$Hom_{Perm}(\mathcal{M}_1, \dots, \mathcal{M}_n; \mathcal{N})$$

is the set (or class) of *multilinear functors*:

$$F : \mathcal{M}_1 \times \dots \times \mathcal{M}_n \longrightarrow \mathcal{N}$$

which are strong monoidal strictly unital in each variable meaning that for each $1 \leq i \leq n$, there is a distributivity structure natural isomorphism:

$$\delta_i : F(x_1, \dots, x_i, \dots, x_n) \otimes F(x_1, \dots, x'_i, \dots, x_n) \rightarrow F(x_1, \dots, x_i \otimes x'_i, \dots, x_n)$$

and strictly unital in that $F(x_1, \dots, 1_{\mathcal{M}}, \dots, x_n) = 1_{\mathcal{N}}$ and $\delta_i = Id$ if any of the x_k or x'_i is $1_{\mathcal{M}}$.

We require analogues of the diagrams in the definition of permutative functors to commute and a third pentagonal diagram relating δ_i, δ_j as in [EM06] pg. 11. Nullary morphisms $() \rightarrow \mathcal{M}$ pick out objects of \mathcal{M} .

When composing we get distributivity morphisms as follows. Given $(g, \{\delta_i^g\}) : (B_1, \dots, B_n) \rightarrow \mathcal{C}$ and $(f_j, \{\delta_i^{f_j}\}) : (A_{j,1}, \dots, A_{j,k_j}) \rightarrow B_j$ their composite is

$$g \circ (f_1, \dots, f_j) := (g \circ (f_1 \times \dots \times f_j), \{\delta_s = \sum g(\delta_i^{f_j}) \circ \delta_j^g\})$$

as described in [EM06] 3.2.

Remark 7.3.10. We are forced to use multicategories when working with permutative categories in this way, as there is no tensor product of permutative categories satisfying the appropriate bilinearity universal property.

Remark 7.3.11. We can instead consider the multicategory of permutative functors and lax monoidal strictly unital multilinear functors. This is the approach taken in [BIO15] and [EM06]. For the most part these behave exactly the same; we will only need the δ s to be isomorphisms in the proof of 14.2.6, but everything in this chapter works identically. We could also consider strict monoidal multifunctors, where the δ are identities, but this is too restrictive for most purposes.

Proposition 7.3.12 ([BIO15]). *Perm is a closed unital multicategory.*

7.4 PC-Categories

Definition 7.4.1. A *PC-category* is a category enriched over *Perm*. A *PC-functor* between PC-categories is a functor enriched in PC-categories. A *PC-natural transformation* is an enriched natural transformation.

Remark 7.4.2. By forgetting the monoidal structure, a PC-category has the underlying structure of a strict 2-category, and a PC-functor has an underlying 2-functor.

Definition 7.4.3. *Perm* is a PC-category with objects permutative categories and in $Perm(\mathcal{C}, \mathcal{D})$ the monoidal product is given by applying the product in \mathcal{D} objectwise:

$$(f + g)(x) := f(x) + g(x)$$

the distributivity morphism of $f \otimes g$ is given by

$$f(x) + g(x) + f(y) + g(y) \xrightarrow{\tau} f(x) + f(y) + g(x) + g(y) \xrightarrow{\delta_f \otimes \delta_g} f(x \otimes y) \otimes g(x \otimes y).$$

We have a composition bilinear bifunctor

$$comp : Perm(B, C) \times Perm(A, B) \rightarrow Perm(A, C)$$

where δ_1 is the identity as $(g \circ f) + (g' \circ f) = (g + g') \circ f$ by definition, and δ_2 is given by $\delta^g : (g \circ f) + (g \circ f') \rightarrow g \circ (f + f')$.

Remark 7.4.4. One can also consider categories enriched in symmetric monoidal categories as opposed to permutative categories, and in many cases these are more natural. However that they have an underlying bicategory 5.0.7 as opposed to an underlying strict 2-category.

Fortunately by Guillou's strictification result [Gui10], all such symmetric-monoidally-enriched categories are suitably equivalent to PC-categories.

Essentially this tells us that by restricting to PC categories we do not lose any important examples or information.

7.5 Spectra and K -theory

In this paper we briefly discuss spectra and G -spectra but never do in-depth calculations with them, so we do not get too deep into their construction or choose a model to work with.

We emphasize that we are generally interested in spectra and G -spectra up to weak homotopy equivalences. We chose to work with the homotopy category as the different constructions on K -theory are only equivalent up to homotopy.

- Sp denotes the (homotopy) category of *spectra*. $Sp_{\geq 0}$ denotes the subcategory of *connective spectra*, those with trivial negative homotopy groups.
- Sp is a closed symmetric monoidal category with the *smash product* \wedge as its monoidal product, the *sphere spectrum* \mathbb{S} as its unit, and $Sp(-, -)$ the *mapping spectrum*.
- Sp_G denotes the (homotopy) category of *genuine G -spectra*, those with desuspensions for all finite G -representations. This is in contrast to $(Sp)^G$, the category of *naive* or *Borel G -spectra*, which are simply G -objects in the category of spectra. $Sp_{G \geq 0}$ denotes the subcategory of *connective G -spectra*, those with trivial negative homotopy Mackey functors.
- Sp_G is similarly a closed symmetric monoidal category with the *smash product* \wedge as its monoidal product, the *equivariant sphere spectrum* \mathbb{S}_G as its unit, and $Sp_G(-, -)$ the *mapping G -spectrum*.

Definition 7.5.1. Given a symmetric or permutative category \mathcal{C} we can define its *algebraic K -theory spectrum* $K(\mathcal{C}) \in Sp_{\geq 0}$.

Intuitively this is a homotopical version of group completing \mathcal{C} with its monoidal structure. For instance $\pi_0(K(\mathcal{C}))$ is the group completion of the monoid of isomorphism classes in \mathcal{C} (for \mathcal{C} a small category).

There are two main constructions of this, which agree up to homotopy. The first uses special Γ -categories, the second uses an operadic construction.

Theorem 7.5.2 (Elmendorf Mandell [EM06]). *K defines a symmetric multifunctor*

$$K : Perm \longrightarrow \widehat{Sp}.$$

Theorem 7.5.3 (Thomason). *Every connective spectrum is equivalent to the K -theory of a permutative category.*

This can be upgraded to show that the homotopy category of permutative categories is equivalent to that of connective spectra.

Theorem 7.5.4. [Man10, Elm21]

There is a multifunctor

$$K^{-1} : (\widehat{Sp}_{\geq 0}) \rightarrow Perm$$

which is a weak inverse (i.e. inverse up to natural equivalence) to K . However it is not strictly symmetric.

Theorem 7.5.5 (Guillou May). [GM11] *There is a zig-zap of Quillen equivalences:*

$$Fun_{Sp}((K_{\bullet}(\mathbf{GE}'))^{op}, Sp) \simeq_Q Sp_G$$

where \mathbf{GE}' is a version of the PC-enriched Burnside category we discuss in 10. We note that in this case we are using the full categories of $(G-)$ spectra, not just their homotopy categories.

Remark 7.5.6. A current area of research is attempting to use multiplicative inverse K -theory and the Guillou-May theorem to prove a version of Theorem 7.5.4 using permutative Mackey functors. Currently the fact that inverse K -theory is not known to be symmetric is the main obstruction as this is needed when taking enriched opposite categories.

Chapter 8

The Category Fin^G

In this chapter we prove several technical results on skeleton category of ordered finite G -sets which will be used in future chapters.

Definition 8.0.1. Fin^G is a skeleton category of ordered finite G -sets and all equivariant maps. For concreteness, we can view this as the category with objects (\underline{n}, α) where $\underline{n} \in \text{Fin}$ and $\alpha : G \rightarrow \Sigma_n$ a group homomorphism.

Remark 8.0.2. We point out that these categories have unordered maps and that there is no compatibility required between the ordering and the G -action. In a sense though restricting to order preserving morphisms does not limit the morphisms we can have. For instance given an arbitrary equivariant map $A \rightarrow B$, by choosing suitable orderings, we can make this be order preserving as well.

The ordering is not mathematically necessary at a moral level, but it is quite usual for bookkeeping and ensuring objects are equal as opposed to isomorphic, which makes many steps simpler. In fact we use ordering for some nice formal properties it gives Fin^G , namely a permutative disjoint union, a permutative product, and a choice of pullbacks satisfying several

key properties.

Notation 8.0.3. We used \rightarrow to denote unordered maps, and \succrightarrow to denote order preserving maps.

Remark 8.0.4. We observe the useful fact that the only order-preserving isomorphisms in \mathbf{Fin}^G or \mathbf{Fin} are identities.

We can also view $\mathbf{Fin} \subseteq \mathbf{Fin}^G$ as the full subcategory of objects with trivial G -action. In this way we let $\underline{n} \in \mathbf{Fin}^G$ denote the (unique ordered) G -set with n elements and trivial G -action. We can also view \mathbf{Fin} as \mathbf{Fin}^G for $G = \{e\}$ the trivial group.

In fact \mathbf{Fin}^G is just the category of G -objects in \mathbf{Fin} .

Given a G -set A , we order A/G based on the least elements in an orbit.

We specify choices of disjoint unions and products in a way that makes \mathbf{Fin}^G permutative with respect to both.

- $A \amalg B$ is the disjoint union where $\iota_A : A \rightarrow A \amalg B$, $\iota_B : B \rightarrow A \amalg B$ are both order preserving and for $a, b \in A \amalg B$, $a < b$. Equivalently in $A \amalg B$, $a < a'$ if and only if $a < a' \in A$; similarly $b < b' \in A \amalg B$ if and only if $b < b' \in A$; and $a < b$.

This makes \mathbf{Fin}^G permutative with \emptyset as the monoidal unit. However it is not strictly symmetric although as objects $A \amalg B = B \amalg A$, the coproduct morphisms are not the same.

For a finite ordered set I or G -set, $\amalg_I A_i$ is the disjoint union with components ordered by I .

- $A \times B$ is the *cartesian product with the lexicographical ordering* or *lex-product*; the unique ordering where $\pi_A : A \times B \rightarrow A$ is order preserving and $\pi_B : A \times B \rightarrow B$ is order

preserving on fibers over $a \in A$. Equivalently it is the ordering where $(a, b) < (a', b')$ if $a < a'$ or if $a = a'$ and $b < b'$. We can characterize it as the ordering on the product where the entry in A take precedence. We let $A^{\amalg n}$ denote $\underline{n} \times A$.

Similarly this makes \mathbf{Fin}^G a permutative but not strictly symmetric category with $(\underline{1}, 1)$ as the monoidal unit. We also note that the product strictly distributes over the disjoint union on the right: $(A \amalg B) \times C = (A \times C) \amalg (B \times C)$. However it is not distributive on the left: $A \times (B \amalg C) \neq (A \times B) \amalg (A \times C)$.

For a finite ordered set I or G -set, $\coprod_I A_i$ is the disjoint union with components ordered by I .

- The *product with colexicographical ordering* or *colex-product* of A and B is simply the categorical product where the role of A and B is reversed.
- We similarly have a preferred choice of pullbacks. Given a cospan $(A \rightarrow X \leftarrow B)$, the *lexicographical pullback* or *lex-pullback* $A \times_X B$ is the subset of $A \times B$ satisfying the pullback universal property. In diagrams we denote the lex-pullback as:

$$\begin{array}{ccc} A \times_X B & \longrightarrow & B \\ q^*f \downarrow & & \downarrow f \\ A & \xrightarrow{q} & X \end{array}$$

In particular we use the convention that the vertical leg is the order-preserving one.

We might also use q^*B to denote $A \times_X B$. We also observe that q^*f is order preserving by definition

We note that the ordering is very important here as $(A \rightarrow X \leftarrow B)$ is not equal to the lex-pullback of $(B \rightarrow X \leftarrow A)$.

Finally, the right distributivity of lex-products over coproducts implies a similar distributivity property for lex-pullbacks and coproducts.

- Our convention in this paper is that *pullback* refers to any categorical pullback satisfying the universal property; we will sometimes refer to them as *categorical pullback* when we want to emphasize the distinction. *Lex-pullback* refers to this specific model, and *chosen pullback* for a choice of pullbacks discussed in depth in Chapter 9.

We use the word *cartesian* for pullback diagrams and for cartesian arrows in the context of a Grothendieck fibration.

- Given a map $f : A \rightarrow B$, we define $A_{\times B}$ to be the lex-pullback of $B = B \xleftarrow{f} A$. As a G -set $A_{\times B}$ is isomorphic to A , but is reordered so that $A_{\times} \rightarrow B$ and the maps on fibers $(A_{\times B})_b \rightarrow A_b$ are order-preserving. Equivalently we can also view $A_{\times B}$ as having the same underlying G -set as A , but ordered lexicographically first by their image in B , then by their original ordering. The intuition is that $A_{\times B}$ is A twisted by the ordering on B .
- We lastly note that as \mathbf{Fin}^G is a skeleton category, $A \amalg B$ and $A \times B$ might not be equal to the obvious point-set interpretation of the disjoint union or product. But this does not pose any issues and the same happens when discussing ordered pullbacks. We can address this using the approach of [GMMO23].

Using our concrete model of \mathbf{Fin}^G , as a set $(\underline{j}, \alpha) \amalg (\underline{k}, \beta) := (\underline{j+k}, \alpha \amalg \beta)$, where $(\alpha \amalg \beta)(g)$ permutes the first j elements of $\underline{j+k}$ as $\alpha(g)$ permutes \underline{j} and $(\alpha \amalg \beta)(g)$ permutes the final k elements of $\underline{j+k}$ as $\beta(g)$ permutes \underline{k} . $\iota_j : (\underline{j}, \alpha) \rightarrow (\underline{j+k}, \alpha \amalg \beta)$ is a bijection on the first j elements of $\underline{j+k}$ and ι_k is a bijection on the final k elements.

As a set $(\underline{j}, \alpha) \times (\underline{k}, \beta) := (\underline{jk}, \alpha \times \beta)$. For $x = nk + r \in \underline{jk}$ where $0 \leq n < j, 0 < r \leq k$ and $g \in G$, $(\alpha \times \beta)(g)(x) = (\alpha(g)(n + 1) - 1)k + \beta(g)(r)$. The ± 1 's appear because σ_j permutes 1 to j whereas n ranges from 0 to $j - 1$. $\pi_j : \underline{jk} \rightarrow \underline{j}$ sends $x \mapsto n + 1$. $\pi_k : \underline{jk} \rightarrow \underline{k}$ sends $x \mapsto r$.

There is no general formula for a pullback $\underline{j} \times_{\underline{n}} \underline{k} \subseteq \underline{j} \times \underline{k}$ of this form, so we must view it as some (\underline{m}, γ) with a suitable inclusion map into $\underline{j} \times \underline{k}$.

8.1 Properties of Lex-Pullbacks

We prove a few basic lemmas about lexicographical pullbacks which will prove useful later. For the most part they tell us that some key facts about categorical pullbacks also apply to lex-pullbacks.

Lemma 8.1.1. *Consider the following diagram in \mathbf{Fin}^G so that both square are lex-pullback squares. Then the full rectangle is a lex-pullback square.*

$$\begin{array}{ccccc} A & \xrightarrow{h} & B & \xrightarrow{k} & C \\ p \downarrow & & q \downarrow & & \downarrow \\ D & \xrightarrow{t} & E & \longrightarrow & F \end{array}$$

We call this result *horizontal pasting* of lex-pullbacks.

Proof. We know that the full rectangle is a categorical pullback by a standard result in category theory so we must show the ordering is correct. As the left square is a lex-pullback square, p is order preserving.

Consider $a < a' \in f^{-1}(d)$; it suffices to show that $kh(a) < kh(a')$. As they are in the same fiber over d , $h(a) < h(a')$; these two are both in the fiber over $tp(a) = tp(a')$, so $kh(a) < kh(a')$

□

Example 8.1.2. In contrast, the analogous statement of vertical pasting is false. For instance in the following diagram it is easy to see that both the top and lower squares are lex-pullbacks, but the rectangle as a whole is not.

$$\begin{array}{ccc}
 \underline{2} & \xrightarrow{(12)} & \underline{2} \\
 \parallel & & \downarrow (12) \\
 \underline{2} & \xlongequal{\quad} & \underline{2} \\
 \downarrow & & \downarrow \\
 \underline{1} & \xlongequal{\quad} & \underline{1}
 \end{array}$$

However vertical pasting does hold in a key circumstance.

Lemma 8.1.3. *Consider the following diagram in \mathbf{Fin}^G so that both squares are lex-pullbacks and q is order preserving. Then the full rectangle is a lex-pullback square.*

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 p \downarrow & & q \downarrow \\
 C & \xrightarrow{g} & D \\
 r \downarrow & & \downarrow \\
 E & \longrightarrow & F
 \end{array}$$

Proof. We know that the full rectangle is a categorical pullback by a standard result in category theory so we must show the ordering is correct.

Let $a < a' \in A$, so that $e = rp(a) = rp(a')$. In the case that $p(a) = p(a')$, then as they are in the same fiber over C and the top square is a lex-pullback, $f(a) < f(a')$. Otherwise, $p(a) < p(a')$; as these are in the same fiber over E and the bottom square is a lex-pullback, $qf(a) = gp(q) < gp(a') = qf(a')$. As q is order preserving, this implies $f(a) < f(a')$. □

Lemma 8.1.4. *Consider the following diagram in Fin^G so that the right square and full rectangle are lex-pullback squares. Then the left square is a lex-pullback square.*

$$\begin{array}{ccccc}
 A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
 p \downarrow & & q \downarrow & & \downarrow \\
 D & \xrightarrow{r} & E & \longrightarrow & F
 \end{array}$$

We call this result *horizontal reverse-pasting* of lex-pullbacks.

Proof. We know that the full rectangle is a categorical pullback by a standard result in category theory so we must show the ordering is correct.

As the full rectangle is a lex-pullback, p is order preserving. Let $a < a' \in A$, both in $p^{-1}(d)$. Suppose for contradiction that $f(a) > f(a')$. Then $qf(a) = rp(a) = rp(a') = qf(a')$; then as the right square is a lex-pullback, $gf(a) > gf(a')$, which contradicts the full rectangle being a lex-pullback.

□

Example 8.1.5. The vertical analogue does not work in general as shown in this example:

$$\begin{array}{ccc}
 \underline{2} & \equiv & \underline{2} \\
 (12) \downarrow & & \downarrow (12) \\
 \underline{2} & \equiv & \underline{2} \\
 \downarrow & & \downarrow \\
 \underline{1} & \equiv & \underline{1}
 \end{array}$$

However, it does hold in a special case.

Lemma 8.1.6. *Consider the following commutative diagram in Fin^G , where the lower square and the full rectangle are lex-pullbacks, and the map q is order preserving. Then the upper square is a lex-pullback.*

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
p \downarrow & & \downarrow q \\
C & \xrightarrow{g} & D \\
r \downarrow & & \downarrow t \\
E & \xlongequal{\quad} & E
\end{array}$$

We call this result *vertical reverse-pasting*.

These conditions initially appear ad-hoc or contrived, but in fact they apply in some important situations we will encounter later

Proof. First we note that g thus f are isomorphisms as they are pullbacks of isomorphisms.

We must first show that f is order preserving when restricted to fibers over $c \in C$. This follows from it being order preserving on fibers over $e \in E$.

Second we must show p is order preserving. Suppose for contradiction there are $a < a' \in A$ so that $p(a) > p(a')$. As rp is order preserving by assumption, $rp(a) \leq rp(a')$. As r is order preserving, the only way for this to happen is if a, a' are in the same fiber over $e = rp(a) = rp(a')$. We know that f is order preserving on fibers over e and an isomorphism so $f(a) < f(a')$. Then $gp(a) = qf(a) < qf(a') = gp(a')$. As g is order preserving on fibers, $p(a) < p(a')$, a contradiction. \square

Lemma 8.1.7. *If f is an isomorphism, then its lex-pullback q^*f is an identity.*

$$\begin{array}{ccc}
A & \xrightarrow{r} & B \\
Id \downarrow & & \downarrow f \\
C & \xrightarrow{q} & D
\end{array}$$

Proof. It is clear that this is a categorical pullback, that Id is order preserving, and that r is order preserving on fibers as those are all singletons. \square

Corollary 8.1.8. *The lex-pullback of an identity map is an identity; $q^*Id = Id$*

We call this result *one-sided unitality* of lex-pullbacks.

Example 8.1.9. In contrast, even in a weaker form, this does not work in the reverse orientation.

Let f be a non-identity isomorphism, then the leg opposite the identity is not an identity.

$$\begin{array}{ccc} A & \xrightarrow{f^{-1}} & B \\ \parallel & & \downarrow f \\ A & \xlongequal{\quad} & A \end{array}$$

Remark 8.1.10. The failure of lex-pullbacks to satisfy the pasting and unitality in both orientations proves to be a major limitation to developing an elegant theory of categories of spans.

In fact by Theorem 9.3.1 we can not get around these issue by choosing pullbacks in some very clever way. This is discussed in much greater length in Chapter 9.

Chapter 9

Chosen Pullbacks

9.1 Chosen Pullback Structures

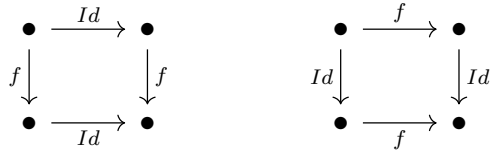
Definition 9.1.1. Let \mathcal{C} be a category with all categorical pullbacks. The structure of a class of *chosen pullbacks* is a choice of a single (categorical) pullback for each cospan.

Example 9.1.2. So far this is a very weak condition. By the axiom of global choice, any category \mathcal{C} with pullbacks of all cospans has a class of chosen pullbacks.

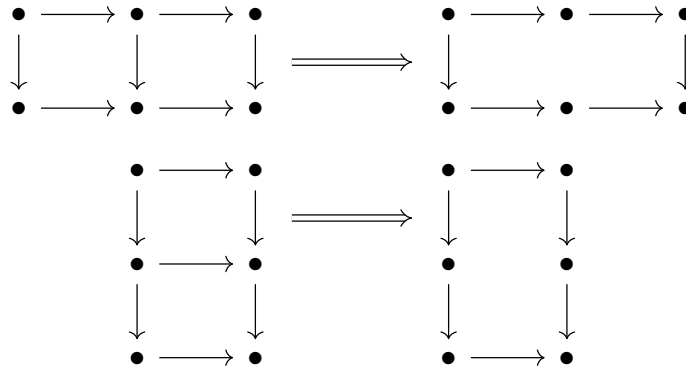
Remark 9.1.3. We note that the choice need not be symmetric, in general the chosen pullback of $(A \rightarrow X \leftarrow B)$ is not equal to the chosen pullback of $(B \rightarrow X \leftarrow A)$.

There are three key properties a class of chosen pullbacks can have that interest us:

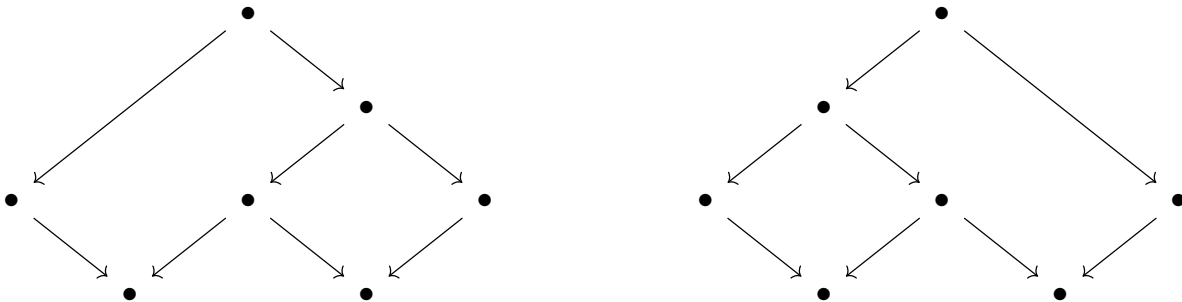
Definition 9.1.4. A class of chosen pullbacks is *unital* if the chosen pullback of an identity morphism is an identity morphism in either orientation



Definition 9.1.5. A class of chosen pullbacks satisfies the *pasting* property if given two chosen pullback squares, the combined rectangle is also a chosen pullback square



Definition 9.1.6. A class of chosen pullbacks is *associative* if given two composable cospans, taking the chosen pullback of the left then the right gives the same span as the right than the left. I.e. the outside spans of the following diagrams agree:



Example 9.1.7. For $\mathcal{C} = \text{Fin}$ or Fin^G , the lexicographical pullback is associative and satisfies one of the unital squares and one part of pasting, but not the other. The colexicographical pullback satisfies the other unital square and other part of pasting

Remark 9.1.8. These three definitions here are symmetric so any statement about horizontal pullbacks also applies to vertical ones.

Definition 9.1.9. A class of chosen pullbacks on \mathcal{C} is *draconian* if it is unital, associative, and satisfies the pasting property¹.

As we will see, these are not logically independent.

Example 9.1.10. If \mathcal{C} is a group there is a draconian class of chosen pullbacks. We define the chosen pullback as follows:

$$\begin{array}{ccc} \bullet & \xrightarrow{gfg^{-1}} & \bullet \\ g \downarrow & & \downarrow g \\ \bullet & \xrightarrow{f} & \bullet \end{array}$$

Example 9.1.11. On a single category \mathcal{C} there can be multiple draconian chosen pullback structures. For instance given a good chosen pullback structure, we can take the reverse of it, where the new chosen pullback of $(A \rightarrow X \leftarrow B)$ is the old chosen pullback of $(B \rightarrow X \leftarrow A)$.

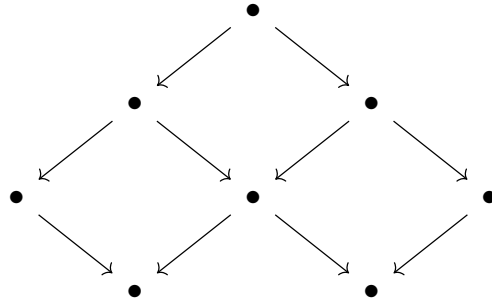
In the example of groups we have an alternative choice given by

$$\begin{array}{ccc} \bullet & \xrightarrow{f} & \bullet \\ fgf^{-1} \downarrow & & \downarrow g \\ \bullet & \xrightarrow{f} & \bullet \end{array}$$

Lemma 9.1.12. *If a chosen pullback structure satisfies the pasting property it is also associative.*

Proof. We consider the two composable cospans at the bottom of the diagram. We first take the pullbacks of those two cospans, then the pullback of the new cospan they generate. By pasting, the left and top squares glue to a chosen pullback so the large span is equal to that of left one in the associativity definition. Similarly, by pasting, the right and top squares glue to a chosen pullback so the large span is equal to that of right one in the associativity definition.

¹We would like to thank Kate Ponto for advice on creative terminology

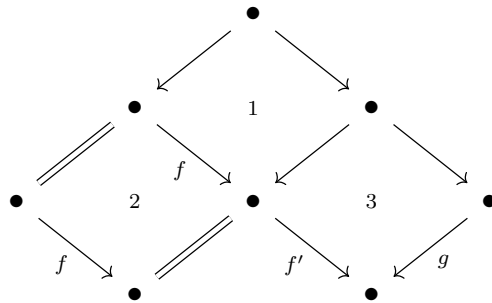


□

Lemma 9.1.13. *If a chosen pullback structure satisfies unitality and associativity that it satisfies the pasting property.*

Proof. Consider the following diagram where we assume that squares 1 and 3 are chosen pullbacks; we aim to show that the rectangle combining 1 and 3 is also a chosen pullback. By unitality, square 2 is a chosen pullback. Then the rectangle formed by 1 and 2 is equal to square 1 so it is also a chosen pullback. So then the outside span is equal to the span around the rectangle of 1 and 3. By associativity, the outside span is equal to the span of the chosen pullback of the cospan around the rectangle of 1 and 3. This shows that pasting is satisfied.

Showing pasting in the other direction is identical.



□

Lemma 9.1.14. *If a chosen pullback structure satisfies unitality and associativity then it is draconian .*

9.2 Span Categories

Definition 9.2.1. Let \mathcal{C} be any category with all pullbacks. The *classical category of spans in \mathcal{C}* , also called the *Lindner category of \mathcal{C}* , $Span(\mathcal{C})$, is defined as the category with the same objects as \mathcal{C} and morphisms are isomorphism classes of spans between objects in \mathcal{C} . Where two spans are isomorphic if there is an isomorphism as follows:

$$\begin{array}{ccc}
 & X & \\
 & \swarrow & \searrow \\
 A & \longleftarrow X' & \longrightarrow B \\
 & \downarrow \cong & \\
 & X &
 \end{array}$$

Composition of two spans $(A \leftarrow X \rightarrow B)$ and $(B \leftarrow Y \rightarrow C)$ is formed by outer span in the below diagram where Z is a pullback of $(X \rightarrow B \leftarrow C)$ in the below diagram.

$$\begin{array}{ccccc}
 & & Z & & \\
 & & \swarrow & \searrow & \\
 & X & & & Y \\
 & \swarrow & & \swarrow & \searrow \\
 A & & B & & C
 \end{array}$$

As we are only interested in isomorphism classes of spans composition is well defined despite the fact that pullbacks are not strictly unique. The isomorphism class represented by span $(A \leftarrow A \rightarrow A)$ is the identity with respect to composition.

Remark 9.2.2. $Span(\mathcal{C})$ is equal to its dual.

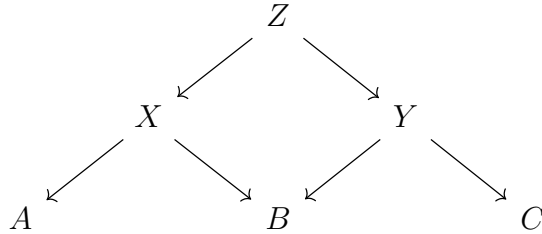
Remark 9.2.3. The categorical product in $Span(\mathcal{C})$ of $X, Y \in \mathcal{C}$ is given by the coproduct $X \amalg Y$ in \mathcal{C} if it exists.

Remark 9.2.4. If \mathcal{C} also has all finite coproducts, then $Span(\mathcal{C})$ is enriched in monoids:

$$(A \leftarrow X \rightarrow B) + (A \leftarrow X' \rightarrow B) := (A \leftarrow X \amalg X' \rightarrow B)$$

Definition 9.2.5. For a category with finite coproducts, the homset-wise group completion of $\text{Span}(\mathcal{C})$ is the *Burnside category* which is much better known. (Though some authors [Bar17, BGS20] use slightly different conventions).

Definition 9.2.6. Let \mathcal{C} be a category with a draconian chosen pullback structure. We define its *new pan 1-Category*, $\text{Span}'(\mathcal{C})$, as the category with the same objects as \mathcal{C} and morphisms are spans between objects in \mathcal{C} . Composition of two spans $(A \leftarrow X \rightarrow B)$ and $(B \leftarrow Y \rightarrow C)$ is formed by outer span in the below diagram where Z is the chosen pullback of $(X \rightarrow B \leftarrow C)$ in the below diagram.



The span $(A = A = A)$ is the identity with respect to composition.

Unitality implies that the span $(A = A = A)$ is the identity with respect to composition. Associativity is equivalent to composition being associative. This motivates the term *draconian*. Note that $\text{Span}'(\mathcal{C})$ is not in general equal to its dual.

$\text{Span}'(\mathcal{C})$ is not equivalent to $\text{Span}(\mathcal{C})$ as homsets in the former consist of spans whereas in the latter they are isomorphism classes of spans. In a sense though, both are 1-categorical versions of the same 2-category.

Definition 9.2.7. Let \mathcal{C} be a category with a chosen pullback structure. We define the *(2,1)*

span category of \mathcal{C} denoted $Span_2(\mathcal{C})$ as the bicategory with object the same as those of \mathcal{C} . $Hom(A, B)$ is equal to the maximal groupoid of spans over A and B . Composition is formed by taking the chosen pullback of the middle cospan.

In the case that \mathcal{C} has a draconian chosen pullback structure, unitality and associativity ensure this is a strict 2-category, however in the more general case, associativity and unitality only hold up to isomorphism.

Lemma 9.2.8 ([Lur23] Ex. 2.2.6.13). *Given two different chosen pullback structures on \mathcal{C} , we have two versions of $Span_2(\mathcal{C})$ which are distinct, but equivalent as 2-categories.*

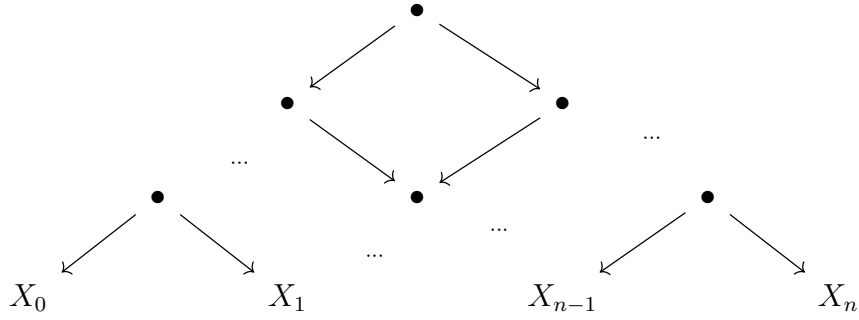
Example 9.2.9. The bicategory GE of [GM11] is defined as $Span_2(\mathbf{Fin}^G)$ with the lex-pullback. We discuss it more in Chapter 10.

By taking the homotopy 1-category of the (2,1) span category we get the classical span category. By taking the underlying 1-category we get the new span category. Although the former is the more “morally correct” version, the latter is often more useful and appears in relevant literature, namely [BIO15].

This is easily generalized to a 2-category as opposed to a (2,1)-category by including all morphisms of spans, not just isomorphisms. However I do not know of any uses for this so it is not discussed.

In the world of ∞ -categories, this is all much more natural as we are not concerned with a distinction between “strict” and “up to coherent equivalences”.

Definition 9.2.10 ([Bar17, BGS20]). For a category or ∞ -category \mathcal{C} with pullbacks, the *effective Burnside category* $A^{eff}(\mathcal{C})$ is the ∞ -category, viewed as a quasicategory with n -simplices given by diagrams in \mathcal{C} of the form:



where all squares are pullbacks. Intuitively, this encodes all of the possible ways to compose the spans from object X_0 to X_n .

Remark 9.2.11. For a 1-category \mathcal{C} with pullbacks it is suspected that $N^D(\text{Span}_2(\mathcal{C})) \simeq \text{A}^{eff}(\mathcal{C})$. I believe that a proof might involve [Lur23] Cor. 8.1.3.12.

9.3 Chosen Pullbacks on Fin

Theorem 9.3.1. *There is no class of draconian chosen pullbacks on Fin.*

Unfortunately the proof is tedious and unilluminating so we leave it for the end of the chapter.

Remark 9.3.2. We believe this result to be widely suspected or considered common sense but it is not explicitly stated anywhere nor is a proof written.

Corollary 9.3.3. *There is no class of draconian chosen pullbacks on Fin^G .*

Proof. of corollary.

Suppose we have a draconian chosen pullback system on Fin^G . If we restrict to objects with a trivial G -action this gives us a draconian chosen pullback system on Fin^G because categorical pullbacks of trivial G -objects are also trivial.

□

Remark 9.3.4. As draconian classes of chosen pullbacks deal with strict equality of objects and morphisms, as opposed to uniqueness up to unique isomorphism, standard category theoretic techniques do not always work and we are forced to work very concretely. For instance, although a class of chosen pullbacks is preserved under an equivalence of categories, this is not known to be the case for a draconian class of chosen pullbacks. Additionally, although we suspect this result also applies in *Top* and the non-skeletal category of finite sets and *G*-sets, our proof heavily relies upon our use of a skeleton category and it is not clear how we could generalize it.

9.4 Proof of Theorem 9.3.1

We first prove a few lemmas.

Lemma 9.4.1. *Assume there is a draconian pullback structure on \mathbf{Fin} . For any surjection in \mathbf{Fin} , $s : X \rightarrow Y$, if $k : Z \rightarrow Y$ is such that $s^*k = Id_X$, then $Z = Y$ and $k = Id_Y$.*

Proof. Let t be the right inverse to s . Then by unitality $t^*Id_X = Id_Y$. But by pasting and unitality, that's also $(st)^*k = Id_Y^*k = k$.

$$\begin{array}{ccccc}
 Z & \longrightarrow & X & \longrightarrow & Z \\
 \downarrow & & \parallel & & \downarrow k \\
 Y & \xrightarrow{t} & X & \xrightarrow{s} & Y \\
 & \searrow & & \nearrow & \\
 & & Id_Y & &
 \end{array}$$

□

Lemma 9.4.2. *Assume there is a draconian pullback structure on \mathbf{Fin} . For any injection in \mathbf{Fin} $t : Y \rightarrow X$, and $k : Z \rightarrow Y$, then there is a $h : W \rightarrow Y$ st $k = t^*h$.*

Proof. Let s be a left inverse to t . We take the chosen pullback of $(X \xrightarrow{t} Y \xleftarrow{k} Z)$ which we denote W and draw as the right square. Then we take the pullback of $(Y \xrightarrow{t} X \xleftarrow{s} Y)$. By pasting and unitality, $t^*h = k$.

$$\begin{array}{ccccc}
 Z & \longrightarrow & W & \longrightarrow & Z \\
 k \downarrow & & h \downarrow & & k \downarrow \\
 Y & \xrightarrow{t} & X & \xrightarrow{s} & Y \\
 & \searrow & \text{Id}_Y & \nearrow & \\
 & & & &
 \end{array}$$

□

Lemma 9.4.3. *Let this be a chosen pullback square in a category with a draconian pullback structure*

$$\begin{array}{ccc}
 \bullet & \xrightarrow{f} & \bullet \\
 p \downarrow & & q \downarrow \\
 \bullet & \xrightarrow{g} & \bullet
 \end{array}$$

If f, g are invertible then this is a chosen pullback square

$$\begin{array}{ccc}
 \bullet & \xrightarrow{f^{-1}} & \bullet \\
 q \downarrow & & p \downarrow \\
 \bullet & \xrightarrow{g^{-1}} & \bullet
 \end{array}$$

If p, q are invertible then this is a chosen pullback square

$$\begin{array}{ccc}
 \bullet & \xrightarrow{g} & \bullet \\
 p^{-1} \downarrow & & q^{-1} \downarrow \\
 \bullet & \xrightarrow{f} & \bullet
 \end{array}$$

We call this process *flipping*.

Proof. We just prove the first case, the second one is identical. Consider the diagram:

$$\begin{array}{ccccc}
 \bullet & \xrightarrow{f^{-1}} & \bullet & \xrightarrow{f} & \bullet \\
 q \downarrow & & p \downarrow & & q \downarrow \\
 \bullet & \xrightarrow{g^{-1}} & \bullet & \xrightarrow{g} & \bullet
 \end{array}$$

The right square is a pullback square by assumption and the full rectangle is by unitality.

That implies $g^{-1}q = p$ and as f is invertible the top arrow is f^{-1} .

□

Lemma 9.4.4. *As before, let the following be a chosen pullback square of a category with draconian choice of pullbacks, and we assume that g and q are non-identity isomorphisms. Then f and p are also not identities.*

Proof. Suppose for contradiction that p is an identity. Then we have the following diagram and by flipping, the left square is a chosen pullback.

$$\begin{array}{ccccc}
 \bullet & \xrightarrow{f^{-1}} & \bullet & \xrightarrow{f} & \bullet \\
 q \downarrow & & p \downarrow & & q \downarrow \\
 \bullet & \xrightarrow{g^{-1}} & \bullet & \xrightarrow{g} & \bullet
 \end{array}$$

However, p is an identity and by unitality q is as well, a contradiction. The case for g is identical.

□

We now are ready to procede to proving Theorem 9.3.1.

Proof. of Theorem 9.3.1

We begin by assuming for contradiction that there is a draconian chosen pullback system on \mathbf{Fin} . In this section all squares shown in diagrams will be chosen pullback squares. We focus on endomorphisms of the set $\underline{3} = \{1, 2, 3\}$.

Let e be the endomorphism of $\underline{3}$ given by $1 \mapsto 2, 2 \mapsto 2, 3 \mapsto 3$.

We define $r : \underline{3} \rightarrow \underline{2}$ by $1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 2$ and $i : \underline{2} \rightarrow \underline{3}$ by $1 \mapsto 2, 2 \mapsto 3$.

First we state some basic facts about these three morphisms.

- e is idempotent.
- $e(12) = e$.
- There are 6 endomorphisms conjugate to e .
- $ri = Id_{\underline{2}}$.
- $ir = e$.

We define $f : \underline{3} \rightarrow \underline{3}$ as $r^*(12)$ and we consider the following diagram:

$$\begin{array}{ccccccc}
 3 & \longrightarrow & 2 & \longrightarrow & 3 & \longrightarrow & 2 \\
 \downarrow f & & \downarrow (12) & & \downarrow f & & \downarrow (12) \\
 3 & \xrightarrow{r} & 2 & \xrightarrow{i} & 3 & \xrightarrow{r} & 2 \\
 & \searrow e & & \nearrow Id & & &
 \end{array}$$

By assumption the left and right squares are chosen pullbacks. As $Id^*(12) = (12)$ the composite of the middle and right squares are chosen pullbacks, so $i^*f = (12)$ and the middle square is a chosen pullback as well.

- We know f is an isomorphism as a pullback of an isomorphism is an isomorphism; as Fin is a skeleton category it must also be an automorphism of $\underline{3}$.
- $f \neq Id_{\underline{3}}$ as $r^*Id_{\underline{3}} = Id_{\underline{2}} \neq (12) = r^*f$.
- As shown in the diagram, $e^*f = f$.

Throughout this section we let f denote such an isomorphism and we call it a morphism fixed by e^* .

- Let e' be the conjugate of e via (12); $e' : 1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 3$. Via direct calculation we see that e' is idempotent, $ee' = e, e'e = e'$. $(12)e = e'$, and $(12)e' = e$.
- If f is fixed by e^* then it is also fixed by e'^* .

$$\begin{array}{ccccc}
 \underline{3} & \longrightarrow & \underline{3} & \longrightarrow & \underline{3} \\
 \downarrow f & & \downarrow f & & \downarrow f \\
 \underline{3} & \xrightarrow{e'} & \underline{3} & \xrightarrow{e} & \underline{3} \\
 & \searrow e & & \nearrow e & \\
 & & & &
 \end{array}$$

Again the right square and full rectangle are chosen pullbacks, pasting implies the left must be as well.

- By the same argument, if f' is any automorphism of $\underline{3}$ fixed by e'^* then it is also fixed by e^* .
- If f is fixed by e^* , then $(12)^*f = f$,

$$\begin{array}{ccccc}
 \underline{3} & \longrightarrow & \underline{3} & \longrightarrow & \underline{3} \\
 \downarrow f & & \downarrow f & & \downarrow f \\
 \underline{3} & \xrightarrow{(12)} & \underline{3} & \xrightarrow{e} & \underline{3} \\
 & \searrow e & & \nearrow e & \\
 & & & &
 \end{array}$$

Similarly the right square and full rectangle are chosen pullbacks, pasting implies the left must be as well.

- f is fixed by e'^* .

$$\begin{array}{ccccc}
 \underline{3} & \longrightarrow & \underline{3} & \longrightarrow & \underline{3} \\
 \downarrow f & & \downarrow f & & \downarrow f \\
 \underline{3} & \xrightarrow{e} & \underline{3} & \xrightarrow{(12)} & \underline{3} \\
 & \searrow e' & & \nearrow e' & \\
 & & & &
 \end{array}$$

We know both squares are chosen pullbacks so pasting the full rectangle is as well.

Recall that there are 6 idempotents conjugate to e of which e' is one. These are the endomorphisms which fix two elements and send the other one to one of the fixed points.

We let $f_{12} := f$ as f corresponds to the idempotents sending 1 and 2 to the same point. By symmetry, there are other non-identity isomorphisms fixed by the other 4 idempotents of that type. We call them f_{13}, f_{23} based on which idempotents they fix.

So far we know that they are non-identity isomorphisms, but we have not shown yet that they are distinct.

Also note that everything we have proved about f applies to the other two, though with the numbers suitably changed. For instance They are pullbacks along maps conjugate to r of $(12) \in \Sigma_2$. And $(32)^* f_{13} = f_{13}$, $(23)^* f_{23} = f_{23}$.

We will prove they are distinct though this is a bit tricky. Let $c : \underline{3} \rightarrow \underline{1}$ be the unique map, and $i_1 : \underline{1} \rightarrow \underline{3}$ send $1 \mapsto 1$, similarly for $i_2, i_3 : \underline{1} \rightarrow \underline{3}$. Let $c_1 : \underline{3} \rightarrow \underline{3}$ send everything to 1, similarly for c_2, c_3 . Then $ci_j = Id_{\underline{1}}, i_jc = c_j$.

Then $c_j^* f = Id_{\underline{3}}$, and similarly for f_{23}, f_{13}

$$\begin{array}{ccccc}
 \underline{3} & \longrightarrow & \underline{1} & \longrightarrow & \underline{3} \\
 Id \downarrow & & Id \downarrow & & f \downarrow \\
 \underline{3} & \xrightarrow{c} & \underline{1} & \xrightarrow{i_j} & \underline{3} \\
 & & \searrow c_j & & \nearrow
 \end{array}$$

The right square is the chosen pullback of $(\underline{1} \xrightarrow{i_j} \underline{3} \xleftarrow{f} \underline{3})$, and $i_j^* f = Id_{\underline{1}}$ as it is the only automorphism of $\underline{1}$. The left square is a chosen pullback by unitality. By pasting the full rectangle is a chosen pullback.

We now prove $f_{12} \neq f_{23}$ and by the same argument we can show that the other pairs are distinct as well.

Let $e_{23} : \underline{3} \rightarrow \underline{3}$ be the idempotent sending $1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 2$. It is one of the 6 conjugates of e . By symmetry $e_{23}^* f_{23} = f_{23}$.

By direct computation $e_{23}e = c_2$. Assume for contradiction that $f_{12} = f_{23}$ then we would have:

$$\begin{array}{ccccc}
 \underline{3} & \longrightarrow & \underline{3} & \longrightarrow & \underline{3} \\
 ? \downarrow & & f_{23} \downarrow & & f_{23} \downarrow \\
 \underline{3} & \xrightarrow{e} & \underline{3} & \xrightarrow{e_{23}} & \underline{3} \\
 & \searrow & & \nearrow & \\
 & & c_2 & &
 \end{array}$$

We know the right square is a chosen pullback. As f_{23} is an isomorphism, we know that $f = f_{12} = e^* f_{23}$ so the left vertical arrow is f . By pasting the full rectangle is a chosen pullback, so the left vertical arrow is $c_2^* f_{23} = Id_3$, a contradiction.

We claim that in $Aut(\underline{2})$ $(12)^*(12) = (12)$. This is as $(12)^*(12)$ is an isomorphism, but cannot be Id_2 by the lemma. Thus this is a chosen pullback square:

$$\begin{array}{ccc}
 \underline{2} & \xrightarrow{(12)} & \underline{2} \\
 (12) \downarrow & & \downarrow (12) \\
 \underline{2} & \xrightarrow{(12)} & \underline{2}
 \end{array}$$

So $f = (r(12))^*(12)$.

We also note that there are 6 distinct surjections $\underline{3} \rightarrow \underline{2}$ including r and r' . Also by symmetry, f_{13}, f_{23} are the pullbacks of (12) along these surjections, where each of $\{f, f_{13}, f_{23}\}$ comes from two of the surjections.

Consider the following diagram. By Lemma 9.4.2 there is a k so that the right square is a chosen pullback.

$$\begin{array}{ccccccc}
\underline{3} & \longrightarrow & \underline{3} & \longrightarrow & \underline{3} & \longrightarrow & \underline{3} \\
\downarrow f_{23} & & \downarrow f & & \downarrow f & & \downarrow k \\
\underline{3} & \xrightarrow{(13)} & \underline{3} & \xrightarrow{e} & \underline{3} & \xrightarrow{(13)} & \underline{3} \\
& & & \searrow e_{23} & & \nearrow & \\
& & & & & &
\end{array}$$

By direct computation, the bottom row is e_{23} . So now we know $(13)^*f = f_{23}$. By similar arguments $(23)^*f = f_{13}$ and $(12)^*f_{13} = f_{23}$.

We claim that $f = f_{12}, f_{23}, f_{13}$ are transpositions. Consider the following diagram where the top square is from flipping

$$\begin{array}{ccc}
\underline{3} & \xrightarrow{r} & \underline{2} \\
f^{-1} \downarrow & & \downarrow (12) \\
\underline{3} & \xrightarrow{r'} & \underline{2} \\
f \downarrow & & \downarrow (12) \\
\underline{3} & \xrightarrow{r} & \underline{2}
\end{array}$$

By examination, r' is a surjection conjugate to r so $f^{-1} \in \{f, f_{13}, f_{23}\}$ By symmetry, $f_{23}^{-1}, f_{13}^{-1} \in \{f = f_{12}, f_{23}, f_{13}\}$ as well. Thus they come in inverse pairs, so at least one must be its own inverse, thus a transposition.

Without loss of generality, let f_{12} be the transposition. Note that this is the first time we have made assumptions about it different from those of f_{13} and f_{23} . We have:

$$\begin{array}{ccc}
\underline{3} & \xrightarrow{r} & \underline{2} \\
f \downarrow & & \downarrow (12) \\
\underline{3} & \xrightarrow{r'} & \underline{2} \\
f \downarrow & & \downarrow (12) \\
\underline{3} & \xrightarrow{r} & \underline{2}
\end{array}$$

Assume for contradiction that $r = r'$. By direct computation, there is no transposition $f \in \Sigma_3$ so that $(12)r = rf$, contradicting the above diagram. Thus $r \neq r'$

Thus $r' = (12)r$ as we know $((12)r)^*(12) = f$. By direct computation, the only transposition f satisfying $rf = r$ is $f = (12)$.

We will show f_{23} must also be a transposition. If not its either (123) or (132) . Then for the left square to commute, the top arrow is Id , contradicting Lemma 9.4.4.

If it is (132) we know the following are chosen pullback squares:

$$\begin{array}{cccc}
 \underline{\mathfrak{Z}} \xrightarrow{(12)} \underline{\mathfrak{Z}} & \underline{\mathfrak{Z}} \xrightarrow{(123)} \underline{\mathfrak{Z}} & \underline{\mathfrak{Z}} \xrightarrow{(132)} \underline{\mathfrak{Z}} & \underline{\mathfrak{Z}} \xrightarrow{(13)} \underline{\mathfrak{Z}} \\
 (12)\downarrow & \downarrow & \downarrow & \downarrow \\
 \underline{\mathfrak{Z}} \xrightarrow{(12)} \underline{\mathfrak{Z}} & \underline{\mathfrak{Z}} \xrightarrow{(13)} \underline{\mathfrak{Z}} & \underline{\mathfrak{Z}} \xrightarrow{(23)} \underline{\mathfrak{Z}} & \underline{\mathfrak{Z}} \xrightarrow{(23)} \underline{\mathfrak{Z}}
 \end{array}$$

For the first we had already proven $(12)^*f = f$, the second as $(13)^*f = f_{23}$, the third we get by making the same argument with (23) as f_{13} to show that $(23)^*f = f_{13} = (123)$ and as we had assumed $f_{13} = f_{23}^{-1}$, the fourth as by assumption $(23)^*f_{23} = f_{23}$.

$$\begin{array}{cccc}
 \underline{\mathfrak{Z}} \xrightarrow{(13)} \underline{\mathfrak{Z}} & \underline{\mathfrak{Z}} \xrightarrow{(123)} \underline{\mathfrak{Z}} & \underline{\mathfrak{Z}} \xrightarrow{(23)} \underline{\mathfrak{Z}} & \underline{\mathfrak{Z}} \xrightarrow{(13)} \underline{\mathfrak{Z}} \\
 (123)\downarrow & \downarrow & \downarrow & \downarrow \\
 \underline{\mathfrak{Z}} \xrightarrow{(123)} \underline{\mathfrak{Z}} & \underline{\mathfrak{Z}} \xrightarrow{(23)} \underline{\mathfrak{Z}} & \underline{\mathfrak{Z}} \xrightarrow{(13)} \underline{\mathfrak{Z}} & \underline{\mathfrak{Z}} \xrightarrow{(132)} \underline{\mathfrak{Z}}
 \end{array}$$

Now we glue squares 1 and 2 of this set vertically and glue squares 3 and 4 vertically to get the following two chosen pullbacks:

$$\begin{array}{cc}
 \underline{\mathfrak{Z}} \xrightarrow{(13)} \underline{\mathfrak{Z}} & \underline{\mathfrak{Z}} \xrightarrow{(23)} \underline{\mathfrak{Z}} \\
 (23)\downarrow & \downarrow \\
 \underline{\mathfrak{Z}} \xrightarrow{(23)} \underline{\mathfrak{Z}} & \underline{\mathfrak{Z}} \xrightarrow{(132)} \underline{\mathfrak{Z}}
 \end{array}$$

to get $(23) = (23)^*(13)$

Then glue the 3d and 4th square here vertically to get $(132)^*(132) = (23)$. But flipping the second one vertically contradicts the earlier square 2. Thus $f_{23} \neq (132)$ and it must a

transposition. So then f_{13} is as well. Using the same argument as with f_{12} we show they are (23), (13) respectively.

We had shown that $(13)^*f = f_{23}$, and $(12)^*f_{13} = f_{23}$

So we have the squares:

$$\begin{array}{ccc}
 \underline{3} & \xrightarrow{(13)} & \underline{3} & & \underline{3} & \xrightarrow{(12)} & \underline{3} \\
 (23)\downarrow & & \downarrow(12) & & (23)\downarrow & & \downarrow(13) \\
 \underline{3} & \xrightarrow{(13)} & \underline{3} & & \underline{3} & \xrightarrow{(12)} & \underline{3}
 \end{array}$$

But all the arguments we made in this section could also be made switching the two legs of a cospan when taking the chosen pullback. That is to say switching the horizontal and vertical axes.

But these two squares are not switched versions of each other. Thus we have a contradiction so \mathbf{Fin} cannot have a draconian chosen pullback structure. □

Chapter 10

GE' and GE_{ord}

In this section we consider three versions of an enriched Burnside category on \mathbf{Fin}^G

Definition 10.0.1. GE is defined as the PC-bicategory with objects those of \mathbf{Fin}^G ,

$\mathbf{GE}(A, B)$ is the category of spans $(A \leftarrow X \rightarrow B)$ and isomorphisms of spans. $\mathbf{GE}(A, B)$ is permutative with Π as its monoidal product. The unit span in $\mathbf{GE}(A, A)$ is $(A = A = A)$ Composition $\mathbf{GE}(B, C) \times \mathbf{GE}(A, B) \rightarrow \mathbf{GE}(A, C)$ is defined by taking the outer span where the square is the lex-pullback:

$$\begin{array}{ccccc}
 & & g^*Y & & \\
 & & \swarrow & \searrow & \\
 & & X & & Y \\
 & \swarrow & \downarrow g^*h & \downarrow h & \searrow \\
 & A & & B & C \\
 & \swarrow f & & \swarrow h & \searrow k \\
 & & & &
 \end{array}
 \tag{10.0.1}$$

Composition is strictly associative, however as the lex-pullback is only unital on one side, GE is not strictly unital but is only a bicategory.

We recall that the distributivity isomorphism δ_1 relates addition of Y, Y' to composition and δ_2 relates addition of X, X' to composition, though this may initially be counterintuitive.

We have that δ_1 is a non-identity isomorphism via the reordering isomorphism $g^*(Y \amalg Y') = (g^*Y \amalg g^*Y')_{\times X} \cong g^*Y \amalg g^*Y'$ which is natural. And δ_2 is the identity as the ordered disjoint union right distributes over the lex-product and thus lex-pullback: $(g \amalg g')^*Y = g^*Y \amalg g'^*Y$.¹

As Theorem 9.3.1 tells us, we cannot simply get around this issue by using another choice of pullback instead of the lex-pullback. Admittedly we have not shown that there can be no clever way of choosing a different pullback for each pair of spans to make it work but this remains unlikely. So we must resort to nastier measures.

Definition 10.0.2 ([GM11], more details in [JY22]). \mathbf{GE}' is the PC-category whose objects are the same as \mathbf{Fin}^G . For $A \neq B$ or $A = B$ and $|A| \leq 1$ we let $\mathbf{GE}'(A, B) := \mathbf{GE}(A, B)$ with the same permutative structure.

For $|A| \geq 2$, $\mathbf{GE}'(A, A)$ has objects spans $(A \leftarrow X \rightarrow B)$ and a whiskered compositional unit we denote Id_A . The morphisms in $\mathbf{GE}'(A, A)$ are generated by the isomorphisms of spans and a unique isomorphism $Id_A \cong (A = A = A)$.

On the spans the permutative structure of $\mathbf{GE}'(A, A)$ agrees with that of $\mathbf{GE}(A, A)$, $(A \leftarrow \emptyset \rightarrow A) \amalg Id_A := Id_A$ and for all other spans $(A \leftarrow X \rightarrow A) \amalg Id_A := (A \leftarrow X \amalg A \rightarrow A)$.

On spans composition is defined using the lex-pullback as in \mathbf{GE} . And Id_A is a strict unit with respect to composition.

For spans δ_1 is the reordering isomorphism and δ_2 is an equality, both as in \mathbf{GE} .

For the whiskered unit, $Id_B \circ (A \leftarrow X \rightarrow B) = (A \leftarrow X \rightarrow B) = (B = B = B) \circ (A \leftarrow X \rightarrow B)$ and we recall that Id_B and $(B = B = B)$ behave the same under addition so there is nothing new to check here for δ_1 . On the other side is it more subtle.

¹We could also phrase this in terms of right (resp. left) composition being strong (resp. strict) monoidal, but this phrasing can be less readable.

However as $(A = A = A) \circ (A \leftarrow X \rightarrow B) \neq (A \leftarrow X \rightarrow B)$, composition is only strong monoidal; $\delta_2 \neq Id$.

We calculate:

$$((A \leftarrow X \rightarrow A) + Id_A) \circ (A \leftarrow Y \rightarrow B) = (A \leftarrow Z \amalg (Y \times_A) \rightarrow B)$$

whereas

$$(A \leftarrow X \rightarrow A) \circ (A \leftarrow Y \rightarrow B) + Id_A \circ (A \leftarrow Y \rightarrow B) = (A \leftarrow Z \amalg Y \rightarrow B)$$

where Z is the lex-pullback of $(X \rightarrow A \leftarrow Y)$. The distributivity isomorphism then comes from the reordering isomorphism $Y \xrightarrow{\cong} Y \times_A$ over A .

Thus in general δ_1, δ_2 are non-identity isomorphisms and composition in \mathbf{GE}' is strong but not strict bilinear.

Lemma 10.0.3. *\mathbf{GE} and \mathbf{GE}' are biequivalent.*

Proof. We have an inclusion pseudofunctor $i : \mathbf{GE} \rightarrow \mathbf{GE}'$, which is the identity on objects, morphisms (spans), and 2-morphisms (isomorphisms between spans). It is easy to see that it induces monoidal equivalences on hom-categories, preserves composition up to natural isomorphism, and sends units to objects isomorphic to Id_A . \square

\mathbf{GE}' is adequate for constructing spectral or permutative Mackey functors as in [GM, BO], however the whiskered unit is somewhat unwieldy, and does not interact particularly nicely with \mathbf{Fin}^G itself. For that reason we introduce a different PC-category to replace \mathbf{GE}

Definition 10.0.4. We define \mathbf{GE}_{ord} as the PC-cat whose objects are again those of \mathbf{Fin}^G , in $\mathbf{GE}_{ord}(A, B)$ objects are spans whose left leg is order preserving: $(A \leftarrow X \rightarrow B)$ and morphisms are isomorphisms of spans.

As before we compose spans by taking the outer span resulting using the lex-pullback which is strictly associative. It is strictly unital with $(A = A = A) \in \mathbf{GE}_{ord}(A, A)$ the unit as

$$\begin{array}{ccc} X & \xlongequal{\quad} & X \\ \downarrow & & \downarrow \\ A & \xlongequal{\quad} & A \end{array}$$

is a lex-pullback square.

Its permutative structure is more complicated. We define

$$(A \leftarrow X \rightarrow B) + (A \leftarrow Y \rightarrow B) := (A \leftarrow (X \amalg Y)_{\times A} \rightarrow B).$$

It is clear this is strictly unital with $(A \leftarrow \emptyset \rightarrow A)$ as the additive unit.

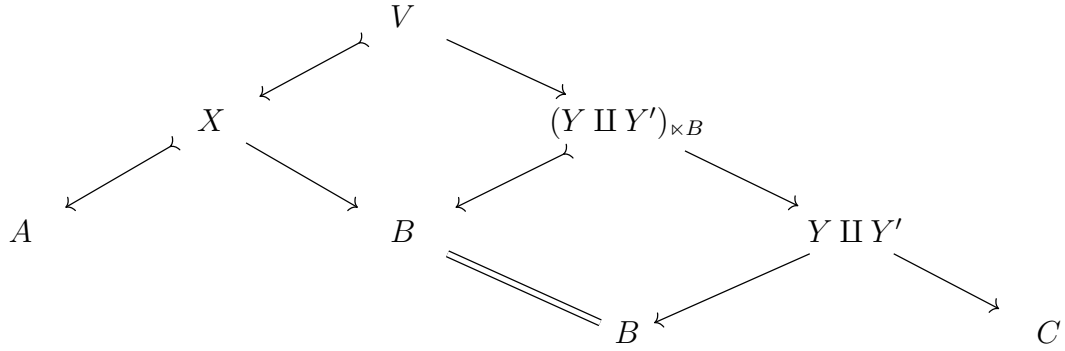
In both $((X \amalg Y)_{\times A} \amalg Z)_{\times A}$ and $(X \amalg (Y \amalg Z)_{\times A})_{\times A}$ elements are ordered lexicographically first by which element of A they lie over, then whether they come from X, Y , or Z , then based on the original ordering in those sets. This can also be shown via a diagram chase. This shows those are the same as ordered G -sets, so associativity is strict.

We now describe the distributivity isomorphisms δ_1, δ_2 .

On the other side it is only strong monoidal; δ_1 is a non-identity isomorphism. We can display

$$((B \leftarrow Y \rightarrow C) + (B \leftarrow Y' \rightarrow C)) \circ (A \leftarrow X \rightarrow B)$$

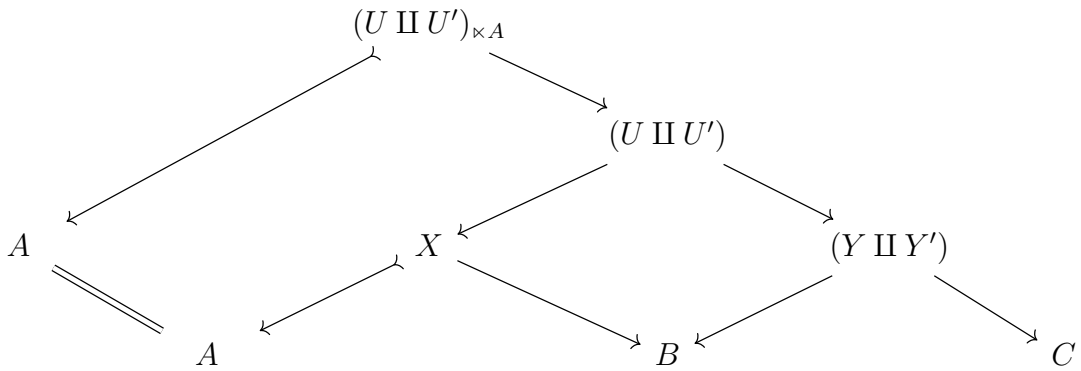
as the following diagram where both squares are lex-pullback and by gluing 8.1.1 the rectangle is as well. V is the lex pullback of $(X \rightarrow B \leftarrow (Y \amalg Y')_{\times B})$.



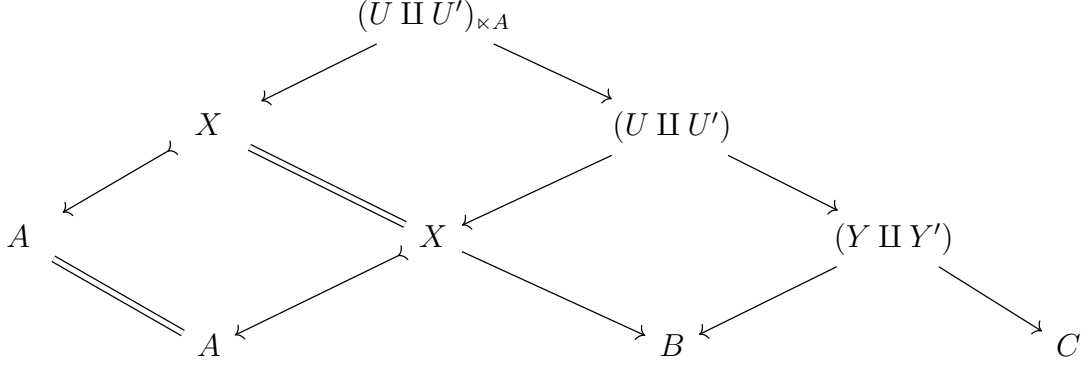
And we can display

$$(B \leftarrow Y \rightarrow C) \circ (A \leftarrow X \rightarrow Y) + (B \leftarrow Y' \rightarrow C) \circ (A \leftarrow X \rightarrow Y)$$

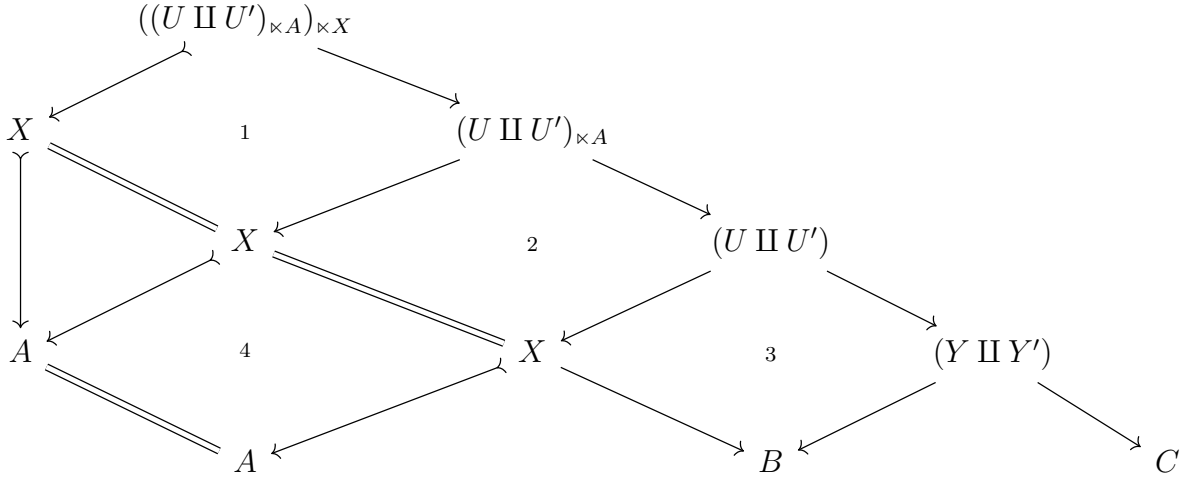
in the following diagram:



where U, U' are the lex-pullbacks of $(X \rightarrow B \leftarrow Y), (X \rightarrow B \leftarrow Y')$ respectively. The middle square is a pullback square but not a lex-pullback. However its legs are order preserving when restricting to Z, Z', Y, Y' its legs are order preserving. The leg $(U \amalg U')_{\times A} \rightarrow A$ factors through X so we can draw:



where the lower left square is a lex-pullback, but the top square is not a lex-pullback. We claim that $((U \amalg U')_{\times A})_{\times X} = Z'''$, and that their maps to A and C agree as well.



This reduces to showing that the composite rectangle of squares 1, 2, and 3 is a lex-pullback. We know the left leg $((U \amalg U')_{\times A})_{\times X} \rightarrow X$ is order preserving, so we consider $u < u' \in ((U \amalg U')_{\times A})_{\times X}$ lying over $x \in X$ and we prove $hgf(u) < hgf(u')$ in $Y \amalg Y'$. First we have that $f(u) < f(u')$ as f is the top square is a lex-pullback. As u, u' are in the same fiber over x , they are also in the same fiber over A , so $gf(u) < gf(u')$, and g is the top leg of the lex-pullback square given by the composite of squares 2 and 4. Now $gf(u), gf(u')$ are still in the same fiber over $x \in X$. If both are in U then as U is the lex-pullback of $(X \rightarrow B \leftarrow Y)$, h is order preserving when restricted to U so $hgf(u) < hgf(u')$. This same

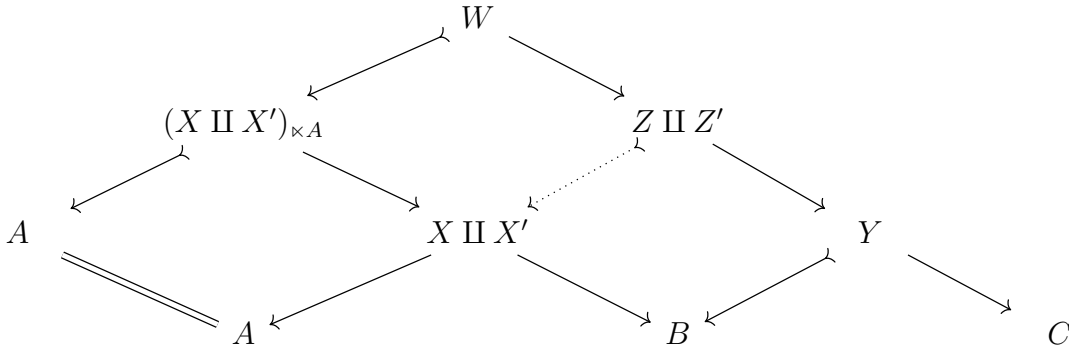
argument holds if both are in U' . The other case is that $gf(u) \in U, gf(u') \in U'$ in this case as $h(U) \subseteq Y, h(U') \subseteq Y'$, we must have that $hgf(u) < hgf(u')$.

So now we have a natural distributivity isomorphism $\delta_1 : V = ((U \amalg U')_{\times A})_{\times X} \cong (U \amalg U')_{\times A}$

Consider spans $(A \leftarrow X \rightarrow B), (A \leftarrow X' \rightarrow B), (B \leftarrow Y \rightarrow C)$ Let Z, Z' be the lex pullbacks of $(X \rightarrow B \leftarrow Y), (X' \rightarrow B \leftarrow Y)$ respectively.

$$(B \leftarrow Y \rightarrow C) \circ ((A \leftarrow X \rightarrow B) + (A \leftarrow X' \rightarrow B))$$

can be drawn as the diagram:



where W is the lex-pullback of $((X \amalg X')_{\times Z} \rightarrow B \leftarrow Y)$. We observe that the leg $W \rightarrow Y$ factors through $Z \amalg Z'$, and that the lower right square with the dotted arrow is a lex pullback.

By reverse pasting the top square is as well. By pasting the rectangle formed by the top square and lower left square is a lex-pullback; thus $W = (Z \amalg Z')_{\times A}$.

We also have by definition that

$$(B \leftarrow Y \rightarrow C) \circ (A \leftarrow X \rightarrow B) + (B \leftarrow Y \rightarrow C) \circ (A \leftarrow X' \rightarrow B) = (A \leftarrow (Z \amalg Z')_{\times A} \rightarrow C)$$

so composition on this side is strict monoidal, $\delta_2 = Id$.

Theorem 10.0.5. *There is a biequivalence $R : \mathbf{GE}' \rightarrow \mathbf{GE}_{ord}$*

Proof. We start by constructing R as a 2-functor, then show it is in fact a PC-functor. It acts as the identity on objects and on hom-categories we define:

$$R_{A,B} : \text{Hom}_{\mathbf{GE}'}(A, B) \longrightarrow \text{Hom}_{\mathbf{GE}_{ord}}(A, B)$$

$$(A \xleftarrow{f} X \rightarrow B) \longmapsto (A \xleftarrow{Id^* f} X_{\times A} \xrightarrow{g} B)$$

where g is determined uniquely by the reordering isomorphism $X \cong X_{\times A}$. $R(Id_A) := (A = A = A)$. We see in the diagram how R acts on a morphism η between spans:

This is well-defined by the universal property of pullbacks and it is clear that it preserves identity morphisms and composition. Thus $R_{A,B}$ defines a functor.

By construction R sends the composition units of \mathbf{GE}' to those of \mathbf{GE}_{ord} .

Next we show that it preserves composition which amounts to $Z_{\times C} = Z''$ in the diagram, where Z is the lex-pullback of $(X \rightarrow B \leftarrow Y)$, and Z'' is the lex-pullback of $(X_{\times A} \rightarrow B \leftarrow Y_{\times B})$. To better keep track of the maps, we use “ \mapsto ” here to denote the top arrows in a lex-pullback square.

$$\begin{array}{c}
 Z'' \\
 \vdots \\
 Z_{\times A} \\
 \downarrow \\
 Z \\
 \swarrow \quad \searrow \\
 X_{\times A} \quad Y_{\times B} \\
 \downarrow \quad \downarrow \\
 A \quad X \quad B \quad Y \quad C \\
 \parallel \quad \quad \parallel \quad \quad \parallel \\
 A \quad \quad \quad B \quad \quad \quad C
 \end{array}
 \tag{10.0.3}$$

The squares 1 and 2 shown below are both lex-pullbacks, so using Lemma 8.1.1 on horizontal pasting, the composite square 3 is also a lex-pullback.

$$\begin{array}{ccc}
 Z'' & \longrightarrow & Y_{\times B} & \longrightarrow & Y \\
 \downarrow & & \downarrow & & \downarrow \\
 X_{\times A} & \longrightarrow & B & \xlongequal{\quad} & B
 \end{array}
 \quad
 \begin{array}{ccc}
 Z'' & \longrightarrow & Y \\
 \downarrow & & \downarrow \\
 X_{\times A} & \longrightarrow & B
 \end{array}$$

We now return to diagram 10.0.3, which we have simplified by substituting in square 3 in place of squares 1 and 2 and omitting C as it is no longer relevant.

$$\begin{array}{c}
 Z'' \\
 \vdots \\
 Z_{\times A} \\
 \downarrow \\
 Z \\
 \swarrow \quad \searrow \\
 X_{\times A} \quad Y \\
 \downarrow \quad \downarrow \\
 A \quad X \quad B \\
 \parallel \\
 A
 \end{array}$$

Next, the arrow $Z_{\times A} \rightarrow A$ factors through $X_{\times A}$ which follows from $X_{\times A} \rightarrow X$ being invertible.

This implies the lex-pullback square 8 shown below is the composite of squares 4 and 5. Using the fact that square 5 is a lex-pullback square, we can reverse gluing of Lemma 8.1.6 to see

that square 4 is a lex-pullback. We then use that square 6 is a lex-pullback to apply horizontal pasting of Lemma 8.1.1 to show that square 7, the composite of 4 and 6, is also a lex-pullback.

$$\begin{array}{cccc}
\begin{array}{ccc}
Z_{\times A} \longmapsto Z \\
\downarrow \quad \quad \downarrow \\
A \longequal{\quad} A
\end{array} & \begin{array}{ccc}
Z_{\times A} \longmapsto Z \\
\downarrow \quad \quad \downarrow \\
X_{\times A} \longmapsto X \\
\downarrow \quad \quad \downarrow \\
A \longequal{\quad} A
\end{array} & \begin{array}{ccccc}
Z_{\times A} \longmapsto Z \longmapsto Y \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
X_{\times A} \longrightarrow X \longrightarrow B
\end{array} & \begin{array}{ccc}
Z_{\times A} \longmapsto Y \\
\downarrow \quad \quad \downarrow \\
X_{\times Z} \longrightarrow B
\end{array} \\
\text{8} & \text{4} & \text{4} \quad \text{6} & \text{7}
\end{array}$$

This completes the proof that both Z'' and $Z_{\times A}$ are lex-pullbacks of $(X_{\times A} \rightarrow B \leftarrow Y)$. We already showed that R sends units to units so we have already shown composition is preserved in the case that either 1-cell is and Id . Showing that composition is a bifunctor comes from a lengthy but straightforward diagram chase.

This completes the proof that R is a strict 2-functor.

We now need to show that it is PC-enriched. By direct checking we see that

$$(X \amalg Y)_{\times A} = (X_{\times A} \amalg Y_{\times A})_{\times A} \text{ meaning that } R_{A,B} \text{ is strict monoidal.}$$

Next we need to show that in the following diagram commutes in the sense that both ways around the diagram give the same bilinear bifunctor of permutative categories. We have already shown that as bifunctors they are the same so it remains to show that the distributivity isomorphism δ_1, δ_2 agree.

$$\begin{array}{ccc}
GE'(B, C) \times GE'(A, B) & \xrightarrow{comp} & GE'(A, C) \\
R \downarrow & & \downarrow R \\
GE_{ord}(B, C) \times GE_{ord}(A, B) & \xrightarrow{comp} & GE_{ord}(A, C)
\end{array}$$

As $R_{A,B}$ is strict monoidal, we only need to consider the distributivity morphisms of composition.

We consider spans with X, X', Y, Y' as before. For δ_1 going across we have $g^*Y \amalg g^*Y' \cong$

$(g^*Y \amalg g^*Y')_{\times X} = g^*(Y \amalg Y')$ going down we then have $(g^*Y \amalg g^*Y')_{\times A} \cong ((g^*Y \amalg g^*Y')_{\times X})_{\times A}$.

We observe that the RHS is equal to $(g^*Y \amalg g^*Y')_{\times X}$ which is the same as the δ_1 of \mathbf{GE}_{ord} .

We recall that on the whiskered unit δ_1 is defined the same as on the span $(B = B = B)$ so there is nothing new to check there.

For δ_2 we first consider spans in \mathbf{GE}' . Going across δ_2 is equality, so then applying R we once again get equality. As δ_2 is an equality in \mathbf{GE}_{ord} these agree. On the whiskered unit, we recall that in \mathbf{GE}' δ_2 is the reordering $Z \amalg Y_{\times A} \cong Z \amalg Y$. Applying R to this we get $(Z \amalg Y_{\times A})_{\times A} = (Z \amalg Y)_{\times A}$ so this is again an equality, agreeing with the δ_2 of \mathbf{GE}_{ord} .

Next we show it is a biequivalence of 2-categories.

By definition it is surjective on objects, so we must show $R_{A,B}$ is an equivalence of categories. In the case of $A = B$ we note that $\mathbf{GE}'(A, A) \simeq \mathbf{GE}(A, A)$ and R factors through the retract so we do not need to treat this case any differently.

$R_{A,B}$ is surjective on objects because $Hom_{\mathbf{GE}_{ord}}(A, B)$ is a subcategory of $Hom_{\mathbf{GE}'}(A, B)$ and $R_{A,B}$ is a retract.

Alternative explanation is that $Hom_{\mathbf{GE}_{ord}}(A, B)$ already has all spans whose left side is order preserving, and $R_{A,B}$ leaves these unchanged.

For fully faithfulness we consider the square in diagram 10.0.2 and note that all the maps in the middle are isomorphisms, in particular so η and $R_{A,B}(\eta)$ uniquely determine each other.

□

Remark 10.0.6. There is no strict 2-functor $\mathbf{GE}_{ord} \rightarrow \mathbf{GE}'$. To see this we note that in \mathbf{GE}' Id_A cannot be expressed as a composition of two non-identity spans, whereas in \mathbf{GE}_{ord} the identity $(A = A = A)$ can be.

Definition 10.0.7 ([BIO15]). A *permutative Mackey functor* is a PC-functor $(\mathbf{GE}')^{op} \rightarrow \mathbf{Perm}$, or from a PC-category biequivalent to $(\mathbf{GE}')^{op}$. This forms a 2-category with PC-natural transformations and modifications.

We note that the original definition used lax monoidal functor whereas we use strong monoidal ones.

Proposition 10.0.8. *PMFs with \mathbf{GE}' are equivalent (how?) to those with \mathbf{GE}_{ord} which we denote PMF_{ord}*

$$Fun_{PC}(\mathbf{GE}'^{op}, \mathbf{Perm}) \simeq Fun_{PC}(\mathbf{GE}_{ord}^{op}, \mathbf{Perm})$$

Proof. As R is a biequivalence, it has a weak inverse we denote $S : \mathbf{GE}_{ord} \rightarrow \mathbf{GE}'$, which is a pseudofunctor, and monoidal pseudonatural equivalences $RS \simeq Id_{\mathbf{GE}_{ord}}$, $SR \simeq Id_{\mathbf{GE}'}$.

we have a precomposition functors $R^{op*} : PMF' \rightarrow PMF_{ord}$, $S^{op*} : PMF_{ord} \rightarrow PMF'$, which are a strict 2-functor and pseudofunctor respectively. Similarly, the monoidal pseudonatural equivalences induce natural equivalences $S^{op*}R^{op*} \simeq Id_{PMF_{ord}}$ and $R^{op*}S^{op*} \simeq Id_{PMF'}$ making this an equivalence of 2-categories.

□

By this result we chose to use to work with PMFs from \mathbf{GE}_{ord} instead.

Part II

$\hat{\Sigma}_G$ -Algebras

Chapter 11

Fin^G -categories and $\hat{\Sigma}_G$

SGAs were first introduced by Peter Bonventre and Luis Pereira in an unpublished draft. Bonventre later refined the definition to its current version for which he defined Fin^G -categories. The work in this and the following chapters builds on their work and fills in technical lemmas in their work. This work is largely based on joint work with Peter Bonventre.

11.1 Fin^G -Categories

Definition 11.1.1. Let $\underline{\mathcal{C}} : (\text{Fin}^G)^{op} \rightarrow \text{Cat}$ be a strictly unital pseudofunctor whose image does not include the empty category. Equivalently we can view this as a fibration over Fin^G whose fibers are non-empty along with a normal cleavage (i.e. a strictly unital choice of cartesian lifts such that). In the case that $\underline{\mathcal{C}}$ is a functor $\underline{\mathcal{C}} \rightarrow \text{Fin}^G$ is a split fibration.

Remark 11.1.2. By the equivalence of weak 2-categories between contravariant functors to Cat and categorical fibrations, we know that everything done in this chapter could also be

done in terms of pseudofunctors $(\mathbf{Fin}^G)^{op} \rightarrow \mathit{Cat}$, however we believe that some parts are more convenient when phrased in terms of fibrations, paraphrasing the slogan attributed to Elden Elmanto,

“*Functors are better for intuition, fibrations are better for calculations.*”

Definition 11.1.3. For $f : A \rightarrow B$ in \mathbf{Fin}^G , we denote associated functor $f^* : \underline{\mathcal{C}}(B) \rightarrow \underline{\mathcal{C}}(A)$ which we call the *restriction* along f .

We say $\underline{\mathcal{C}}$ is *weakly additive* (resp. *strongly, strictly*) if for all $A, B \in \mathbf{Fin}^G$ the natural map

$$\underline{\mathcal{C}}(A \amalg B) \xrightarrow{\iota_A^* \times \iota_B^*} \underline{\mathcal{C}}(A) \times \underline{\mathcal{C}}(B) \quad (11.1.1)$$

is an adjoint equivalence of categories (resp. isomorphism, equality). We denote the weak inverse λ and associated natural isomorphisms η, ϵ :

$$\eta : Id_{\underline{\mathcal{C}}(A \amalg B)} \xrightarrow{\cong} \lambda(\iota_A^* \times \iota_B^*), \quad \epsilon : (\iota_A^* \times \iota_B^*)\lambda \xrightarrow{\cong} Id_{\underline{\mathcal{C}}(A) \times \underline{\mathcal{C}}(B)} \quad (11.1.2)$$

Note that we can replace η or ϵ with another natural isomorphism to make this the case if it is not already, so we can expand the \mathbf{Fin}^G -categories to include those that merely have equivalences. As this is an adjoint equivalence, both $\lambda, \iota_A^* \times \iota_B^*$ are left and right adjoints. Our choice of η and ϵ as the unit and counit respectively as opposed to their inverses (which would be counit and unit respectively) is thus arbitrary.

We next require that ϵ is suitably preserved by restriction along isomorphisms. Spelled out this means given isomorphisms $A \rightarrow A', B \rightarrow B'$ the whiskering diagrams agree:

$$\begin{array}{ccc}
V(A) \times V(A') & & V(A) \times V(A') \\
\downarrow r & & \parallel \\
V(B) \times V(B') & \xrightarrow{\lambda} & V(A \amalg A') \\
\parallel & \xleftarrow{\epsilon} & \\
V(B) \times V(B') & \xleftarrow{\iota^*} &
\end{array}
\quad
\begin{array}{ccc}
V(A) \times V(A') & & V(A) \times V(A') \\
\parallel & \xrightarrow{\lambda} & V(A \amalg A') \\
\leftarrow \epsilon & & \\
V(A) \times V(A') & \xleftarrow{\iota^*} & \\
\downarrow r & & \\
V(B) \times V(B') & &
\end{array}
\tag{11.1.3}$$

Implicit in this we are requiring restriction along isomorphism to strictly commute with λ .

A quick diagram chase using the triangle identities shows that this is equivalent to a similar condition on η .

We next require that they are associative and commutative in the sense that the following diagrams strictly commutes for all A, B, C , and that they are suitably compatible.

$$\begin{array}{ccc}
\underline{\mathcal{C}}(A) \times \underline{\mathcal{C}}(B) \times \underline{\mathcal{C}}(C) & \xrightarrow{\lambda \times Id} & \underline{\mathcal{C}}(A \amalg B) \times \underline{\mathcal{C}}(C) \\
Id \times \lambda \downarrow & & \downarrow \lambda \\
\underline{\mathcal{C}}(A) \times \underline{\mathcal{C}}(B \amalg C) & \xrightarrow{\lambda} & \underline{\mathcal{C}}(A \amalg B \amalg C)
\end{array}
\tag{11.1.4}$$

$$\begin{array}{ccc}
\underline{\mathcal{C}}(A) \times \underline{\mathcal{C}}(B) & \xrightarrow{\lambda} & \underline{\mathcal{C}}(A \amalg B) \\
\tau \downarrow & & \downarrow \tau^* \\
\underline{\mathcal{C}}(B) \times \underline{\mathcal{C}}(A) & \xrightarrow{\lambda} & \underline{\mathcal{C}}(B \amalg A)
\end{array}
\tag{11.1.5}$$

where τ denotes both twist maps.

These additional conditions mean we have a well defined $\lambda : \prod_i \underline{\mathcal{C}}(A_i) \rightarrow \underline{\mathcal{C}}(\amalg_i A_i)$ which is equivariant with respect to action of Σ_n .

This is a weak inverse to $\prod_i \iota_{A_i}^*$, however a priori we can form the (co)units by various compositions of the η s and ϵ s.

We finally require that the various compositions yield the same natural isomorphism.

Consequently we have an equivalence $\lambda : \prod_i \underline{\mathcal{C}}(A_i) \rightarrow \underline{\mathcal{C}}(\coprod_i A_i) : \prod_i \iota_{A_i}^*$ with specified (co)units, which we also denote η, ϵ .

Remark 11.1.4. By lemma 4.0.5 if we have η 's and ϵ 's forming these equivalences and we replace the η 's (or ϵ 's) so as to form adjoint equivalences, they still are suitably compatible as required.

Notation 11.1.5. For concision we will occasionally use ι^* to denote $\iota_A^* \times \iota_B^*$, and r for a restriction. We will sometimes use $\lambda_{\underline{\mathcal{C}}}$ for clarity if multiple fibrations are present.

Throughout this paper for clarity we will generally prove results only for λ, ι^* in the case of two objects, however by the required well-definition and compatibility, the same proofs work just as well for multiple objects. And we will appeal to some results in the case of multiple objects.

Lemma 11.1.6. *Weak (resp. strong) additivity implies that $\underline{\mathcal{C}}(\emptyset) \simeq *$ (resp. $\underline{\mathcal{C}}(\emptyset) \cong *$).*

We call this *weak unitality* (resp. *strong*).

Proof. We observe that applying the natural map 11.1.1 to $A = B = \emptyset$ we obtain:

$$\underline{\mathcal{C}}(\emptyset) = \underline{\mathcal{C}}(\emptyset \amalg \emptyset) \xrightarrow{\Delta} \underline{\mathcal{C}}(\emptyset) \times \underline{\mathcal{C}}(\emptyset).$$

As $\Delta = \iota_{\emptyset}^* \times \iota_{\emptyset}^*$. In the weak case, as Δ is essentially surjective, for $(a, b) \in \underline{\mathcal{C}}(\emptyset) \times \underline{\mathcal{C}}(\emptyset)$ there must be a $d \in \underline{\mathcal{C}}(\emptyset)$ so that $(a, b) \cong (d, d)$ in $\underline{\mathcal{C}}(\emptyset) \times \underline{\mathcal{C}}(\emptyset)$. This implies $a \cong d \cong b$ in $\underline{\mathcal{C}}(\emptyset)$. Thus all objects are isomorphic.

Next, by full faithfulness, $\Delta : \text{End}_{\underline{\mathcal{C}}(\emptyset)}(a, a) \cong \text{End}_{\underline{\mathcal{C}}(\emptyset) \times \underline{\mathcal{C}}(\emptyset)}((a, a), (a, a)) \cong \text{End}_{\underline{\mathcal{C}}(\emptyset)}(a, a) \times \text{End}_{\underline{\mathcal{C}}(\emptyset)}(a, a)$. Thus $\text{End}_{\underline{\mathcal{C}}(\emptyset)}(a, a)$ is a singleton. Finally as we assume $\underline{\mathcal{C}}(\emptyset)$ is not the empty category, it must be equivalent to $*$.

In the strict case, singletons are the only non-empty sets (or proper classes) which are in bijection with their product with themselves via the diagonal, so $\underline{\mathcal{C}}(\emptyset)$ has a single object. As it is equivalent to $*$, it now must be isomorphic to $*$ as well.

This is the only occasion we use the assumption that $\underline{\mathcal{C}}$ only takes values in non-empty categories. □

Lemma 11.1.7. *Restrictions and λ s commute up to isomorphism, and strictly if $\underline{\mathcal{C}}$ is strongly additive.*

Proof. They commute up to the 2-cell Λ defined by the pasting diagram:

$$\begin{array}{ccc}
 \underline{\mathcal{C}}(A) \times \underline{\mathcal{C}}(A') & & \\
 \parallel & \searrow \lambda & \\
 & \underline{\mathcal{C}}(A \amalg A') & \\
 \longleftarrow \epsilon & & \downarrow r \\
 \underline{\mathcal{C}}(A) \times \underline{\mathcal{C}}(A') & \swarrow \iota^* & \underline{\mathcal{C}}(B \amalg B') \\
 \downarrow r & & \parallel \\
 \underline{\mathcal{C}}(B) \times \underline{\mathcal{C}}(B') & \longleftarrow \eta & \underline{\mathcal{C}}(B \amalg B') \\
 & \searrow \lambda & \\
 & \underline{\mathcal{C}}(B \amalg B') &
 \end{array} \tag{11.1.6}$$

The center square commutes strictly if $\underline{\mathcal{C}}$ is a split fibration and up to isomorphism if not.

The unit and counit η and ϵ are trivial in the case of strict additivity. □

Lemma 11.1.8. *Λ transitive along restrictions. Explicitly we mean that given $A \xrightarrow{f} B \xrightarrow{g} C$ and $A' \xrightarrow{f'} B' \xrightarrow{g'} C'$ the first and last diagram in 11.1.7 agree.*

Proof. The following pasting diagrams agree.

$$\begin{array}{ccc}
\underline{\mathcal{C}}(C) \times \underline{\mathcal{C}}(C') \xrightarrow{\lambda} \underline{\mathcal{C}}(C \amalg C') & & \underline{\mathcal{C}}(C) \times \underline{\mathcal{C}}(C') \xrightarrow{\lambda} \underline{\mathcal{C}}(C \amalg C') \\
g^* \times g'^* \downarrow \swarrow \Lambda \Downarrow & & g^* \times g'^* \downarrow \swarrow \iota^* \Leftarrow \eta = \parallel \\
\underline{\mathcal{C}}(B) \times \underline{\mathcal{C}}(B') \xrightarrow{\lambda} \underline{\mathcal{C}}(B \amalg B') & & \underline{\mathcal{C}}(B) \times \underline{\mathcal{C}}(B') \xrightarrow{\lambda} \underline{\mathcal{C}}(B \amalg B') \\
f^* \times f'^* \downarrow \swarrow \Lambda \Downarrow & & \parallel \Leftarrow \epsilon = \iota^* \swarrow \downarrow (g \amalg g')^* \\
\underline{\mathcal{C}}(A) \times \underline{\mathcal{C}}(A') \xrightarrow{\lambda} \underline{\mathcal{C}}(A \amalg A') & & \underline{\mathcal{C}}(B) \times \underline{\mathcal{C}}(B') \xrightarrow{\lambda} \underline{\mathcal{C}}(B \amalg B') \\
& & f^* \times f'^* \downarrow \swarrow \iota^* \Leftarrow \eta = \parallel \\
& & \underline{\mathcal{C}}(A) \times \underline{\mathcal{C}}(A') \xrightarrow{\lambda} \underline{\mathcal{C}}(A \amalg A') \\
& & \parallel \Leftarrow \epsilon = \iota^* \swarrow \downarrow (f \amalg f')^* \\
& & \underline{\mathcal{C}}(A) \times \underline{\mathcal{C}}(A') \xrightarrow{\lambda} \underline{\mathcal{C}}(A \amalg A') \tag{11.1.7}
\end{array}$$

$$\begin{array}{ccc}
\underline{\mathcal{C}}(C) \times \underline{\mathcal{C}}(C') \xrightarrow{\lambda} \underline{\mathcal{C}}(C \amalg C') & & \underline{\mathcal{C}}(C) \times \underline{\mathcal{C}}(C') \xrightarrow{\lambda} \underline{\mathcal{C}}(C \amalg C') \\
(gf)^* \times (g'f')^* \downarrow \swarrow \iota^* \Leftarrow \eta = \parallel & & (gf)^* \times (g'f')^* \downarrow \swarrow \Lambda \Downarrow \\
\underline{\mathcal{C}}(A) \times \underline{\mathcal{C}}(A') \xrightarrow{\lambda} \underline{\mathcal{C}}(A \amalg A') & & \underline{\mathcal{C}}(A) \times \underline{\mathcal{C}}(A') \xrightarrow{\lambda} \underline{\mathcal{C}}(A \amalg A') \\
\parallel \Leftarrow \epsilon = \iota^* \swarrow \downarrow (gf \amalg g'f')^* & & \\
\underline{\mathcal{C}}(A) \times \underline{\mathcal{C}}(A') \xrightarrow{\lambda} \underline{\mathcal{C}}(A \amalg A') & &
\end{array}$$

To go from the first to the second we expand the Λ s. From the second to the third we apply the triangle identities and use the functoriality of restrictions. From the third to the fourth we contract Λ . □

Definition 11.1.9. (Bonventre) Finally, we define a \mathbf{Fin}^G -category as a weakly additive split fibration over \mathbf{Fin}^G , with non-empty fibers.

A *pseudo- \mathbf{Fin}^G -category* is then a weakly additive fibration over \mathbf{Fin}^G with a chosen normal cleavage and non-empty fibers.

We call these *strong* (resp. *strict*) if it is strongly (resp. strictly) additive.

Remark 11.1.10. At first glance the additivity condition imposed by λ appears to be a sort of Segal condition and sufficient to create a monoidal structure on $\underline{\mathcal{C}}$. The rough intuition

behind this is that a monoidal structure on $\underline{\mathcal{C}}(A)$ is induced by

$$\underline{\mathcal{C}}(A) \times \underline{\mathcal{C}}(A) \xrightarrow{\sim} \underline{\mathcal{C}}(A \amalg A) \dashrightarrow \underline{\mathcal{C}}(A)$$

where the dashed arrow is induced by the fold map. However this is misleading, as $\mathbf{Fin}^G\text{-Cats}$ have *contravariant* functors from \mathbf{Fin}^G , such a map does not exist. We will see later in Proposition 13.0.16, that the *covariant* transfer maps in an SGA induce such a structure.

Instead the proper way to view this condition is that it tells us that a \mathbf{Fin}^G -category is essentially determined by its values on \mathcal{O}_G up to equivalence of categories (resp. isomorphism) in a coherent way.

Definition 11.1.11. A *morphism of \mathbf{Fin}^G -categories* is a morphism of fibrations over \mathbf{Fin}^G , meaning a functor over \mathbf{Fin}^G sending cartesian arrows to cartesian arrows.

A *split morphism of \mathbf{Fin}^G -categories* is a morphism of split fibrations over \mathbf{Fin}^G , sending chosen cartesian arrows to chosen cartesian arrows.

We recall that this induces pseudofunctors between the respective fibers, and functors in the case of a split morphism.

Definition 11.1.12. An *additive morphism of \mathbf{Fin}^G -categories* is one which is natural with respect to the respective λ s. I.e. the diagram 11.1.8 strictly commutes.

$$\begin{array}{ccc} \underline{\mathcal{C}}(A) \times \underline{\mathcal{C}}(B) & \xrightarrow{\lambda_{\underline{\mathcal{C}}}} & \underline{\mathcal{C}}(A \amalg B) \\ F \downarrow & & \downarrow F \\ \underline{\mathcal{D}}(A) \times \underline{\mathcal{D}}(B) & \xrightarrow{\lambda_{\underline{\mathcal{D}}}} & \underline{\mathcal{D}}(A \amalg B) \end{array} \quad (11.1.8)$$

Note that in order for Φ to be natural with respect to A, B, η and ϵ must be suitably natural as well.

Lemma 11.1.13. *The additivity square 11.1.8 commutes up to isomorphism.*

Proof. We define Φ as the following pasting diagram, where the central natural isomorphism comes from morphisms of (non-split) fibrations commuting with restriction up to isomorphism.

$$\begin{array}{ccc}
 \underline{\mathcal{C}}(A) \times \underline{\mathcal{C}}(B) & & \\
 \parallel & \searrow \lambda & \\
 & \xRightarrow{\eta} & \underline{\mathcal{C}}(A \amalg B) \\
 & \swarrow \iota^* & \\
 \underline{\mathcal{C}}(A) \times \underline{\mathcal{C}}(B) & & \downarrow F \\
 & \xRightarrow{\cong} & \underline{\mathcal{D}}(A \amalg B) \\
 \downarrow F & & \parallel \\
 \underline{\mathcal{D}}(A) \times \underline{\mathcal{D}}(B) & \xRightarrow{\epsilon} & \underline{\mathcal{D}}(A \amalg B) \\
 & \swarrow \lambda_D & \\
 & & \parallel \\
 & & \underline{\mathcal{D}}(A \amalg B)
 \end{array}
 \tag{11.1.9}$$

□

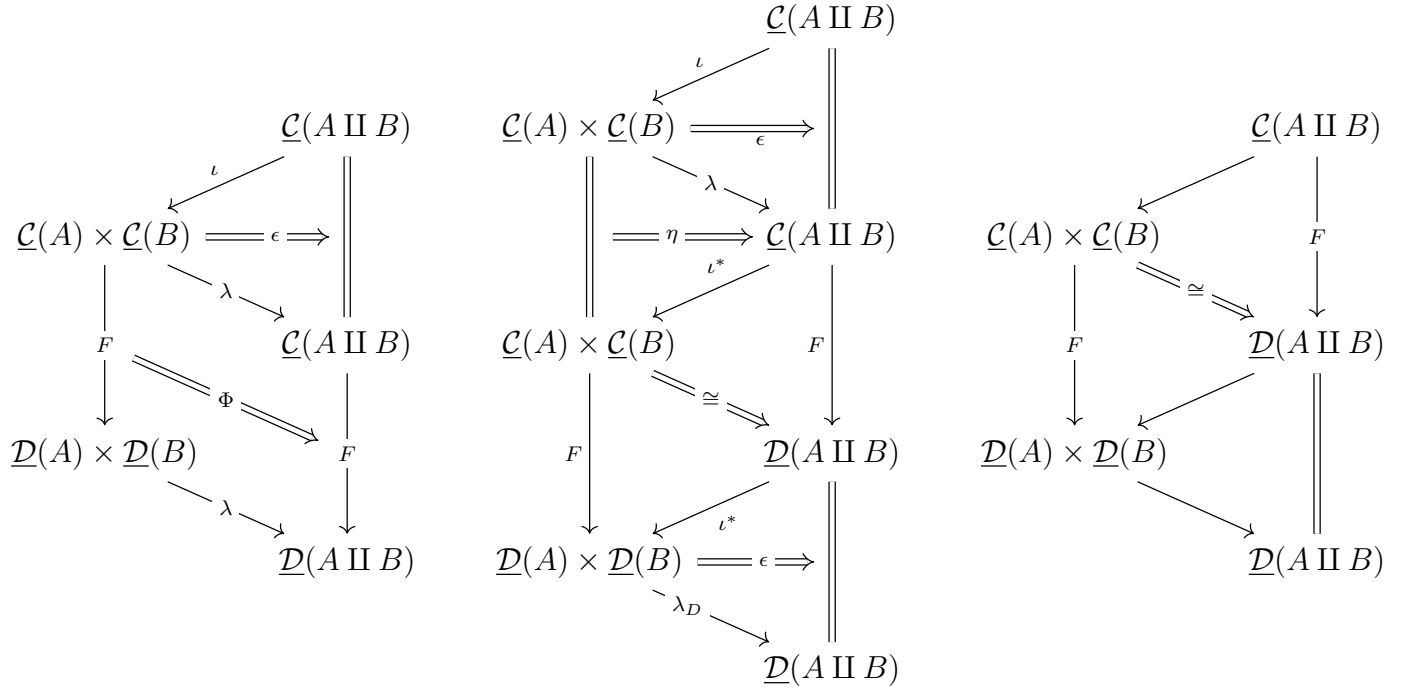
Corollary 11.1.14. *If \mathcal{C}, \mathcal{D} are strongly additive and F is split, then F is automatically additive.*

Lemma 11.1.15. *The following three pasting diagrams are equal:*

$$\begin{array}{ccc}
 \underline{\mathcal{C}}(A) \times \underline{\mathcal{C}}(B) & & \underline{\mathcal{C}}(A) \times \underline{\mathcal{C}}(B) & & \underline{\mathcal{C}}(A) \times \underline{\mathcal{C}}(B) \\
 \downarrow F & \searrow \lambda_C & \parallel & \searrow \lambda & \parallel & \searrow \lambda_C \\
 \underline{\mathcal{D}}(A) \times \underline{\mathcal{D}}(B) & \xRightarrow{\Phi} & \underline{\mathcal{C}}(A) \times \underline{\mathcal{C}}(B) & \xRightarrow{\eta} & \underline{\mathcal{D}}(A) \times \underline{\mathcal{D}}(B) & \xRightarrow{\eta} & \underline{\mathcal{C}}(A \amalg B) \\
 \parallel & \searrow \lambda_D & \downarrow F & \swarrow \iota^* & \downarrow F & \swarrow \iota^* & \downarrow F \\
 \underline{\mathcal{D}}(A) \times \underline{\mathcal{D}}(B) & \xRightarrow{\eta} & \underline{\mathcal{D}}(A \amalg B) & \xRightarrow{\cong} & \underline{\mathcal{D}}(A \amalg B) & \xRightarrow{\cong} & \underline{\mathcal{D}}(A \amalg B) \\
 \parallel & \swarrow \iota^* & \parallel & \swarrow \iota^* & \parallel & \swarrow \iota^* & \parallel \\
 \underline{\mathcal{D}}(A) \times \underline{\mathcal{D}}(B) & \xRightarrow{\eta} & \underline{\mathcal{D}}(A) \times \underline{\mathcal{D}}(B) & \xRightarrow{\epsilon} & \underline{\mathcal{D}}(A) \times \underline{\mathcal{D}}(B) & \xRightarrow{\eta} & \underline{\mathcal{D}}(A \amalg B) \\
 \parallel & \swarrow \lambda_D & \parallel & \swarrow \lambda_D & \parallel & \swarrow \lambda_D & \parallel \\
 \underline{\mathcal{D}}(A) \times \underline{\mathcal{D}}(B) & \xRightarrow{\eta} & \underline{\mathcal{D}}(A) \times \underline{\mathcal{D}}(B) & \xRightarrow{\eta} & \underline{\mathcal{D}}(A) \times \underline{\mathcal{D}}(B) & \xRightarrow{\eta} & \underline{\mathcal{D}}(A \amalg B) \\
 \parallel & \swarrow \iota^* & \parallel & \swarrow \iota^* & \parallel & \swarrow \iota^* & \parallel \\
 \underline{\mathcal{D}}(A) \times \underline{\mathcal{D}}(B) & \xRightarrow{\eta} & \underline{\mathcal{D}}(A) \times \underline{\mathcal{D}}(B) & \xRightarrow{\eta} & \underline{\mathcal{D}}(A) \times \underline{\mathcal{D}}(B) & \xRightarrow{\eta} & \underline{\mathcal{D}}(A \amalg B)
 \end{array}$$

Proof. To go from the first to second we simply expand Φ . To go from the second to the third, we see that the bottom two cells for the triangle identity of the ι, λ adjunction, so they paste to the identity 2-cell. □

Lemma 11.1.16. *The following three diagrams are equal:*



Proof. Again we expand Φ then use the triangle identity. □

We can also display this by saying these two triangular prism-shaped 2-diagrams commute, where their rear face is the identity 2-cell:

$$\begin{array}{ccc}
\underline{\mathcal{C}}(A) \times \underline{\mathcal{C}}(B) & \xrightarrow{\quad \epsilon \quad} & \underline{\mathcal{C}}(A) \times \underline{\mathcal{C}}(B) \\
\downarrow F & \searrow \lambda_{\underline{\mathcal{C}}} & \nearrow \iota^* \\
& \underline{\mathcal{C}}(A \amalg B) & \\
& \downarrow F & \\
\underline{\mathcal{D}}(A) \times \underline{\mathcal{D}}(B) & \xrightarrow{\quad \epsilon \quad} & \underline{\mathcal{D}}(A) \times \underline{\mathcal{D}}(B) \\
& \searrow \lambda_{\underline{\mathcal{D}}} & \nearrow \iota^* \\
& \underline{\mathcal{D}}(A \amalg B) &
\end{array}
\quad
\begin{array}{ccc}
\underline{\mathcal{C}}(A \amalg B) & \xrightarrow{\quad \eta \quad} & \underline{\mathcal{C}}(A \amalg B) \\
\downarrow F & \searrow \iota^* & \nearrow \lambda_{\underline{\mathcal{C}}} \\
& \underline{\mathcal{C}}(A) \times \underline{\mathcal{C}}(B) & \\
& \downarrow F & \\
\underline{\mathcal{D}}(A \amalg B) & \xrightarrow{\quad \eta \quad} & \underline{\mathcal{D}}(A \amalg B) \\
& \searrow \iota^* & \nearrow \lambda_{\underline{\mathcal{D}}} \\
& \underline{\mathcal{D}}(A) \times \underline{\mathcal{D}}(B) &
\end{array}$$

(11.1.10)

Example 11.1.17.

$$Id : \mathbf{Fin}^G \rightarrow \mathbf{Fin}^G$$

is a strong \mathbf{Fin}^G -category corresponding to the constant functor $(\mathbf{Fin}^G)^{op} \rightarrow * \subset \mathbf{Cat}$.

Notation 11.1.18. When viewing \mathbf{Fin}^G in this way as a \mathbf{Fin}^G -category, we denote it \ast .

Example 11.1.19 (Bonventre). Given a *coefficient system* $\underline{\mathcal{C}} : \mathcal{O}_G^{op} \rightarrow \mathbf{Cat}$ we can extend it to a strict \mathbf{Fin}^G -category by $\underline{\mathcal{C}}^+(A) := \prod_{U \in A/G} \underline{\mathcal{C}}(U)$. Restrictions f^* are given by the composite

$$\prod_{B/G} \underline{\mathcal{C}}(V) \longrightarrow \prod_{A/G} \underline{\mathcal{C}}(f(U)) \xrightarrow{(f_U^*)} \prod_{A/G} \underline{\mathcal{C}}(U)$$

with $f_U : U \rightarrow f(U)$.

Example 11.1.20. Let \mathcal{C} be a category with a G action; by taking fixed points it defines a coefficient system $G/H \mapsto \mathcal{C}^H$, with restrictions given by inclusion. Using the above example it then gives a strict \mathbf{Fin}^G category.

11.2 $\hat{\Sigma}_G$ and $\hat{\Sigma}_G \wr (-)$

Definition 11.2.1. Let $\hat{\Sigma}_G \rightarrow \text{Fin}^G$ be the fibration given by the Grothendieck construction on the functor $B \mapsto \text{Fin}_{/B, \text{ord}}^G$ the subcategory of the slice category over B consisting of only order-preserving arrows $A \twoheadrightarrow B$. On morphisms $f : B' \rightarrow B$ it gives the functor sending $(p : A \twoheadrightarrow B) \mapsto f^*p : A \times_B B' \twoheadrightarrow B$. Horizontal pasting and partial unitality suffice to make this functorial.

So then $\hat{\Sigma}_G$ has objects $A \twoheadrightarrow B$ and morphisms are pullback squares.

Lemma 11.2.2. $\hat{\Sigma}_G$ is a strong Fin^G -category.

Proof. This is a split fibration as it comes from a Grothendieck construction.

Showing it is strongly additive takes a bit more work. Let $A, B \in \text{Fin}^G$. The functor $\hat{\Sigma}_G \wr \underline{\mathcal{C}}(A \amalg B) \xrightarrow{\iota_A^* \times \iota_B^*} \hat{\Sigma}_G \wr \underline{\mathcal{C}}(A) \times \hat{\Sigma}_G \wr \underline{\mathcal{C}}(B)$ sends

$$(p : X \twoheadrightarrow A \amalg B) \mapsto ((p^{-1}(A \twoheadrightarrow A), (p^{-1}(B) \twoheadrightarrow B)).$$

Its inverse λ is

$$((X \twoheadrightarrow A), (Y \twoheadrightarrow B)) \mapsto (X \amalg Y \twoheadrightarrow A \amalg B).$$

Associativity and commutativity of λ is clear. $\text{Fin}_{/\emptyset}^G$ is the subcategory of Fin^G consisting of only $(Id : \emptyset \twoheadrightarrow \emptyset)$. □

Definition 11.2.3. For $\underline{\mathcal{C}} \in \text{Fin}^G - \text{Cat}$, we define the wreath product $\hat{\Sigma}_G \wr \underline{\mathcal{C}}$ to be the

following 1-pullback:

$$\begin{array}{ccc}
 \hat{\Sigma}_G \wr \underline{\mathcal{C}} & \longrightarrow & \underline{\mathcal{C}} \\
 \downarrow & & \downarrow \\
 \hat{\Sigma}_G & \xrightarrow{s} & \mathbf{Fin}^G
 \end{array} \tag{11.2.1}$$

where s is the source map $(A \twoheadrightarrow C) \mapsto A$. Objects are tuples $(A \twoheadrightarrow B, x)$ with $x \in \underline{\mathcal{C}}(A)$, and arrows pullback squares in \mathbf{Fin}^G and a map $x' \rightarrow x$ in $\underline{\mathcal{C}}$ over $A' \rightarrow A$.

Proposition 11.2.4. *This is again a \mathbf{Fin}^G -category via the composite*

$$\hat{\Sigma}_G \wr \underline{\mathcal{C}} \rightarrow \hat{\Sigma}_G \xrightarrow{t} \mathbf{Fin}^G$$

Furthermore, this construction is functorial in $\underline{\mathcal{C}}$ with respect to (split, additive) morphisms of fibrations. In the case that $\underline{\mathcal{C}}$ is a pseudo- \mathbf{Fin}^G -category then $\hat{\Sigma}_G \wr \underline{\mathcal{C}}$ is as well.

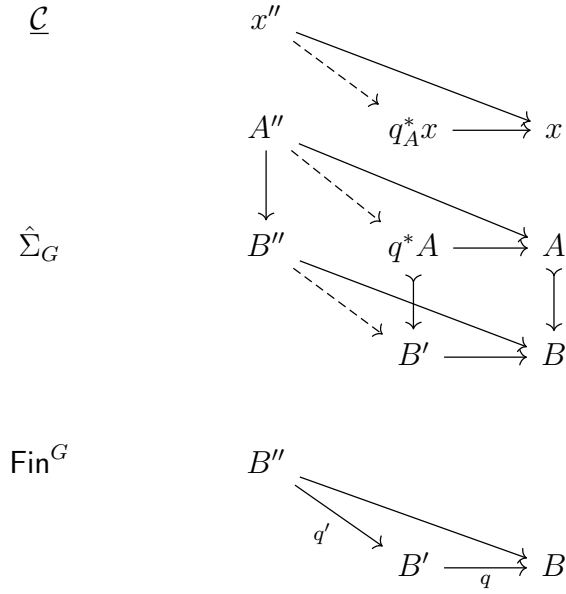
Proof. This clearly defines a functor $\hat{\Sigma}_G \wr \underline{\mathcal{C}} \rightarrow \mathbf{Fin}^G$. For $q : B' \rightarrow B$ in \mathbf{Fin}^G and $(A \twoheadrightarrow B, x)$ in $\hat{\Sigma}_G \wr \underline{\mathcal{C}}(B)$ its chosen cartesian lift is

$$q^*(A \twoheadrightarrow B, x) := (q^*A \twoheadrightarrow B', q_A^*x) \tag{11.2.2}$$

where q_A^*x is the source of the chosen cartesian lift of $\underline{\mathcal{C}}$ of $q_A : q_A^* \rightarrow x$. We can also display this visually in a diagram:

$$\begin{array}{ccc}
 (q_A)^* : q_A^*x & \longrightarrow & x \\
 \\
 q^*A & \longrightarrow & A \\
 \downarrow & & \downarrow \\
 B' & \xrightarrow{q} & B
 \end{array} \tag{11.2.3}$$

Checking that this is in fact cartesian, we consider $q' : B'' \rightarrow B'$ and a pullback/commuting square A'', A, B'', B above qq'' as well as $x'' \in \underline{\mathcal{C}}(B'')$ and a map in $\underline{\mathcal{C}}$ $x'' \rightarrow x$ above qq' .



Now to confirm existence and uniqueness, q' is the only map $B'' \rightarrow B'$ here; by the universal property of the pullback, $A'' \rightarrow q^*A$ is uniquely determined. $x'' \rightarrow x$ then exists and is unique by virtue of $\underline{\mathcal{C}}$ being a fibration.

We confirm that this is a split fibration and $(Id)^* = Id$ as $\underline{\mathcal{C}}$ is assumed to be a normal split fibration. and because lex-pullback is suitably unital and transitive. Unitality is also easy to check.

In the case of $\underline{\mathcal{C}}$ a pseudo- Fin^G -category, this gives us a normal cleavage instead of a split one.

Finally, this is weakly (resp. strongly) additive whenever $\underline{\mathcal{C}}$ is as in $\hat{\Sigma}_G \wr \underline{\mathcal{C}}$, $\iota_A^* \times \iota_B^*$ is given by

$$(p : X \twoheadrightarrow A \amalg B, x) \mapsto ((p^{-1}(A) \twoheadrightarrow A, \iota_{\underline{\mathcal{C}}, A}^* x), (p^{-1}(B) \twoheadrightarrow B, \iota_{\underline{\mathcal{C}}, B}^* x)).$$

With (resp. strict) inverse $\lambda_{\hat{\Sigma}_G \wr \underline{\mathcal{C}}}$

$$((X \twoheadrightarrow A, x), (Y \twoheadrightarrow B, y)) \mapsto (X \amalg Y \twoheadrightarrow A \amalg B, \lambda_{\underline{\mathcal{C}}}(x, y)).$$

For $\hat{\Sigma}_G \wr \underline{\mathcal{C}}$, η and ϵ are defined as:

$$\eta_x : (p : X \twoheadrightarrow A \amalg B, x) \xrightarrow{Id, \eta_{\underline{\mathcal{C}}}} (p : X \twoheadrightarrow A \amalg B, \lambda_{\underline{\mathcal{C}}}(\iota_{\underline{\mathcal{C}}, A}^* x, \iota_{\underline{\mathcal{C}}, B}^* x))$$

$$\epsilon_{(x, y)} : ((X \twoheadrightarrow A, \iota_{\underline{\mathcal{C}}, A}^*(\lambda_{\underline{\mathcal{C}}}(x, y))), (Y \twoheadrightarrow B, \iota_{\underline{\mathcal{C}}, B}^*(\lambda_{\underline{\mathcal{C}}}(x, y)))) \xrightarrow{Id, \epsilon_{\underline{\mathcal{C}}}} ((X \twoheadrightarrow A, x), (Y \twoheadrightarrow B, y))$$

These satisfy the adjunction triangle identities, following directly from the fact that $\eta_{\underline{\mathcal{C}}}$ and $\epsilon_{\underline{\mathcal{C}}}$ do; similarly for being preserved by isomorphisms.

Associativity and commutativity of λ follow from that of $\lambda_{\underline{\mathcal{C}}}$, as do the coherence conditions on the η, ϵ .

The fiber over \emptyset is $\{(Id : \emptyset \twoheadrightarrow \emptyset)\} \times \underline{\mathcal{C}}(\emptyset)$ which is nonempty.

For $F : \underline{\mathcal{C}} \rightarrow \underline{\mathcal{D}}$ a morphism (resp. split, additive), gives $\hat{\Sigma}_G \wr F : \hat{\Sigma}_G \wr \underline{\mathcal{D}} \rightarrow \hat{\Sigma}_G \wr \underline{\mathcal{C}}$ is a morphism (resp. split, additive).

This sends $(A \twoheadrightarrow B, x) \mapsto (A \twoheadrightarrow B, F(x))$ on objects, and on morphisms $f : x' \rightarrow x$ to the identity on squares and $F(f) : F(x') \rightarrow F(x)$. It is clear this is all over Fin^G . As F is a morphism of (split) fibrations $\hat{\Sigma}_G \wr F$ is as well. To see this is additive, we observe from the description of λ that it commutes with $\hat{\Sigma}_G \wr F$ when F is additive.

□

Example 11.2.5. As a simple example we have $\hat{\Sigma}_G \wr \mathbf{Fin}^G = \hat{\Sigma}_G$.

Example 11.2.6. $\hat{\Sigma}_G \wr \hat{\Sigma}_G$ consists of composable order-preserving G -maps $(A_1 \twoheadrightarrow A_0 \twoheadrightarrow A_{-1})$, and morphisms are stacks of pullback squares. The left vertical and top horizontal maps of 11.2.1 delete A_1 and A_{-1} , respectively.

Example 11.2.7. $\hat{\Sigma}_G^{\wr k+1} \wr \underline{\mathcal{C}} = \hat{\Sigma}_G \wr (\hat{\Sigma}_G^{\wr k} \wr \underline{\mathcal{C}})$ consists of pairs $(A_k \twoheadrightarrow \dots \twoheadrightarrow A_0 \twoheadrightarrow A_{-1}, x \in \underline{\mathcal{C}}(A_k))$ of a string of maps and an element of the appropriate fiber; morphisms are stacks of pullback squares and a map $x' \rightarrow q_k^*x$. Our convention is that $\hat{\Sigma}_G^{\wr 0} \wr \underline{\mathcal{C}} = \underline{\mathcal{C}}$.

Remark 11.2.8. The intuition behind this construction is that the ordered maps can be thought of as operations.

Proposition 11.2.9. (*Bonventre*) *The iterated wreath products $\hat{\Sigma}_G^{\wr \bullet+1} \wr \underline{\mathcal{C}}$ form a coaugmented cosimplicial object in $\mathbf{Fin}^G\text{-Cat}$ (resp. with split, additive morphisms). This also holds for pseudo- \mathbf{Fin}^G -categories. Its cofaces are*

$$\delta_i: \hat{\Sigma}_G^{\wr k+1} \wr \underline{\mathcal{C}} \rightarrow \hat{\Sigma}_G^{\wr k+2} \wr \underline{\mathcal{C}}$$

inserting equalities $A_i = A_i$ for $-1 \leq i \leq k$, and codegeneracies

$$\sigma_i: \hat{\Sigma}_G^{\wr k+1} \wr \underline{\mathcal{C}} \rightarrow \hat{\Sigma}_G^{\wr k} \wr \underline{\mathcal{C}}$$

deleting A_i and composing the adjacent maps for $0 \leq i \leq k$.

Moreover, $\hat{\Sigma}_G \wr \sigma_i = \sigma_{i+1}$ and $\hat{\Sigma}_G \wr \delta_i = \delta_{i+1}$.

Furthermore this is functorial in $\underline{\mathcal{C}}$ with (split, additive) morphisms of (pseudo-) \mathbf{Fin}^G -categories.

Corollary 11.2.10.

$$(\hat{\Sigma}_G \wr (-), \sigma_0, \delta_{-1})$$

forms a monad in (pseudo-)Fin^G-categories (resp. with split, additive morphisms).

For explicitness, we observe that $\delta_{-1} : \underline{\mathcal{C}} \rightarrow \hat{\Sigma}_G \wr \underline{\mathcal{C}}$ sends $x \in \underline{\mathcal{C}}(A)$ to $(Id : A \rightarrow A, x) \in \hat{\Sigma}_G \wr \underline{\mathcal{C}}(A)$.

Chapter 12

Defining $\hat{\Sigma}_G$ -algebras

Definition 12.0.1. (Bonventre) A $\hat{\Sigma}_G$ -algebra or SGA¹ is a pseudoalgebra over the monad $\hat{\Sigma}_G \wr (-)$ in the category of Fin^G -categories.

First we write this out in full detail. An SGA consists of a Fin^G -category $\underline{\mathcal{V}} \rightarrow \text{Fin}^G$, a *multiplication* map of fibrations $\otimes : \hat{\Sigma}_G \wr \underline{\mathcal{V}} \rightarrow \underline{\mathcal{V}}$

$$\begin{array}{ccc} \hat{\Sigma}_G \wr \underline{\mathcal{V}} & \xrightarrow{\otimes} & \underline{\mathcal{V}} \\ & \searrow & \swarrow \\ & \text{Fin}^G & \end{array}$$

We denote the image of $(f : A \mapsto B, x)$ in $\hat{\Sigma}_G \wr \underline{\mathcal{C}}(B)$ under \otimes by $\otimes(f, x)$. Along with an associativity natural isomorphism of fibrations α over Fin^G :

$$\begin{array}{ccc} \hat{\Sigma}_G^2 \wr \underline{\mathcal{V}} & \xrightarrow{\hat{\Sigma}_G \wr \otimes} & \hat{\Sigma}_G \wr \underline{\mathcal{V}} \\ \sigma_0 \downarrow & \swarrow \alpha & \downarrow \otimes \\ \hat{\Sigma}_G \wr \underline{\mathcal{V}} & \xrightarrow{\otimes} & \underline{\mathcal{V}} \end{array} \tag{12.0.1}$$

¹We choose this name for clarity and to disambiguate it from terms like “genuine equivariant symmetric monoidal category”.

and a unitality natural isomorphism of fibrations ² ω over \mathbf{Fin}^G :

$$\begin{array}{ccc}
 \underline{\mathcal{V}} & & \\
 \delta_{-1} \downarrow & \searrow \omega & \\
 \hat{\Sigma}_G \wr \underline{\mathcal{V}} & \xrightarrow{\otimes} & \underline{\mathcal{V}}
 \end{array} \tag{12.0.2}$$

We recall that α, ω being natural transformations of fibrations means that objectwise lie over identities in \mathbf{Fin}^G . They are also subject to the relations that these pasting diagrams are equal:

$$\begin{array}{ccc}
 \hat{\Sigma}_G^{\wr 3} \wr \underline{\mathcal{V}} & \xrightarrow{\hat{\Sigma}_G^{\wr 2} \wr \otimes} & \hat{\Sigma}_G^{\wr 2} \wr \underline{\mathcal{V}} & \xrightarrow{\hat{\Sigma}_G \wr \otimes} & \hat{\Sigma}_G \wr \underline{\mathcal{V}} & \xrightarrow{\otimes} & \underline{\mathcal{V}} \\
 \sigma_0 \downarrow & & \sigma_0 \downarrow & \swarrow \alpha & & & \parallel \\
 \hat{\Sigma}_G^{\wr 2} \wr \underline{\mathcal{V}} & \xrightarrow{\hat{\Sigma}_G \wr \otimes} & \hat{\Sigma}_G \wr \underline{\mathcal{V}} & \xrightarrow{\otimes} & \underline{\mathcal{V}} & & \parallel \\
 \sigma_0 \downarrow & & & \swarrow \alpha & & & \parallel \\
 \hat{\Sigma}_G \wr \underline{\mathcal{V}} & \xrightarrow{\otimes} & \underline{\mathcal{V}} & & \underline{\mathcal{V}} & & \parallel
 \end{array}
 \qquad
 \begin{array}{ccc}
 \hat{\Sigma}_G^{\wr 3} \wr \underline{\mathcal{V}} & \xrightarrow{\hat{\Sigma}_G^{\wr 2} \wr \otimes} & \hat{\Sigma}_G^{\wr 2} \wr \underline{\mathcal{V}} & \xrightarrow{\hat{\Sigma}_G \wr \otimes} & \hat{\Sigma}_G \wr \underline{\mathcal{V}} & \xrightarrow{\otimes} & \underline{\mathcal{V}} \\
 \sigma_1 \downarrow & & & \swarrow \hat{\Sigma}_G \wr \alpha & & & \parallel \\
 \hat{\Sigma}_G^{\wr 2} \wr \underline{\mathcal{V}} & \xrightarrow{\hat{\Sigma}_G \wr \otimes} & \hat{\Sigma}_G \wr \underline{\mathcal{V}} & \xrightarrow{\otimes} & \underline{\mathcal{V}} & & \parallel \\
 \sigma^0 \downarrow & & & \swarrow \alpha & & & \parallel \\
 \hat{\Sigma}_G \wr \underline{\mathcal{V}} & \xrightarrow{\otimes} & \underline{\mathcal{V}} & & \underline{\mathcal{V}} & & \parallel
 \end{array}$$

and that these two are equal as well:

$$\begin{array}{ccc}
 \hat{\Sigma}_G \wr \underline{\mathcal{V}} & \xrightarrow{\otimes} & \underline{\mathcal{V}} & \xrightarrow{\omega} & \underline{\mathcal{V}} \\
 \delta_0 \downarrow & & \delta_0 \downarrow & \swarrow \omega & \parallel \\
 \hat{\Sigma}_G^{\wr 2} \wr \underline{\mathcal{V}} & \xrightarrow{\hat{\Sigma}_G \wr \otimes} & \hat{\Sigma}_G \wr \underline{\mathcal{V}} & \xrightarrow{\otimes} & \underline{\mathcal{V}} \\
 \sigma_0 \downarrow & & & \swarrow \alpha & \parallel \\
 \hat{\Sigma}_G \wr \underline{\mathcal{V}} & \xrightarrow{\otimes} & \underline{\mathcal{V}} & & \underline{\mathcal{V}}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \hat{\Sigma}_G \wr \underline{\mathcal{V}} & \xrightarrow{\omega} & \hat{\Sigma}_G \wr \underline{\mathcal{V}} & \xrightarrow{\otimes} & \underline{\mathcal{V}} \\
 \delta_{-1} \downarrow & & & \swarrow \omega & \parallel \\
 \hat{\Sigma}_G^{\wr 2} \wr \underline{\mathcal{V}} & \xrightarrow{\otimes} & \hat{\Sigma}_G \wr \underline{\mathcal{V}} & \xrightarrow{\otimes} & \underline{\mathcal{V}} \\
 \sigma_0 \downarrow & & & \swarrow \alpha & \parallel \\
 \hat{\Sigma}_G \wr \underline{\mathcal{V}} & \xrightarrow{\otimes} & \underline{\mathcal{V}} & & \underline{\mathcal{V}}
 \end{array}$$

Definition 12.0.2. There are many possible strengthenings and weakenings of this structure:

(a) We say an SGA is *split* (resp. *additive*) if the action map $\hat{\Sigma}_G \wr \underline{\mathcal{V}} \xrightarrow{\otimes} \underline{\mathcal{V}}$ is split (resp. additive) as a map of \mathbf{Fin}^G -categories.

(b) We say an SGA is *strict* if it is in fact an algebra over $\hat{\Sigma}_G \wr (-)$.

²We call it ω as η is taken.

(c) If the underlying \mathbf{Fin}^G -category of $\underline{\mathcal{V}}$ is strongly (resp. strictly) additive, we say $\underline{\mathcal{V}}$ is *strongly* (resp. *strictly*) *additive as a \mathbf{Fin}^G category*.³

(d) If $\underline{\mathcal{V}}$ is split, strict, and additive we call it *permutative*.

(e) In the case that $\underline{\mathcal{V}}$ is a pseudo- \mathbf{Fin}^G -category we call it a *weak SGA*.

Remark 12.0.3. If $\underline{\mathcal{V}}$ is split and strongly additive as a \mathbf{Fin}^G -category then it is additive.

Remark 12.0.4. Intuitively, \otimes is a way of turning genuine equivariant operations combinatorially encoded by maps of G -sets $B \mapsto A$ into functors $\underline{\mathcal{V}}(B) \rightarrow \underline{\mathcal{V}}(A)$. As before, the restriction to order-preserving maps does not reflect any deeper meaning, it is merely so technical parts work out cleanly.

Conjecture 12.0.5. Given an SGA we can strictify it to an equivalent permutative SGA.

As a rough sketch we first make \otimes additive, which we could do by making $\underline{\mathcal{V}}$ and thus $\hat{\Sigma}_G \wr \underline{\mathcal{V}}$ strongly additive as \mathbf{Fin}^G -categories. We can attempt to do this by first restricting to the fibers over \mathcal{O}_G , then expanding to strongly additive ones via the $(-)^+$ construction. Next we modify \otimes into a split morphism, perhaps by replacing $\hat{\Sigma}_G \wr \underline{\mathcal{V}}$ or $\underline{\mathcal{V}}$ with an equivalent fibration with a different choice of cleavage. Finally we strictify from pseudo-algebras to algebras, perhaps using results of [Lac02] or [BKP89].

³We recall that all \mathbf{Fin}^G -categories are at least weakly additive.

12.1 Examples

Example 12.1.1. Given a (split, additive) \mathbf{Fin}^G -category $\underline{\mathcal{C}}$, $\hat{\Sigma}_G \wr \underline{\mathcal{C}}$ is also a (split, additive) strict SGA. This we view as the free SGA on a \mathbf{Fin}^G -category as it is left adjoint to the forgetful functor from SGAs to \mathbf{Fin}^G -categories.

Example 12.1.2. $\underline{\mathbf{Fin}}^G := \hat{\Sigma}_G \wr \mathbf{Fin}^G = \hat{\Sigma}_G$ is an example of this.

This is a prototypical example in the sense that \mathbf{Fin} is the prototypical symmetric monoidal category. Intuitively it should be thought of as the equivariant symmetric monoidal category \mathbf{Fin}^G viewed as an SGA. Indeed $\underline{\mathbf{Fin}}^G(G/H) \simeq \mathbf{Fin}^H$, with restriction being restriction of subgroups $K \leq H$ and transfers being $X \mapsto X \times_K H$.

Intuitively, this is the free SGA on a single generator.

Example 12.1.3. $Id : \mathbf{Fin}^G \rightarrow \mathbf{Fin}^G$ of Example 11.1.17 is a permutative SGA where

$\otimes : \hat{\Sigma}_G \wr \mathbf{Fin}^G = \hat{\Sigma}_G \xrightarrow{p} \mathbf{Fin}^G$ where p is the fibration map. We denote this by $\underline{*}$ as each fiber is the singleton category.

To see that α is an equality, we observe that for $(A \twoheadrightarrow B \twoheadrightarrow C) \in \hat{\Sigma}_G^2 \wr \mathbf{Fin}^G$ both ways around the diagram ?? give C . For unitality $\delta_{-1} : \mathbf{Fin}^G \rightarrow \hat{\Sigma}_G$ sends $A \mapsto (Id : A \twoheadrightarrow A)$ making it a section which shows ω is an equality.

This is the terminal object in the category of \mathbf{Fin}^G categories. It is also the initial object (in a 2-categorical sense) in the category of SGAs as morphisms of SGAs are unital.

Definition 12.1.4. We say a map of fibrations $\underline{*} \rightarrow \underline{\mathcal{V}}$ is an *object of $\underline{\mathcal{V}}$* . In particular we note it picks out an object of each fiber, which are sent to one another by restrictions. We note that it is equivalent to specify an object of $\underline{\mathcal{V}}(G/G)$ as it is initial in $(\mathbf{Fin}^G)^{op}$.

Lemma 12.1.5. *To define a strict SGA it suffices to give a categorical coefficient system $F : (\mathcal{O}_G)^{op} \rightarrow Perm$ and for each $f : A \rightarrow B$ a functor $f_* : F(A) \rightarrow F(B)$ which is functorial and satisfies a strict double coset formula for lex-pullbacks.*

This defines \otimes , as it strictly satisfies a double coset formula, it defines a natural transformation from $\hat{\Sigma}_G \wr F \Rightarrow F$, viewed a coefficient systems, which then via the Grothendieck construction corresponds to a split map of \mathbf{Fin}^G -categories. As $F, \hat{\Sigma}_G \wr F$ are strongly additive, \otimes is as well.

Remark 12.1.6. We suspect that we can weaken this to defining a pseudo-SGA given a pseudofunctor $F : (\mathcal{O}_G)^{op} \rightarrow SymMon$ and for each $f : A \rightarrow B$ a functor $f_* : F(A) \rightarrow F(B)$ which is pseudofunctorial and satisfies a natural double coset isomorphism.

Remark 12.1.7. We suspect that we can define Top as the pseudo-SGA given via $G/H \mapsto Top^H$. Restriction along $G/K \rightarrow G/H$ is by restriction $BK \hookrightarrow BH \rightarrow Top$. Transfer along $G/K \rightarrow G/H$ is by $X \mapsto X \times_K H$.

Example 12.1.8 ([Hav18]). Mack is defined as the Grothendieck construction applied to $G/H \mapsto Mack_H$. Restrictions agree with $Res_H^G : Mack_G \rightarrow Mack_H$ induced by the forgetful functor $\mathbf{Fin}^G \rightarrow \mathbf{Fin}^H$ and transfers agree with the norm $N_H^G : Mack_H \rightarrow Mack_G$. Using results 3.1.3, 3.1.15, and 3.1.16 of [Hav18] this is a pseudo-SGA.

Remark 12.1.9. I believe that most of [Hav18] could be redone using lex-pullbacks, and the added strictness would allow us to define Mack as an SGA as opposed to a pseudo-SGA.

Example 12.1.10. Sp_G is the pseudo-SGA of genuine G -spectra; $\underline{Sp}_G(G/H) = Sp_H$. Restriction is given by the usual restriction and transfers are given by the HHR norm. By [BH17] §9, we have that the double coset formula is suitably natural.

Remark 12.1.11. As [BH17] work with ∞ -categories, we can only apply their results to the homotopy category of G and H -spectra. So as we have been doing throughout this paper we work with homotopy categories. Similarly, it is not clear how easily we could strictify this, as originally everything is defined up to homotopy.

Example 12.1.12. For a permutative category \mathcal{C} with a G action by strict symmetric monoidal functors. We have an SGA $\underline{\mathcal{C}}$ given by applying the Grothendieck construction to $G/H \mapsto \mathcal{C}^H$. Restrictions are via precomposition. For $f : G/K \rightarrow G/H$, $\otimes f(x)$ is given by $\otimes_{[h_i] \in H/K} h_i(x)$.

Given a symmetric monoidal category with G action by strictly unital strong symmetric monoidal functors, we have a SGA with $G/H \mapsto \mathcal{C}^{hH}$, using Remark 1.0.2.

Example 12.1.13. Consider a semi-Mackey functor $\underline{M} = (M^*, M_*)$ which we view using the second definition.

We define \underline{M} as the Grothendieck construction applied to $M^* : \mathbf{Fin}^G \rightarrow \mathbf{Cat}$ where we identify sets (monoids) and function with discrete categories, (permutative) categories and (strict monoidal) functors. This is a strongly additive \mathbf{Fin}^G -category.

We define $\otimes(f : A \rightarrow B, x)$ as $M_*(f)(x)$. This is a map of \mathbf{Fin}^G -categories following from the double coset formula of Mackey functors and additive as $\underline{M}, \hat{\Sigma}_G \wr \underline{M}$ are strong additive.

Example 12.1.14. For $X \in \mathbf{Fin}^G$ we define $\Sigma_G \wr X$ as the Grothendieck construction applied to $A \mapsto \mathbf{GE}_{ord}(A, X)$ where for $f : B \rightarrow A$ we define $f^* : \mathbf{GE}_{ord}(A, X) \rightarrow \mathbf{GE}_{ord}(B, X)$ as composition with $(B = B \xrightarrow{f} A)$.

We can show this is a strong \mathbf{Fin}^G category using essentially the same proof of Lemma 11.2.2 where we showed $\hat{\Sigma}_G = \Sigma_G \wr *$ is a strong \mathbf{Fin}^G -category; only now we also have maps to X .

We now construct its SGA structure. For $g : A \mapsto A'$, we define $\otimes(g, (A \leftarrow Y \rightarrow X))$ as

the composition $(A \leftarrow Y \rightarrow X) \circ (A' \xleftarrow{g} A = A)$ in \mathbf{GE}_{ord} .

This is a map of split fibrations as it satisfies the double coset formula by composition in \mathbf{GE}_{ord} . As $\Sigma_G \wr X$ is a strong \mathbf{Fin}^G category, $\hat{\Sigma}_G \wr (\Sigma_G \wr X)$ is as well, so additivity of \otimes is automatic. Strict associativity and unitality follow from \mathbf{GE}_{ord} being a strict 2-category.

Example 12.1.15. $\mathbf{Fin}^G \wr \mathcal{C}$ for a permutative category \mathcal{C} is given by the Grothendieck construction on $A \mapsto \prod_{a \in A} \mathcal{C}$. \otimes is given by summing as in \mathcal{C} , in the order given by the ordering of A .

Chapter 13

Structure of SGAs

Notation 13.0.1. For the remainder of this chapter $(\underline{\mathcal{V}}, \otimes, \alpha, \omega)$ will be a fixed SGA. $\underline{\mathcal{V}}$ will be reserved for SGAs and $\underline{\mathcal{C}}$ will denote a generic Fin^G -category.

In this section we prove several key results on the structure of SGAs. The different levels and different types of strength (*pseudo-, strict, split, additive*) will affect the strength of these results. We summarize the main results:

- For an order preserving map $F : A \rightarrow B$ we have a covariant map $f_* : \underline{\mathcal{V}}(A) \rightarrow \underline{\mathcal{V}}(B)$ we call a *transfer* (Definition 13.0.2).
- There is a natural double coset isomorphism (identity for lex-pullbacks if $\underline{\mathcal{V}}$ split) (Proposition 13.0.10, Lemma 13.0.11).
- $\underline{\mathcal{V}}$ is levelwise symmetric monoidal (permutative if additive, split, and strict) (Proposition 13.0.16).
- Restrictions are functorial and strong monoidal (strict monoidal if $\underline{\mathcal{V}}$ split and additive as a Fin^G -category) (Proposition 13.0.19).

- Transfers are pseudofunctorial (functorial if $\underline{\mathcal{V}}$ is strict) and strong monoidal (Lemma 13.0.9).
- For $\underline{\mathcal{V}}$ the double coset isomorphism is a monoidal natural isomorphism (Proposition 13.0.21).
- For $\underline{\mathcal{V}}$ permutative, the double coset isomorphism suitably commutes with λ , restrictions, and transfers (Lemma 13.0.15).

Definition 13.0.2. Given an order preserving map $f : A \rightarrow B$ in \mathbf{Fin}^G we have a function $\gamma_f : \underline{\mathcal{V}}(A) \rightarrow \hat{\Sigma}_G \wr \underline{\mathcal{V}}(B)$ defined by $x \mapsto (f, x)$ on objects. On morphisms it sends $x \rightarrow x'$ to the pullback square shown below along with $x \rightarrow x'$.

$$\begin{array}{ccc} A & \xlongequal{\quad} & A \\ f \downarrow & & \downarrow f \\ B & \xlongequal{\quad} & B \end{array}$$

Remark 13.0.3. We note that this is not a functor over \mathbf{Fin}^G and that it is a fully faithful inclusion.

Lemma 13.0.4. γ is pseudonatural in the sense that given a pullback square in \mathbf{Fin}^G ,

$$\begin{array}{ccc} A' & \xrightarrow{p} & A \\ f \downarrow & & \downarrow g \\ B' & \xrightarrow{q} & B \end{array}$$

There is a natural isomorphism Γ making the following diagram commute up to isomorphism. Γ is also suitably natural. In the case of a lex-pullback square, Γ is the identity and the diagram strictly commutes.

$$\begin{array}{ccc}
V(A) & \xrightarrow{p^*} & \underline{\mathcal{V}}(A') \\
\gamma_g \downarrow & \swarrow \Gamma & \downarrow \gamma_f \\
\hat{\Sigma}_G \wr \underline{\mathcal{V}}(B) & \xrightarrow{\hat{\Sigma}_G \wr (q^*)} & \hat{\Sigma}_G \wr \underline{\mathcal{V}}(B')
\end{array} \tag{13.0.1}$$

Furthermore, Γ is preserved by morphisms of fibrations.

Proof. For $x \in \underline{\mathcal{V}}(A)$, going down then across gives us q^*g, q_A^*x . This is the source of the chosen cartesian morphism in $\hat{\Sigma}_G \wr \underline{\mathcal{V}}$ over q , as displayed in Diagram 11.2.3.

Going across then down yields f, p^*x . In the case the original pullback was a lex-pullback, this is equal to q^*g, q_A^*x . This is the source of the morphism in $\hat{\Sigma}_G \wr \underline{\mathcal{V}}$

$$p^*x \longrightarrow x$$

$$\begin{array}{ccc}
A' & \xrightarrow{p} & A \\
f \downarrow & & g \downarrow \\
B' & \xrightarrow{q} & B
\end{array}$$

which also lies above q . In fact we can show it is a cartesian lift of q , using an argument identical to the one we used to show the first map is cartesian in the proof of Proposition 11.2.4. By the universal property of cartesian lifts there is a unique isomorphism between these two, which is how we define Γ_x .

Furthermore, this is natural in the sense that that given two pullback squares:

$$\begin{array}{ccccc}
A'' & \xrightarrow{p'} & A' & \xrightarrow{p} & A \\
h \downarrow & & f \downarrow & & g \downarrow \\
B'' & \xrightarrow{q'} & B' & \xrightarrow{q} & B
\end{array}$$

the following diagrams agree.

$$\begin{array}{ccc}
V(A) & \xrightarrow{p^*} & V(A') & \xrightarrow{p'^*} & V(A'') \\
\gamma_g \downarrow & \swarrow \Gamma & \downarrow \gamma_f & \swarrow \Gamma & \downarrow \gamma_h \\
SV(B) & \xrightarrow{S(q^*)} & SV(B') & \xrightarrow{S(q'^*)} & SV(B'')
\end{array}
\qquad
\begin{array}{ccc}
V(A) & \xrightarrow{(pp')^*} & V(A'') \\
\gamma_g \downarrow & \swarrow \Gamma & \downarrow \gamma_h \\
SV(B) & \xrightarrow{S((qq')^*)} & SV(B'')
\end{array}$$

This follows from the fact that all three ways to trace through the pasting diagram give sources of cartesian lifts of qq' , there are unique isomorphisms between them and the uniqueness implies the pasted morphism agrees with that of the right hand diagram.

Lastly, a morphism of fibrations $F : \underline{\mathcal{C}} \rightarrow \underline{\mathcal{D}}$ sends cartesian lifts to cartesian lifts, so it must preserve the unique isomorphisms between their sources. □

Definition 13.0.5. Given an order preserving map $f : A \rightarrow B$ in \mathbf{Fin}^G , we have a *transfer* $f_* : \underline{\mathcal{V}}(A) \rightarrow \underline{\mathcal{V}}(B)$ defined as the composite:

$$f_* : \underline{\mathcal{V}}(A) \xrightarrow{\gamma_f} \hat{\Sigma}_G \wr \underline{\mathcal{V}}(B) \xrightarrow{\otimes} \underline{\mathcal{V}}(B) \quad (13.0.2)$$

Remark 13.0.6. We remark that we do not define transfer only for order-preserving map for a deep reason. Rather we want them to interact well with disjoint unions, products, and pullbacks of objects in \mathbf{Fin}^G .

Lemma 13.0.7. *Transfers commute with λ s up to isomorphism (resp. strictly if $\underline{\mathcal{V}}$ is additive).*

First we spell out exactly what we mean by this. Given order preserving maps $f : A \rightarrow A', g : B \rightarrow B'$ then the following diagram commutes up to an isomorphism we call Ψ :

$$\begin{array}{ccc}
\underline{\mathcal{V}}(A) \times \underline{\mathcal{V}}(B) & \xrightarrow{\lambda} & \underline{\mathcal{V}}(A \amalg B) \\
f_* \times g_* \downarrow & \swarrow \Psi & \downarrow (f \amalg g)_* \\
\underline{\mathcal{V}}(A') \times \underline{\mathcal{V}}(B') & \xrightarrow{\lambda} & \underline{\mathcal{V}}(A' \amalg B')
\end{array}$$

Proof. We expand the above diagram and define Ψ as the composite:

$$\begin{array}{ccc}
 \underline{\mathcal{V}}(A) \times \underline{\mathcal{V}}(B) & \xrightarrow{\lambda} & \underline{\mathcal{V}}(A \amalg B) \\
 \downarrow & & \downarrow \\
 \hat{\Sigma}_G \wr \underline{\mathcal{V}}(A') \times \hat{\Sigma}_G \wr \underline{\mathcal{V}}(B') & \xrightarrow{\lambda} & \hat{\Sigma}_G \wr \underline{\mathcal{V}}(A' \amalg B') \\
 \otimes \downarrow & \swarrow \Phi & \otimes \downarrow \\
 \underline{\mathcal{V}}(A') \times \underline{\mathcal{V}}(B') & \xrightarrow{\lambda} & \underline{\mathcal{V}}(A' \amalg B')
 \end{array} \tag{13.0.3}$$

By assumption in the bottom square \otimes commutes with λ up to isomorphism (resp. strictly) so it reduces to showing that in the top square $\underline{\mathcal{V}}(A) \hookrightarrow \hat{\Sigma}_G \wr \underline{\mathcal{V}}(A')$ does as well; in fact it actually commutes strictly:

Given $(x, y) \in \underline{\mathcal{V}}(A) \times \underline{\mathcal{V}}(B)$ if we go right then down we get $(f \amalg g, \lambda_{\underline{\mathcal{V}}}(x, y))$. Going down gives $((f, x), (g, y))$, then across we also get $(f \amalg g, \lambda_{\underline{\mathcal{V}}}(x, y))$ by Lemma 11.2.4.

□

We have a version of the prism lemmas relating λ and transfers.

Lemma 13.0.8. *Consider order preserving maps $f : A \rightarrow A', g : B \rightarrow B'$. The following prism diagrams commute, where the vertical arrows are those of Definition 13.0.5:*

$$\begin{array}{ccc}
 \underline{\mathcal{V}}(A) \times \underline{\mathcal{V}}(B) & \xrightarrow{\lambda} & \underline{\mathcal{V}}(A \amalg B) \\
 \downarrow \gamma_{f \times \gamma_g} & \searrow \epsilon & \downarrow \gamma_{f \times \gamma_g} \\
 \hat{\Sigma}_G \wr \underline{\mathcal{V}}(A') \times \hat{\Sigma}_G \wr \underline{\mathcal{V}}(B') & \xrightarrow{\lambda} & \hat{\Sigma}_G \wr \underline{\mathcal{V}}(A' \amalg B') \\
 \downarrow \gamma_{f \amalg g} & \searrow \epsilon & \downarrow \gamma_{f \amalg g} \\
 \hat{\Sigma}_G \wr \underline{\mathcal{V}}(A') \times \hat{\Sigma}_G \wr \underline{\mathcal{V}}(B') & \xrightarrow{\lambda} & \hat{\Sigma}_G \wr \underline{\mathcal{V}}(A' \amalg B')
 \end{array}
 \qquad
 \begin{array}{ccc}
 \underline{\mathcal{V}}(A \amalg B) & \xrightarrow{\lambda} & \underline{\mathcal{V}}(A \amalg B) \\
 \downarrow \gamma_{f \amalg g} & \searrow \eta & \downarrow \gamma_{f \amalg g} \\
 \underline{\mathcal{V}}(A) \times \underline{\mathcal{V}}(B) & \xrightarrow{\lambda} & \underline{\mathcal{V}}(A \amalg B) \\
 \downarrow \gamma_{f \times \gamma_g} & \searrow \eta & \downarrow \gamma_{f \times \gamma_g} \\
 \hat{\Sigma}_G \wr \underline{\mathcal{V}}(A') \times \hat{\Sigma}_G \wr \underline{\mathcal{V}}(B') & \xrightarrow{\lambda} & \hat{\Sigma}_G \wr \underline{\mathcal{V}}(A' \amalg B')
 \end{array}$$

Proof. We recall from the proof of Lemma 13.0.7 that the squares with λ s strictly commute. And the rear squares do as well. Next, the Γ s are identities by Lemma 13.0.4 as they arise

from pullback squares similar to

$$\begin{array}{ccc} A & \xrightarrow{\iota_A} & A \amalg B \\ f \downarrow & & \downarrow f \amalg g \\ A' & \xrightarrow{\iota_{A'}} & A' \amalg B' \end{array}$$

which are lex-pullbacks.

We observe that $\iota_{A'}^*(f \amalg g) = f$ and $\iota_{B'}^*(f \amalg g) = g$ proving that these are the same.

Showing that the square commutes on morphisms uses the same argument.

Now we show directly that the prisms commute. As all the squares are identity 2-cells, it suffices to show the following pairs of whiskering diagrams agree.

$$\begin{array}{ccc} \underline{\mathcal{V}}(A) \times \underline{\mathcal{V}}(B) & \xlongequal{\quad} & \underline{\mathcal{V}}(A) \times \underline{\mathcal{V}}(B) & \longrightarrow & \hat{\Sigma}_G \wr \underline{\mathcal{V}}(A') \times \hat{\Sigma}_G \wr \underline{\mathcal{V}}(B') \\ \lambda \downarrow & \nearrow \epsilon & \nearrow \iota^* & & \\ \underline{\mathcal{V}}(A \amalg B) & & & & \end{array}$$

$$\begin{array}{ccc} \underline{\mathcal{V}}(A) \times \underline{\mathcal{V}}(B) & \longrightarrow & \hat{\Sigma}_G \wr \underline{\mathcal{V}}(A') \times \hat{\Sigma}_G \wr \underline{\mathcal{V}}(B') & \xlongequal{\quad} & \hat{\Sigma}_G \wr \underline{\mathcal{V}}(A') \times \hat{\Sigma}_G \wr \underline{\mathcal{V}}(B') \\ & & \lambda_C \downarrow & \nearrow \epsilon & \nearrow \iota^* \\ & & \hat{\Sigma}_G \wr \underline{\mathcal{V}}(A' \amalg B') & & \end{array}$$

$$\begin{array}{ccc} \underline{\mathcal{V}}(A \amalg B) & \xlongequal{\quad} & \underline{\mathcal{V}}(A \amalg B) & \longrightarrow & \hat{\Sigma}_G \wr \underline{\mathcal{V}}(A' \amalg B') \\ \iota^* \downarrow & \nwarrow \eta & \nwarrow \lambda_C & & \\ \underline{\mathcal{V}}(A) \times \underline{\mathcal{V}}(B) & & & & \end{array}$$

$$\begin{array}{ccc} & \hat{\Sigma}_G \wr \underline{\mathcal{V}}(A') \times \hat{\Sigma}_G \wr \underline{\mathcal{V}}(B') & \\ & \iota^* \uparrow & \nwarrow \eta \\ \underline{\mathcal{V}}(A \amalg B) & \longrightarrow & \hat{\Sigma}_G \wr \underline{\mathcal{V}}(A' \amalg B') & \xlongequal{\quad} & \hat{\Sigma}_G \wr \underline{\mathcal{V}}(A' \amalg B') \\ & & & \nearrow \lambda_D & \end{array}$$

This now comes down to checking definitions. In the first diagram for (x, y) in $\underline{\mathcal{V}}(A) \times \underline{\mathcal{V}}(B)$

the composite natural transformation is

$$\left((f, \iota_A^* \lambda(x, y)), (g, \iota_B^* \lambda(x, y)) \right) \rightarrow \left((f, x), (g, y) \right).$$

Is given by (ϵ_x, ϵ_y) and the pullback squares:

$$\begin{array}{ccc} A & \xlongequal{\quad} & A \\ \downarrow f & & \downarrow f \\ A' & \xlongequal{\quad} & A' \end{array} \quad \begin{array}{ccc} B & \xlongequal{\quad} & B \\ \downarrow g & & \downarrow g \\ B' & \xlongequal{\quad} & B' \end{array}$$

Tracing through the second diagram we have this morphism as well.

For $z \in \underline{\mathcal{V}}(A \amalg B)$ in the third pasting diagram we have

$$(f \amalg g, z) \longrightarrow (f \amalg g, \lambda^* z)$$

which is given by η_z and the pullback square:

$$\begin{array}{ccc} A \amalg B & \xlongequal{\quad} & A \amalg B \\ f \amalg g \downarrow & & \downarrow f \amalg g \\ A' \amalg B' & \xlongequal{\quad} & A' \amalg B' \end{array}$$

Tracing through the fourth diagram we have this morphism as well.

□

Proposition 13.0.9. *Transfers are pseudofunctorial. If $\underline{\mathcal{V}}$ is strict, they are functorial.*

This means $(gf)_* \cong g_* f_*$ and that $(Id)_* \cong Id$.

Proof. This essentially come down to associativity and unitality of $\underline{\mathcal{V}}$ as a monad.

Consider order preserving maps $A \xrightarrow{f} B \xrightarrow{g} C$ and $x \in \underline{\mathcal{V}}(A)$.

We view this as $(A \xrightarrow{f} B \xrightarrow{g} C, x) \in \hat{\Sigma}_G^{\wr 2} \wr \underline{\mathcal{V}}(C)$ and we trace around the associativity diagram ??.

Applying $\hat{\Sigma}_G \wr \otimes$ we get $(B \xrightarrow{g} C, \otimes(f, x)) \in \hat{\Sigma}_G \wr \underline{\mathcal{V}}(C)$. Applying \otimes to this we get $\otimes(g, \otimes(f, x)) = g_*(f_*(x))$

Applying σ_0 we get $(A \xrightarrow{gf} C, x) \in \hat{\Sigma}_G \wr \underline{\mathcal{V}}(C)$ applying \otimes we get $\otimes(A \xrightarrow{gf} C, x) = (gf)_*(x)$.

By associativity these are naturally isomorphic (resp. equal).

Now for $x \in \underline{\mathcal{V}}(A)$, applying σ_0 we get $(Id : A \rightarrow A, x) \in \hat{\Sigma}_G \wr \underline{\mathcal{V}}(A)$. Then applying \otimes we get $\otimes(Id, x) \in \underline{\mathcal{V}}(A)$. By unitality this is naturally isomorphic (resp. equal) to x

The fact that α, ω are natural transformations of fibrations is needed to imply that the isomorphisms between elements of $\underline{\mathcal{V}}(C)$ they create are not merely morphisms in $\underline{\mathcal{V}}$, but in the fiber over C . The coherence diagrams relating α and ω are needed to imply that these pseudofunctoriality isomorphisms of transfers are suitably coherent.

□

Similarly to classical Mackey functors, we have a version of the double coset formula

Proposition 13.0.10. *Consider a pullback square in \mathbf{Fin}^G with its vertical arrows order preserving*

$$\begin{array}{ccc} A' & \xrightarrow{p} & A \\ \downarrow g & & \downarrow f \\ B' & \xrightarrow{q} & B \end{array} \quad (13.0.4)$$

Then for an SGA $\underline{\mathcal{V}}$, we have a natural isomorphism of functors $\underline{\mathcal{V}}(A) \rightarrow \underline{\mathcal{V}}(B')$

$$\Theta_{q,f} : g_* p^* \cong q^* f_* \quad (13.0.5)$$

Furthermore if this is a lex-pullback square $A' = q^ A$ and $\underline{\mathcal{V}}$ is split, then $\Theta_{q,f}$ is the identity and the two are equal.*

We refer to this result and Θ as the *double coset formula*.

Proof. We define $\Theta_{q,f}$ as the composite given by the pasting diagram, where the bottom square comes from \otimes being a map of fibrations.

$$\begin{array}{ccc}
\underline{\mathcal{V}}(A) & \xrightarrow{p^*} & \underline{\mathcal{V}}(A') \\
\gamma_f \downarrow & \swarrow \Gamma & \downarrow \gamma_g \\
\hat{\Sigma}_G \wr \underline{\mathcal{V}}(B) & \xrightarrow{\hat{\Sigma}_G \wr (q^*)} & \hat{\Sigma}_G \wr \underline{\mathcal{V}}(B') \\
\otimes \downarrow & \swarrow \cong & \otimes \downarrow \\
\underline{\mathcal{V}}(B) & \xrightarrow{q^*} & \underline{\mathcal{V}}(B')
\end{array} \tag{13.0.6}$$

In the case of a lex-pullback the top square commutes strictly. In the case of $\underline{\mathcal{V}}$ split, the bottom square commutes strictly. With a simple diagram chase, this implies that naturality as in the pasting diagrams of Lemma 13.0.11 both ways around the diagram given cartesian lifts so the resulting isomorphisms between them must be the same.

□

We note that nothing in this proof relied on $\underline{\mathcal{V}} \rightarrow \mathbf{Fin}^G$ being a split fibration, so indeed this result applies to pseudo- \mathbf{Fin}^G -categories as well.

We will see later in Proposition 13.0.21 that Θ is a monoidal natural transformation.

Lemma 13.0.11. *Θ is natural in the sense that when gluing pullbacks either horizontally or vertically, the corresponding pasting diagrams of Θ equal the isomorphism of the full pullback.*

Proof. For the pullback squares in \mathbf{Fin}^G :

$$\begin{array}{ccccc}
A'' & \xrightarrow{r} & A' & \xrightarrow{p} & A \\
h \downarrow & & g \downarrow & & f \downarrow \\
B'' & \xrightarrow{s} & B' & \xrightarrow{q} & B
\end{array}$$

the two pasting diagrams are equal:

$$\begin{array}{ccc}
\underline{\mathcal{V}}(A) & \xrightarrow{(pr)^*} & \underline{\mathcal{V}}(A'') \\
f_* \downarrow & \swarrow \Theta_{qs,f} & \downarrow h_* \\
\underline{\mathcal{V}}(B) & \xrightarrow{(qs)^*} & \underline{\mathcal{V}}(B'')
\end{array} = \begin{array}{ccccc}
\underline{\mathcal{V}}(A) & \xrightarrow{r^*} & \underline{\mathcal{V}}(A') & \xrightarrow{p^*} & \underline{\mathcal{V}}(A'') \\
f_* \downarrow & \swarrow \Theta_{q,f} & \downarrow g_* & \swarrow \Theta_{s,g} & \downarrow h_* \\
\underline{\mathcal{V}}(B) & \xrightarrow{q^*} & \underline{\mathcal{V}}(B') & \xrightarrow{s^*} & \underline{\mathcal{V}}(B'')
\end{array}$$

This follows from the naturality of Γ , and the fact that \otimes is a map of \mathbf{Fin}^G categories, so \cong is also appropriately natural. And for the pullback squares in \mathbf{Fin}^G

$$\begin{array}{ccc} A' & \xrightarrow{p} & A \\ g \downarrow & & f \downarrow \\ B' & \xrightarrow{q} & B \\ k \downarrow & & h \downarrow \\ C' & \xrightarrow{r} & C \end{array}$$

the two pasting diagrams are equal, where the “ \cong ”s come from the pseudofunctoriality of transfers.

$$\begin{array}{ccc} \underline{\mathcal{V}}(A) \xrightarrow{p^*} \underline{\mathcal{V}}(A') & & \underline{\mathcal{V}}(A) \xrightarrow{p^*} \underline{\mathcal{V}}(A') \\ \downarrow \scriptstyle h_* f_* \cong \quad \downarrow \scriptstyle (hf)_* & \swarrow \scriptstyle \Theta_{r,hf} & \downarrow \scriptstyle (kg)_* \\ \underline{\mathcal{V}}(C) \xrightarrow{r^*} \underline{\mathcal{V}}(C') & & \underline{\mathcal{V}}(C) \xrightarrow{r^*} \underline{\mathcal{V}}(C') \end{array} = \begin{array}{ccc} \underline{\mathcal{V}}(A) \xrightarrow{p^*} \underline{\mathcal{V}}(A') & & \underline{\mathcal{V}}(A) \xrightarrow{p^*} \underline{\mathcal{V}}(A') \\ f_* \downarrow \swarrow \scriptstyle \Theta_{q,f} & \downarrow \scriptstyle g_* & \downarrow \scriptstyle g_* \\ \underline{\mathcal{V}}(B) \xrightarrow{q^*} \underline{\mathcal{V}}(B') \cong & & \underline{\mathcal{V}}(B) \xrightarrow{q^*} \underline{\mathcal{V}}(B') \cong \\ h_* \downarrow \swarrow \scriptstyle \Theta_{f,h} & \downarrow \scriptstyle k_* & \downarrow \scriptstyle k_* \\ \underline{\mathcal{V}}(C) \xrightarrow{r^*} \underline{\mathcal{V}}(C') & & \underline{\mathcal{V}}(C) \xrightarrow{r^*} \underline{\mathcal{V}}(C') \end{array}$$

We prove this using the following diagram:

$$\begin{array}{ccccccc} \underline{\mathcal{V}}(A') & \xrightarrow{\quad} & \hat{\Sigma}_G \wr \underline{\mathcal{V}}(B') & \xrightarrow{\quad} & \hat{\Sigma}_{G!} \otimes & \xrightarrow{\quad} & \underline{\mathcal{V}}(B') \\ & \searrow & \uparrow \scriptstyle \Gamma \cong & \downarrow & & & \downarrow \\ & & \underline{\mathcal{V}}(A) & \xrightarrow{\quad} & \hat{\Sigma}_G \wr \underline{\mathcal{V}}(B) & \xrightarrow{\quad} & \hat{\Sigma}_{G!} \otimes & \xrightarrow{\quad} & \underline{\mathcal{V}}(B) \\ & & & & \downarrow \scriptstyle \Gamma \cong & & \downarrow & & \downarrow \scriptstyle \Gamma \cong \\ & & & & \hat{\Sigma}_G^{!2} \wr \underline{\mathcal{V}}(C') & \xrightarrow{\quad} & \hat{\Sigma}_G \wr \underline{\mathcal{V}}(C') & & \hat{\Sigma}_G \wr \underline{\mathcal{V}}(C') \\ & & & & \downarrow & & \downarrow \scriptstyle \hat{\Sigma}_{G!} \otimes & & \downarrow \\ & & & & \hat{\Sigma}_G^{!2} \wr \underline{\mathcal{V}}(C) & \xrightarrow{\quad} & \hat{\Sigma}_G \wr \underline{\mathcal{V}}(C) & & \hat{\Sigma}_G \wr \underline{\mathcal{V}}(C) \\ & & & & \downarrow \scriptstyle \sigma_0 & & \downarrow \scriptstyle \sigma_0 & & \downarrow \scriptstyle \otimes \\ & & & & \hat{\Sigma}_G \wr \underline{\mathcal{V}}(C') & \xrightarrow{\quad} & \underline{\mathcal{V}}(C') & & \underline{\mathcal{V}}(C') \\ & & & & \downarrow \scriptstyle \alpha \cong & & \downarrow \scriptstyle \alpha \cong & & \downarrow \scriptstyle \otimes \\ & & & & \hat{\Sigma}_G \wr \underline{\mathcal{V}}(C) & \xrightarrow{\quad} & \underline{\mathcal{V}}(C) & & \underline{\mathcal{V}}(C) \end{array}$$

Going right then down is applying one transfer than the other. Down then right is applying the transfer of the composite.

The top composite face is Θ , the right composite face is Θ , the left and bottom faces paste to form Θ . The front composite face pastes to form the 2-cell of $(kg)_* \cong k_*g_*$, the back composite face pastes to form the 2-cell of $(hf)_* \cong h_*f_*$. So now it suffices to show this 2-diagram commutes.

The top cube commuting is exactly the fact that Γ commutes with the morphisms of \mathbf{Fin}^G categories, namely $\hat{\Sigma}_G \wr \otimes : \hat{\Sigma}_G \wr \underline{\mathcal{V}} \rightarrow \underline{\mathcal{V}}$.

The bottom cube commuting is exactly the fact that α is a natural isomorphism of fibrations so is natural with respect to the restriction of $C' \rightarrow C$.

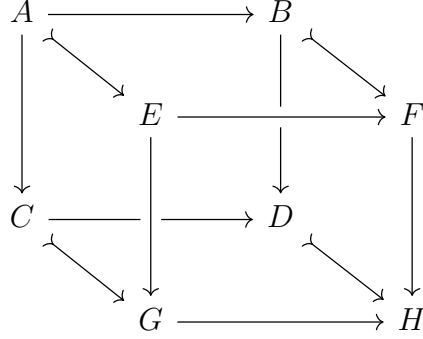
The left prism is a bit trickier, we must do a direct calculation. Starting at $x \in \underline{\mathcal{V}}(A)$ applying γ_f we get $(A \xrightarrow{f} B, x) \in \hat{\Sigma}_G \wr \underline{\mathcal{V}}(B)$. Applying γ_h we get $(A \xrightarrow{f} B \xrightarrow{h} C, x) \in \hat{\Sigma}_G^{\wr 2} \wr \underline{\mathcal{V}}(C)$, applying σ_0 to this yields $(A \xrightarrow{hf} C, x) \in \hat{\Sigma}_G \wr \underline{\mathcal{V}}(C)$. Direct observation tells us this is the same as $\gamma_{hf}(x)$, so the front face strictly commutes. The same argument applies to the back face.

Γ_x is then the unique isomorphism between sources of the cartesian lifts of r , as is the composite of the 2-cells the other way around.

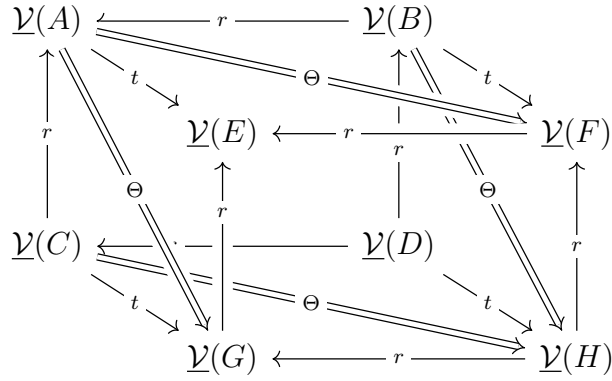
□

Remark 13.0.12 (Bonventre). We note that, even in the case where $A' = q^*A$ is the chosen pullback, we do not have an equality between p_*g^* and f^*q_* . This is due to the necessary choice of an ordering of the pullback, which is not preserved by switching B' and A , yielding that q^*A is isomorphic, but not equal, to f^*B' .

Lemma 13.0.13. Consider the following commutative diagram in \mathbf{Fin}^G where the top, bottom, left, and right sides are pullbacks.

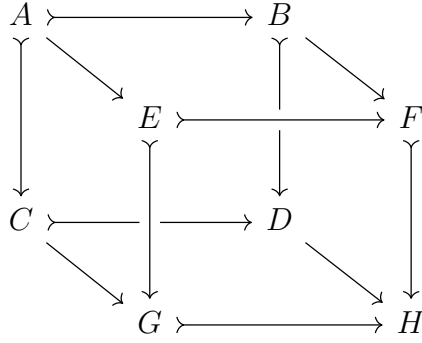


Then the following 2-diagram commutes, where the 2-cells are Θ s and r, t denote restrictions and transfers of the obvious maps:

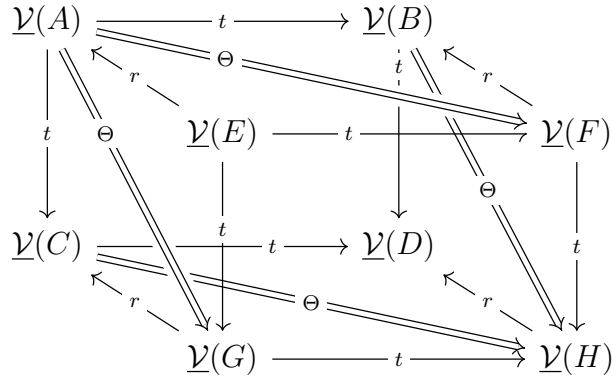


Proof. The front and back squares commute strictly as restrictions commute with restrictions. By pasting, the top and side faces combine to form a single 2-cell from the double coset formula. This is by the naturality of the double coset formula. Similarly for the left and bottom faces. As the front and back faces of the original cube commute, the two pastings are the double coset formula applied to the same maps, so they agree and the cube commutes. \square

Lemma 13.0.14. Consider the following commutative diagram in \mathbf{Fin}^G where the top, bottom, left, and right sides are pullbacks.



Then the following 2-diagram commutes, where the top, bottom, left, and right 2-cells are Θ s and r, t denote restrictions and transfers of the obvious maps, the front and back faces do not strictly commute but only up to the isomorphism coming from the pseudonaturality of transfers.



Proof. The front and back squares commute up to isomorphism as transfers are pseudofunctorial; and commute strictly in the case that $\underline{\mathcal{V}}$ is strict. By pasting, the top and side faces combine to form a single 2-cell from the double coset formula. This is by the naturality of the double coset formula. Similarly for the left and bottom faces. As the front and back faces of the original cube commute, the two pastings are the double coset formula applied to the same maps, so they agree and the cube commutes □

These two results we refer to as the *cube lemmas*.

Lemma 13.0.15. Consider the two pullback squares in Fin^G .

$$\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D
\end{array}
\qquad
\begin{array}{ccc}
A' & \longrightarrow & B' \\
\downarrow & & \downarrow \\
C' & \longrightarrow & D'
\end{array}$$

Then the following cube commutes.

$$\begin{array}{ccccc}
\underline{\mathcal{V}}(A) \times \underline{\mathcal{V}}(A') & \xleftarrow{r} & \underline{\mathcal{V}}(B) \times \underline{\mathcal{V}}(B') & & \\
\downarrow t & \searrow \lambda & \downarrow \Lambda & \searrow \lambda & \\
& \underline{\mathcal{V}}(A \amalg A') & \xleftarrow{r} & \underline{\mathcal{V}}(B \amalg B') & \\
& \downarrow \Psi & \downarrow \Theta & \downarrow \Psi & \downarrow t \\
\underline{\mathcal{V}}(C) \times \underline{\mathcal{V}}(C') & \xleftarrow{r} & \underline{\mathcal{V}}(D) \times \underline{\mathcal{V}}(D') & & \\
& \searrow \lambda & \downarrow \Lambda & \searrow \lambda & \\
& \underline{\mathcal{V}}(C \amalg C') & \xleftarrow{r} & \underline{\mathcal{V}}(D \amalg D') &
\end{array}$$

We refer to this result as λ commuting with the double coset formula.

Proof. We prove this using a 2-diagram pasting argument. Consider the following 2-diagram.

$$\begin{array}{ccccc}
\underline{\mathcal{V}}(A) \times \underline{\mathcal{V}}(A') & \xleftarrow{r} & \underline{\mathcal{V}}(B) \times \underline{\mathcal{V}}(B') & & \\
\downarrow \gamma & \searrow \lambda & \downarrow \Gamma & \searrow \lambda & \\
& \underline{\mathcal{V}}(A \amalg A') & \xleftarrow{r} & \underline{\mathcal{V}}(B \amalg B') & \\
& \downarrow \Gamma & \downarrow \Gamma & \downarrow \Gamma & \downarrow \Gamma \\
\hat{\Sigma}_G \wr \underline{\mathcal{V}}(C) \times \hat{\Sigma}_G \wr \underline{\mathcal{V}}(C') & \xleftarrow{r} & \hat{\Sigma}_G \wr \underline{\mathcal{V}}(C) \times \hat{\Sigma}_G \wr \underline{\mathcal{V}}(C') & & \\
\downarrow \otimes & \searrow \lambda & \downarrow \otimes & \searrow \lambda & \\
& \underline{\mathcal{V}}(C \amalg C') & \xleftarrow{r} & \underline{\mathcal{V}}(C \amalg C') & \\
& \downarrow \Phi & \downarrow \Phi & \downarrow \Phi & \downarrow \Phi \\
\underline{\mathcal{V}}(C) \times \underline{\mathcal{V}}(C') & \xleftarrow{r} & \underline{\mathcal{V}}(D) \times \underline{\mathcal{V}}(D') & & \\
& \searrow \lambda & \downarrow \Gamma & \searrow \lambda & \\
& \underline{\mathcal{V}}(C \amalg C') & \xleftarrow{r} & \underline{\mathcal{V}}(C \amalg C') & \\
& \downarrow \eta & \downarrow \eta & \downarrow \eta & \downarrow \eta
\end{array}$$

First we confirm that its outer faces agree with the ones we seek to show commute. The internal faces paste to form Θ s.

The middle parallelepiped commutes by the cube lemma 13.0.13, the top two prisms commute by Lemma 13.0.8, the bottom two prisms commute by Lemma 11.1.10 applied to \otimes .

□

Proposition 13.0.16. *Let $\underline{\mathcal{V}}$ be a SGA, then for each A , $\underline{\mathcal{V}}(A)$ is a symmetric monoidal category. In the case that $\underline{\mathcal{V}}$ is permutative (additive, strict, and split) $\underline{\mathcal{V}}(A)$ is a permutative category.*

Proof. We construct a multiplication $\text{map}^1 \otimes : \underline{\mathcal{V}}(A)^k \rightarrow \underline{\mathcal{V}}(A)$ as the composite:

$$\otimes : \underline{\mathcal{V}}(A)^k \xrightarrow{\lambda} \underline{\mathcal{V}}(A^{\amalg k}) \xrightarrow{\xi^*} \underline{\mathcal{V}}(A \times \underline{k}) \xrightarrow{\nabla_*} \underline{\mathcal{V}}(A) \quad (13.0.7)$$

where ξ is the reordering isomorphism. For $k = 0$ this is $\nabla_* : \underline{\mathcal{V}}(\emptyset) \rightarrow \underline{\mathcal{V}}(A)$. Recall that $\underline{\mathcal{V}}(\emptyset) \simeq *$ with equality if $\underline{\mathcal{V}}$ is additive. So this defines an object or contractible choice of objects in $\underline{\mathcal{V}}(A)$ which will be the unit.

This is symmetric up to isomorphism as \otimes is pseudonatural with respect to reordering isomorphisms $\chi : \underline{k} \rightarrow \underline{k}$:

$$\begin{array}{ccccccc} \underline{\mathcal{V}}(A)^k & \xrightarrow{\lambda} & \underline{\mathcal{V}}(A^{\amalg k}) & \xrightarrow{\xi^*} & \underline{\mathcal{V}}(A \times \underline{k}) & \xrightarrow{\nabla_*} & \underline{\mathcal{V}}(A) \\ x \downarrow & & x^* \downarrow & & x^* \downarrow & \swarrow \Theta & \parallel \\ \underline{\mathcal{V}}(A)^k & \xrightarrow{\lambda} & \underline{\mathcal{V}}(A^{\amalg k}) & \xrightarrow{\xi^*} & \underline{\mathcal{V}}(A \times \underline{k}) & \xrightarrow{\nabla_*} & \underline{\mathcal{V}}(A) \end{array} \quad (13.0.8)$$

The left commutes strictly by 11.1.5, the middle as restrictions are functorial, and the right commutes up to Θ (strictly if $\underline{\mathcal{V}}$ is additive) as it comes from the lex-pullback square:

$$\begin{array}{ccc} A \times \underline{k} & \xrightarrow{x} & A \times \underline{k} \\ \nabla \downarrow & & \downarrow \nabla \\ A & \xlongequal{\quad} & A \end{array} \cdot$$

¹We use \otimes to denote both the operation of $\underline{\mathcal{V}}$ as a monad over $\hat{\Sigma}_G \wr (-)$ and the monoidal product internal to $\underline{\mathcal{V}}(A)$. In context it should be clear which we are referring to.

To see associativity, consider k_1, \dots, k_n and $k = \sum_i k_i$. We consider the following diagram:

$$\begin{array}{ccccccc}
\prod_n \prod_{k_i} \underline{\mathcal{V}}(A) & \xrightarrow{\lambda} & \prod_n \underline{\mathcal{V}}(A^{\amalg k_i}) & \xrightarrow{\xi^*} & \prod_n \underline{\mathcal{V}}(A \times \underline{k}_i) & \xrightarrow{\nabla_*} & \prod_n \underline{\mathcal{V}}(A) \\
& \searrow \lambda & \downarrow \lambda & \swarrow \Lambda & \downarrow \lambda & \swarrow \Psi & \downarrow \lambda \\
& & \underline{\mathcal{V}}(\underline{k} \times A) & \xrightarrow{\xi^*} & \underline{\mathcal{V}}(\prod_n (A \times \underline{k}_i)) & \xrightarrow{\nabla_*} & \underline{\mathcal{V}}(A^n) \\
& & & \searrow \xi^* & \downarrow \xi^* & \swarrow \Theta & \downarrow \xi^* \\
& & & & \underline{\mathcal{V}}(A \times \underline{k}) & \xrightarrow{\nabla_*} & \underline{\mathcal{V}}(A \times \underline{n}) \\
& & & & & \swarrow \cong & \downarrow \nabla_* \\
& & & & & & \underline{\mathcal{V}}(A)
\end{array} \tag{13.0.9}$$

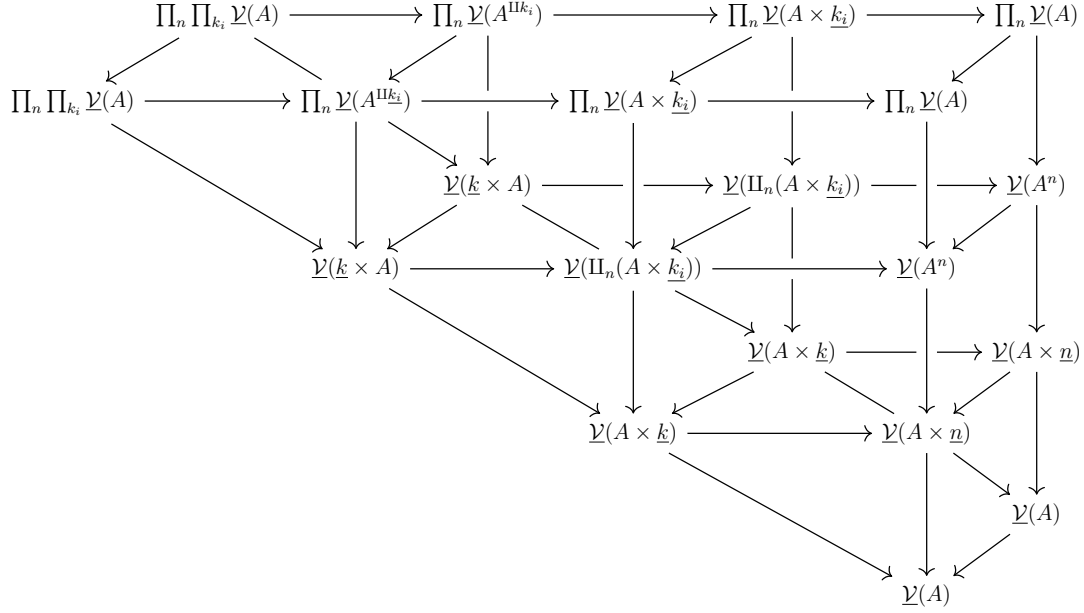
where we note that $\underline{k} \times A = \prod_n (A^{k_i})$.

We go through the cells of this diagram in order.

- The top left triangle commutes as λ is associative 11.1.4.
- The top middle square commutes up to isomorphism given by Λ and strictly if $\underline{\mathcal{V}}$ is strongly additive as a Fin^G category.
- The top right commutes up to isomorphism as transfers commute up to isomorphism with λ and strictly if $\underline{\mathcal{V}}$ is additive (13.0.7).
- The middle triangle commutes as restrictions are functorial.
- The middle right square commutes up to isomorphism by the double coset formula. In the case that $\underline{\mathcal{V}}$ is split, it strictly commutes as it arises from a lex-pullback.
- The bottom triangle commutes up to isomorphism as transfers are pseudofunctorial and strictly if $\underline{\mathcal{V}}$ is strict (13.0.9).

We note that unitality is the special case of associativity when $[k_i] = \emptyset$ for some i ; this also demonstrates compatibility of unitality and associativity.

Compatibility of the symmetry and associativity isomorphism follows from this diagram being pseudonatural with respect to block permutations $\Xi : \Pi_i k_i \rightarrow \Pi_i k_i$. We display this as a large 2-diagram. The arrows coming out of the page are induced by Ξ , the rest are those of the above diagram. We have not drawn the 2-cells for clarity.



We go through the prisms and cubes in order to confirm they commute.

- The top left triangular prism commutes as λ is commutative and associative in a compatible way by 11.1.4 and 11.1.5.
- The top middle cube commutes as Λ is transitive with respect to restrictions 11.1.8.
- λ The top right cube commutes by 13.0.15.
- The middle triangular prism commutes as restrictions are functorial (all the 2-cells here are trivial).
- The middle right cube commutes by the cube lemma 13.0.13.

- The lower right triangular prism commutes by the cube lemma 13.0.14.

□

Remark 13.0.17. We point out that the coherence condition on λ , η , and ϵ of 11.1.4, 11.1.5, 11.1.3 are required for us to have a symmetric monoidal structure.

Remark 13.0.18 (Bonventre). Using this symmetric monoidal structure, we can give an isomorphic description of the additivity inverses λ . Indeed, λ is naturally isomorphic to $(\iota_A)_* + (\iota_B)_*$ as natural transformations $\underline{\mathcal{V}}(A) \times \underline{\mathcal{V}}(B) \rightarrow \underline{\mathcal{V}}(A \amalg B)$, with equality holding if $\underline{\mathcal{V}}$ is permutative.

To see this, consider the following diagram:

$$\begin{array}{ccccccc}
 V(A) \times V(B) & \xrightarrow{\lambda} & V(A \amalg B) & \longrightarrow & V(A \amalg B) & \xlongequal{\quad} & V(A \amalg B) \\
 \downarrow \iota_{A*} \times \iota_{B*} & \swarrow \Psi & \downarrow \iota_{A \amalg B*} & \swarrow \Theta & \downarrow \iota_{A \amalg B*} & \swarrow \cong & \parallel \\
 V(A \amalg B) \times V(A \amalg B) & \xrightarrow{\lambda} & V((A \amalg B)^{\amalg 2}) & \longrightarrow & V((A \amalg B) \times \underline{2}) & \xrightarrow{\nabla_*} & V(A \amalg B)
 \end{array}$$

The left square commutes up to Ψ which is an identity if $\underline{\mathcal{V}}$ is additive, the middle square commutes up to the double coset formula which is an equality if $\underline{\mathcal{V}}$ is split the right square commutes up to isomorphism as transfers are pseudofunctorial by Lemma 13.0.9 and commutes strictly if $\underline{\mathcal{V}}$ is strict. As the bottom composite is multiplication in $\underline{\mathcal{V}}(A \amalg B)$, the result follows.

Proposition 13.0.19. *For $f : A \rightarrow B$, $f^* : \underline{\mathcal{V}}(B) \rightarrow \underline{\mathcal{V}}(A)$ is strong monoidal, and strict in the case that $\underline{\mathcal{V}}$ is split and additive.*

Proof.

$$\begin{array}{ccccccc}
\underline{\mathcal{V}}(B)^k & \xrightarrow{\lambda} & \underline{\mathcal{V}}(B^{\amalg k}) & \xrightarrow{\xi^*} & \underline{\mathcal{V}}(B \times \underline{k}) & \xrightarrow{\nabla_*} & \underline{\mathcal{V}}(B) \\
f_* \downarrow & \swarrow \Lambda & \parallel & f_* \downarrow & \swarrow \Theta & \parallel & \downarrow f_* \\
\underline{\mathcal{V}}(A)^k & \xrightarrow{\lambda} & \underline{\mathcal{V}}(A^{\amalg k}) & \xrightarrow{\xi^*} & \underline{\mathcal{V}}(A \times \underline{k}) & \xrightarrow{\nabla_*} & \underline{\mathcal{V}}(A)
\end{array}$$

$$\begin{array}{ccc}
A \times \underline{k} & \longrightarrow & B \times \underline{k} \\
\downarrow & & \downarrow \\
A & \longrightarrow & B
\end{array}$$

The left square strictly commutes if $\underline{\mathcal{V}}$ is split and the right square strictly commutes if $\underline{\mathcal{V}}$ is additive, using the fact that the bottom square is a lex pullback. \square

Proposition 13.0.20. *For $f : A \rightarrow B$, the transfer $f_* : \underline{\mathcal{V}}(A) \rightarrow \underline{\mathcal{V}}(B)$ is strong monoidal; f_* is strict monoidal in the case that $\underline{\mathcal{V}}$ is additive and strict and f is injective.*

Proof.

$$\begin{array}{ccccccc}
\underline{\mathcal{V}}(A)^k & \xrightarrow{\lambda} & \underline{\mathcal{V}}(A^{\amalg k}) & \xrightarrow{\xi^*} & \underline{\mathcal{V}}(A \times \underline{k}) & \xrightarrow{\nabla_*} & \underline{\mathcal{V}}(A) \\
f_* \downarrow & \swarrow \Psi & \parallel & f_* \downarrow & \swarrow \Theta & \parallel & \downarrow f_* \\
\underline{\mathcal{V}}(B)^k & \xrightarrow{\lambda} & \underline{\mathcal{V}}(B^{\amalg k}) & \xrightarrow{\xi^*} & \underline{\mathcal{V}}(B \times \underline{k}) & \xrightarrow{\nabla_*} & \underline{\mathcal{V}}(B)
\end{array}$$

$$\begin{array}{ccc}
A \times \underline{k} & \xrightarrow{\xi} & A^k \\
f \times Id \downarrow & & \downarrow f^k \\
B \times \underline{k} & \xrightarrow{\xi} & B^k
\end{array}$$

The left square commutes up to the isomorphism Ψ , which is an equality when $\underline{\mathcal{V}}$ is additive. The middle square commutes up to the isomorphism Θ , which is strict when $\underline{\mathcal{V}}$ is additive and the lower square is a lex-pullback in \mathbf{Fin}^G , which is the case if f is injective. The third square commutes up to isomorphism as transfers are pseudofunctorial by Lemma 13.0.9, and strictly if $\underline{\mathcal{V}}$ is strict. \square

Proposition 13.0.21. *For a SGA, Θ of the double coset formula is a monoidal natural transformation.*

Proof. This is another 2-diagram chase. We omit the 2-cells for readability.

$$\begin{array}{ccccccc}
 \mathcal{V}(A)^2 & \xrightarrow{\lambda} & \mathcal{V}(A \amalg A) & \xrightarrow{\xi^*} & \mathcal{V}(A \times \underline{2}) & \xrightarrow{\nabla_*} & \mathcal{V}(A) \\
 \swarrow f_* & & \swarrow f_* & & \swarrow f_* & & \swarrow f_* \\
 \mathcal{V}(B)^2 & \xrightarrow{\lambda} & \mathcal{V}(B \amalg B) & \xrightarrow{\xi^*} & \mathcal{V}(B \times \underline{2}) & \xrightarrow{\nabla_*} & \mathcal{V}(B) \\
 \swarrow p_* & & \swarrow p_* & & \swarrow p_* & & \swarrow p_* \\
 & & \mathcal{V}(A')^2 & \xrightarrow{\lambda} & \mathcal{V}(A' \amalg A') & \xrightarrow{\xi^*} & \mathcal{V}(A' \times \underline{2}) & \xrightarrow{\nabla_*} & \mathcal{V}(A') \\
 \swarrow q_* & & \swarrow q_* & & \swarrow q_* & & \swarrow q_* & & \swarrow q_* \\
 & & \mathcal{V}(B')^2 & \xrightarrow{\lambda} & \mathcal{V}(B' \amalg B') & \xrightarrow{\xi^*} & \mathcal{V}(B' \times \underline{2}) & \xrightarrow{\nabla_*} & \mathcal{V}(B')
 \end{array}$$

The left cube commutes as Θ commutes with λ as in Lemma 13.0.15. The middle cube commutes by the cube Lemma 13.0.13, and the right commutes by the other cube Lemma 13.0.14.

□

Remark 13.0.22. Similar results hold for pseudo-SGAs, although they were not detailed in the interest of concision. The main differences are that as restrictions are pseudofunctorial, we also must include invertible 2-cells to account for composition and unitality. Thus our lemmas and results which used the functoriality of restrictions now must also include these new 2-cells. Generally this means that a commuting square or triangle now has a non-trivial 2-cell.

The main consequence is that everything is weak - fibers are symmetric monoidal instead of permutative, restrictions and transfers are strong symmetric monoidal functors but not strictly unital, and the double coset formula is generally not an equality.

13.1 Morphisms of SGAs

Definition 13.1.1. A morphism of *morphism of SGAs* is a morphism of fibrations $F : \underline{\mathcal{V}} \rightarrow \underline{\mathcal{W}}$ that is also a lax morphism of pseudoalgebras over $\hat{\Sigma}_G \wr (-)$, meaning that we have the 2-cell β :

$$\begin{array}{ccc}
 \hat{\Sigma}_G \wr \underline{\mathcal{V}} & \xrightarrow{\hat{\Sigma}_G \wr F} & \hat{\Sigma}_G \wr \underline{\mathcal{W}} \\
 \downarrow \otimes & \swarrow \beta & \downarrow \otimes \\
 \underline{\mathcal{V}} & \xrightarrow{F} & \underline{\mathcal{W}}
 \end{array}$$

so that the following 2-diagrams commute:

The left diagram is a commutative 2-diagram with nodes $\hat{\Sigma}_G^{\wr 2} \underline{\mathcal{V}}$, $\hat{\Sigma}_G^{\wr 2} \underline{\mathcal{W}}$, $\hat{\Sigma}_G \wr \underline{\mathcal{V}}$, $\hat{\Sigma}_G \wr \underline{\mathcal{W}}$, $\underline{\mathcal{V}}$, and $\underline{\mathcal{W}}$. It features 2-cells α and β , and natural transformations σ_0 and \otimes .

The right diagram is a commutative 2-diagram with nodes $\underline{\mathcal{V}}$, $\hat{\Sigma}_G \wr \underline{\mathcal{V}}$, $\underline{\mathcal{W}}$, and $\hat{\Sigma}_G \wr \underline{\mathcal{W}}$. It features 2-cells δ_{-1} and β , and natural transformations F and \otimes .

There are several possible strengthenings similar to those of SGAs.

- F is *strong* if F is a strong morphism of pseudoalgebras, meaning that β is invertible.
- F is *strict* if F is a strict morphism of pseudoalgebras, meaning that β is the identity.
- F is *split* if F is morphism of split fibrations.
- F is *additive* if F is an additive morphism of Fin^G categories.

Remark 13.1.2. We believe but do not yet have a full proof that lax morphism of SGAs induces a lax symmetric monoidal functor on fibers, strong monoidal if strong, and strict monoidal if strict, split, and additive.

The natural transformation $F(-) \otimes_{\underline{\mathcal{W}}(A)} F(-) \Rightarrow F(- \otimes_{\underline{\mathcal{V}}(A)} -)$ is given by the following 2-cell:

$$\begin{array}{ccccccccc}
\underline{\mathcal{V}}(A)^k & \xrightarrow{\lambda} & \underline{\mathcal{V}}(A^{\amalg k}) & \xrightarrow{\xi^*} & \underline{\mathcal{V}}(A \times \underline{k}) & \xrightarrow{\gamma_{\nabla}} & \hat{\Sigma}_G \wr \underline{\mathcal{V}}(A) & \xrightarrow{\otimes} & \underline{\mathcal{V}}(A) \\
F \downarrow & \swarrow \Phi^{-1} & \downarrow F & \swarrow \cong & \downarrow F & & F \downarrow & \swarrow \beta & \downarrow F \\
\underline{\mathcal{W}}(A)^k & \xrightarrow{\lambda} & \underline{\mathcal{W}}(A^{\amalg k}) & \xrightarrow{\xi^*} & \underline{\mathcal{W}}(A \times \underline{k}) & \xrightarrow{\gamma_{\nabla}} & \hat{\Sigma}_G \wr \underline{\mathcal{W}}(A) & \xrightarrow{\otimes} & \underline{\mathcal{W}}(A)
\end{array}$$

The first square on the left commutes up to Φ^{-1} which is an equality in the case that F is additive. We use Φ^{-1} as opposed to Φ simply to have the correct directionality. The second square commutes up to isomorphism as a map of fibrations is pseudofunctorial in this sense; it is an equality in the case that F is split. We directly check that the third square strictly commutes. Given $x \in \underline{\mathcal{V}}(A)$ both ways around send it to $(\nabla, F(x))$. The fourth square commutes up to β , which is invertible if F is strong and an equality if F is strict. The unit morphism is given by this same cell for $k = 0$. It is clear to see that this cell weakly commutes with restrictions and transfers in $\underline{\mathcal{V}}$ and $\underline{\mathcal{W}}$.

The subtlety arises in showing that this natural transformation satisfies the necessary conditions relating it to the associativity, unitality, and symmetry isomorphisms in $\underline{\mathcal{V}}(A)$ and $\underline{\mathcal{W}}(A)$.

Conjecture 13.1.3. We believe that $\underline{\pi}_0 : \underline{Sp}_G \rightarrow \underline{Mack}_G$ is a lax morphism of SGAs. The strength of $\underline{\pi}_0$ as well as the methods of proving this claim depend heavily on the exact ways in which we construct \underline{Sp}_G and \underline{Mack}_G as SGAs.

Definition 13.1.4. A natural transformations of SGAs is a natural transformation $\alpha : F \Rightarrow G$

of fibrations.

Part III

Comparison of $\hat{\Sigma}_G$ -algebras and other Models

Chapter 14

Permutative Mackey Functors

Definition 14.0.1. We define a *permutative Mackey functor* (PMF) as a PC-functor

$$(\mathbf{GE}_{ord})^{op} \longrightarrow \mathit{Perm}$$

and that they form a 2-category with PC natural transformations and modifications.

Remark 14.0.2. The original version in [BIO15] used \mathbf{GE}' and permutative categories with lax monoidal multifunctors.

14.1 SGAs to PMFs

Theorem 14.1.1. *Given a permutative SGA, $\underline{\mathcal{V}}$, we can construct a PMF $M_{\underline{\mathcal{V}}} : \mathbf{GE}_{ord}^{op} \rightarrow \mathit{Perm}$.*

Let $\underline{\mathcal{V}}$ be a permutative SGA. We first briefly recall that as $\underline{\mathcal{V}}$ is permutative, $\underline{\mathcal{V}}(A)$ is permutative, restrictions are strong monoidal strict unital, transfers are functorial and strong monoidal strictly unital, and the double coset formula is an equality for lex-pullbacks.

We construct the PMF $M_{\underline{\mathcal{V}}} : \mathbf{GE}_{ord}^{op} \rightarrow \mathbf{Perm}^s$ as follows.

- $M_{\underline{\mathcal{V}}}(A) := \underline{\mathcal{V}}(A)$.

- For $\chi = (A \xleftarrow{r} X \xrightarrow{t} B)$, with t order preserving, $M_{\underline{\mathcal{V}}}(\chi)$ is the composition $M_{\underline{\mathcal{V}}}(A) = \underline{\mathcal{V}}(A) \xrightarrow{r^*} \underline{\mathcal{V}}(X) \xrightarrow{t_*} \underline{\mathcal{V}}(B) = M_{\underline{\mathcal{V}}}(B)$.

- For $\xi = (A \xleftarrow{r'} X' \xrightarrow{t'} B)$ and $\eta : \chi \Rightarrow \xi$ we construct $\underline{\mathcal{V}}(\eta)$ as follows. We have the map of

spans in \mathbf{Fin}^G , with η an isomorphism:

$$\begin{array}{ccccc} A & \xleftarrow{r} & X & \xrightarrow{t} & B \\ & \swarrow r' & \downarrow \eta & \searrow t' & \\ & & Y & & \end{array}$$

As η may not be order preserving, so we cannot take transfers along it. But we have the pullback square:

$$\begin{array}{ccc} Y & \xrightarrow{\eta^{-1}} & X \\ t' \downarrow & & \downarrow t \\ B & \xlongequal{\quad} & B \end{array}$$

By the double coset formula, we have a monoidal natural transformation $t'^* \Rightarrow g_* \circ t^*$

It is sent to

$$\begin{array}{ccccc} \underline{\mathcal{V}}(A) & \xrightarrow{r^*} & \underline{\mathcal{V}}(X) & \xrightarrow{t_*} & \underline{\mathcal{V}}(B) \\ & \searrow r'^* & \downarrow \eta^{-1*} & \swarrow t'^* & \parallel \\ & & \underline{\mathcal{V}}(Y) & \xrightarrow{t'_*} & \underline{\mathcal{V}}(B) \end{array}$$

Θ

The left triangle strictly commutes, and the right one is connected by the natural transformation described above. This gives a monoidal natural transformation between the two functors.

We first show that $M_{\underline{\mathcal{V}}}$ is a strict 2-functor, and will later show that it is PC-enriched.

We claim that $M_{\underline{\mathcal{V}}_{A,B}}$ is functorial. This sends Id_{ξ} to $Id_{M_{\underline{\mathcal{V}}}(\xi)}$ as the pullback square is a

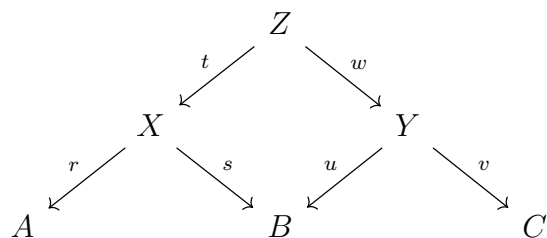
lex-pullback. For composition we consider:

$$\begin{array}{ccc}
 & X & \\
 r \swarrow & & \searrow t \\
 A & \xleftarrow{r'} Y & \xrightarrow{t'} B \\
 r'' \swarrow & & \searrow t'' \\
 & Z & \\
 \eta \downarrow & & \downarrow \eta' \\
 & Y & \\
 \eta^{-1*} \downarrow & & \downarrow \eta'^{-1*} \\
 \mathcal{V}(X) & \xrightarrow{t_*} \mathcal{V}(B) & \\
 r^* \swarrow & \downarrow \eta^{-1*} \Theta & \parallel \\
 \mathcal{V}(A) & \xrightarrow{r'^*} \mathcal{V}(Y) \xrightarrow{t'_*} \mathcal{V}(B) & \\
 r''^* \swarrow & \downarrow \eta'^{-1*} \Theta & \parallel \\
 & \mathcal{V}(Z) \xrightarrow{t''_*} \mathcal{V}(B) &
 \end{array}
 \quad \mapsto \quad
 \begin{array}{ccc}
 & \mathcal{V}(X) & \xrightarrow{t_*} \mathcal{V}(B) \\
 r^* \swarrow & & \parallel \\
 \mathcal{V}(A) & \xrightarrow{(\eta'\eta)^{-1*}} \mathcal{V}(Y) & \xrightarrow{t'_*} \mathcal{V}(B) \\
 r''^* \swarrow & \downarrow (\eta'\eta)^{-1*} \Theta & \parallel \\
 & \mathcal{V}(Z) & \xrightarrow{t''_*} \mathcal{V}(B)
 \end{array}$$

Where the equality is by the naturality of Θ .

To see that compositional unit are preserved we recall that as $Id^* = Id, Id_* = Id$.

On objects (spans) this amounts to showing the two compositions agree:

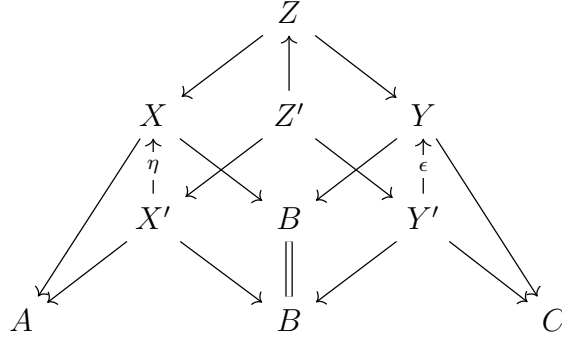


Where Z is the chosen pullback.

$$v_* \circ u^* \circ s_* \circ r^* = v_* \circ w_* \circ t^* \circ r^* = (v \circ w)_* \circ (r \circ t)^*$$

Where the first equality is from the double coset formula.

Now we show that this commutes on morphisms. For the diagram in Fin^G :



we need to show that the two pasting diagrams agree.

$$\begin{array}{ccccccccc}
 V_A & \longrightarrow & V_X & \longrightarrow & V_B & \longrightarrow & V_Y & \longrightarrow & V_C \\
 \parallel & & \downarrow & \swarrow & \parallel & & \downarrow & \swarrow & \parallel \\
 V_A & \longrightarrow & V_{X'} & \longrightarrow & V_B & \longrightarrow & V_{Y'} & \longrightarrow & V_C
 \end{array}$$

$$\begin{array}{ccccccc}
 V_A & \longrightarrow & V_Z & \longrightarrow & V_C \\
 \parallel & & \downarrow & \swarrow & \parallel \\
 V_A & \longrightarrow & V_{Z'} & \longrightarrow & V_C
 \end{array}$$

Using the naturality of Θ , the second diagram is equal to the following one:

$$\begin{array}{ccccccccc}
 V_A & \longrightarrow & V_X & \longrightarrow & V_Z & \longrightarrow & V_Y & \longrightarrow & V_C \\
 \parallel & & \downarrow & & \downarrow & \swarrow & \downarrow & \swarrow & \parallel \\
 V_A & \longrightarrow & V_{X'} & \longrightarrow & V_{Z'} & \longrightarrow & V_{Y'} & \longrightarrow & V_C
 \end{array}$$

By the cube lemma 13.0.13, the composite of the middle two cells in this are equal to the middle two cells of the first diagram, using the fact that Z, Z' are lex-pullbacks. This completes the proof that $M_{\mathcal{V}}$ is a strict 2-functor.

Now we show that it is PC-enriched. The first step is to show that $M_{\mathcal{V},A,B} : \mathbf{GE}_{ord}^{op}(A, B) \rightarrow Perm(\mathcal{V}(A), \mathcal{V}(B))$ is strong monoidal strict unital.

Now we need to show this is monoidal on hom-categories. Given spans $(A \leftarrow X \rightarrow B), (X \leftarrow X' \rightarrow B)$ we have the monoidality isomorphism given by the large pasted 2-cell,

where r, t denote restrictions and transfers, ϕ is the reordering isomorphism. Θ is trivial as the square shown at the bottom left is a lex-pullback.

$$\begin{array}{ccc}
\underline{\mathcal{V}}(A) & \xrightarrow{r} & \underline{\mathcal{V}}(X) \times \underline{\mathcal{V}}(X') \xrightarrow{t} \underline{\mathcal{V}}(B) \times \underline{\mathcal{V}}(B) \\
\downarrow t & & \swarrow \iota^* \quad \searrow \eta^{-1} \quad \downarrow \lambda \\
\underline{\mathcal{V}}((X \amalg X')_{\times B}) & \xrightarrow{\phi^{-1*}} & \underline{\mathcal{V}}(X \amalg X') \xrightarrow{t} \underline{\mathcal{V}}(B^{\amalg 2}) \\
& & \downarrow \phi^* \quad \swarrow \Theta \quad \downarrow \xi^* \\
(X \amalg X')_{\times B} & \xrightarrow{\phi} & X \amalg X' \quad \underline{\mathcal{V}}((X \amalg X')_{\times B}) \xrightarrow{t} \underline{\mathcal{V}}(B \times [2]) \\
\downarrow & & \downarrow t \quad \downarrow \nabla_* \\
B \times 2 & \xrightarrow{\xi} & B^{\amalg 2} \quad \underline{\mathcal{V}}(B) \xrightarrow{=} \underline{\mathcal{V}}(B)
\end{array}$$

Showing this is natural in X, X' comes down to showing we can “stack” this diagram; i.e. the following diagram commutes. Consider spans $(A \leftarrow Y \rightarrow B), (A \leftarrow Y' \rightarrow B)$ and isomorphisms of spans $X \cong Y, X' \cong Y'$.

$$\begin{array}{ccccc}
\underline{\mathcal{V}}(A) & \xrightarrow{\quad} & \underline{\mathcal{V}}(Y) \times \underline{\mathcal{V}}(Y') & \xrightarrow{\quad} & \underline{\mathcal{V}}(B) \times \underline{\mathcal{V}}(B) \\
\downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\
\underline{\mathcal{V}}(A) & \xrightarrow{\quad} & \underline{\mathcal{V}}(X) \times \underline{\mathcal{V}}(X') & \xrightarrow{\quad} & \underline{\mathcal{V}}(B) \times \underline{\mathcal{V}}(B) \\
\downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\
\underline{\mathcal{V}}((Y \amalg Y')_{\times B}) & \xrightarrow{\quad} & \underline{\mathcal{V}}(Y \amalg Y') & \xrightarrow{\quad} & \underline{\mathcal{V}}(B^{\amalg 2}) \\
\downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\
\underline{\mathcal{V}}((X \amalg X')_{\times B}) & \xrightarrow{\quad} & \underline{\mathcal{V}}(X \amalg X') & \xrightarrow{\quad} & \underline{\mathcal{V}}(B^{\amalg 2}) \\
\downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\
\underline{\mathcal{V}}((Y \amalg Y')_{\times B}) & \xrightarrow{\quad} & \underline{\mathcal{V}}(Y \amalg Y') & \xrightarrow{\quad} & \underline{\mathcal{V}}(B \times [2]) \\
\downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\
\underline{\mathcal{V}}((X \amalg X')_{\times B}) & \xrightarrow{\quad} & \underline{\mathcal{V}}(X \amalg X') & \xrightarrow{\quad} & \underline{\mathcal{V}}(B \times [2]) \\
\downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\
\underline{\mathcal{V}}(B) & \xrightarrow{\quad} & \underline{\mathcal{V}}(B) & \xrightarrow{\quad} & \underline{\mathcal{V}}(B) \\
\downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\
\underline{\mathcal{V}}(B) & \xrightarrow{\quad} & \underline{\mathcal{V}}(B) & \xrightarrow{\quad} & \underline{\mathcal{V}}(B)
\end{array}$$

Here we have omitted most labels for readability. The top left trapezoidal prism commutes as all of its faces strictly commute (all 2-cells here are trivial). The top triangular prism commutes by the naturality of η with respect to restrictions along isomorphisms. The top

right cube commutes by Lemma 13.0.15, The middle right cube commutes by Lemma 13.0.13 and the bottom right cube commutes by Lemma 13.0.14.

The additive units are preserved as $\underline{\mathcal{V}}(\emptyset) \cong *$ And this is symmetric with respect to swapping X and Y so this is a strictly unital functor of permutative categories.

Now we show that composition is enriched, in other words the following diagram commutes in \mathbf{Perm} . We have already shown that the underlying functor commutes, so we show that the distributivity morphisms agree as well.

$$\begin{array}{ccc}
 GE_{ord}^{op}(B, C) \times GE_{ord}^{op}(A, B) & \xrightarrow{comp} & GE_{ord}^{op}(A, C) \\
 \downarrow M & & \downarrow M \\
 Perm(\underline{\mathcal{V}}(B), \underline{\mathcal{V}}(C)) \times Perm(\underline{\mathcal{V}}(A), \underline{\mathcal{V}}(B)) & \xrightarrow{comp} & Perm(\underline{\mathcal{V}}(A), \underline{\mathcal{V}}(C))
 \end{array}$$

In \mathbf{Perm} , $\delta_1 = Id$. In GE_{ord} $\delta_2 = Id$, implying that in GE_{ord}^{op} , $\delta_1 = Id$, so all the relevant 2-cells are trivial and monoidality is preserved.

Checking δ_2 is trickier though. First we do some set up. Consider spans $(B \leftarrow Y \rightarrow C), (A \leftarrow X \rightarrow B), (A \leftarrow X' \rightarrow B)$. Let U, U' be the lex-pullbacks of $(Y \rightarrow B \leftarrow X), (Y \rightarrow B \leftarrow X')$ respectively.

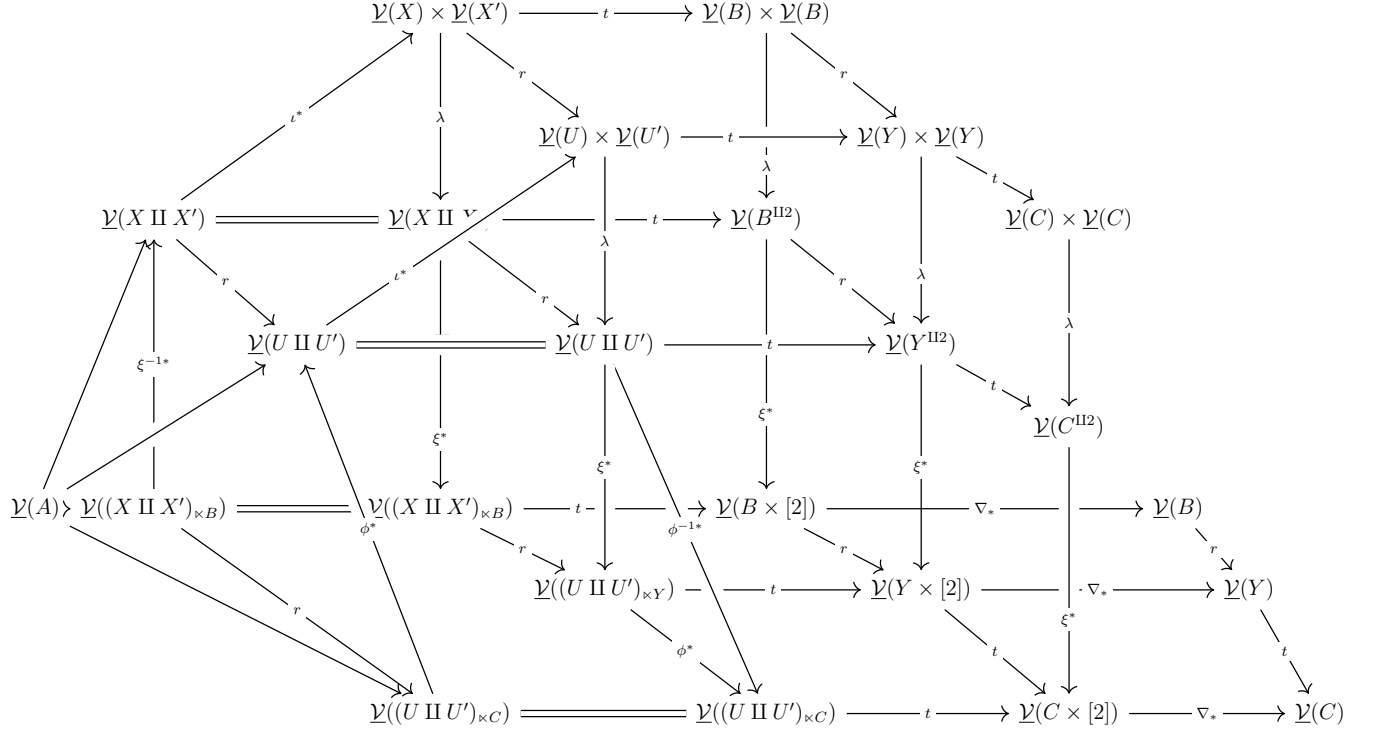
Across then down yields the top diagram, down then across yields the bottom.

$$\begin{array}{ccccccc}
\underline{\mathcal{V}}(A) & \xrightarrow{\quad r^*, r'^* \quad} & \underline{\mathcal{V}}(U) \times \underline{\mathcal{V}}(U') & \xrightarrow{t_*} & \underline{\mathcal{V}}(C) \times \underline{\mathcal{V}}(C) & & \\
\downarrow r^* & \searrow r^* & \downarrow \lambda & & \downarrow \lambda & & \\
\underline{\mathcal{V}}((U \amalg U') \rtimes_C) & \xrightarrow{\phi^{-1*}} & \underline{\mathcal{V}}(U \amalg U') & \xrightarrow{t_*} & \underline{\mathcal{V}}(C^{\amalg 2}) & & \\
\downarrow t_* & & \downarrow \epsilon & & \downarrow \xi^* & & \\
\underline{\mathcal{V}}(Y) & \xrightarrow{t_*} & \underline{\mathcal{V}}(C) & \xrightarrow{t_*} & \underline{\mathcal{V}}(C) & \xrightarrow{\nabla_*} & \underline{\mathcal{V}}(C) \\
\downarrow t_* & & \downarrow \Theta & & \downarrow t_* & & \downarrow \nabla_* \\
\underline{\mathcal{V}}((U \amalg U') \rtimes_Y) & \xrightarrow{\phi^*} & \underline{\mathcal{V}}((U \amalg U') \rtimes_C) & \xrightarrow{t_*} & \underline{\mathcal{V}}(C \times [2]) & & \\
\downarrow t_* & & \downarrow \Theta & & \downarrow \nabla_* & & \\
\underline{\mathcal{V}}(Y) & \xrightarrow{t_*} & \underline{\mathcal{V}}(C) & \xrightarrow{t_*} & \underline{\mathcal{V}}(C) & \xrightarrow{\nabla_*} & \underline{\mathcal{V}}(C)
\end{array}$$

$$\begin{array}{ccccccc}
\underline{\mathcal{V}}(A) & \xrightarrow{\quad r \quad} & \underline{\mathcal{V}}(U) \times \underline{\mathcal{V}}(U') & & & & \\
\downarrow r & \searrow r & \downarrow \lambda & & \downarrow \lambda & & \\
\underline{\mathcal{V}}(X \amalg X') & \xrightarrow{t^*} & \underline{\mathcal{V}}(X \amalg X') & \xrightarrow{t} & \underline{\mathcal{V}}(B^{\amalg 2}) & \xrightarrow{r} & \underline{\mathcal{V}}(Y^{\amalg 2}) & \xrightarrow{t} & \underline{\mathcal{V}}(C^{\amalg 2}) \\
\downarrow \xi^* & & \downarrow \epsilon & & \downarrow \xi^* & & \downarrow \xi^* & & \downarrow \xi^* \\
\underline{\mathcal{V}}((X \amalg X') \rtimes_B) & \xrightarrow{\xi^{-1*}} & \underline{\mathcal{V}}((X \amalg X') \rtimes_B) & \xrightarrow{t} & \underline{\mathcal{V}}(B \times [2]) & \xrightarrow{r} & \underline{\mathcal{V}}(Y \times [2]) & \xrightarrow{t} & \underline{\mathcal{V}}(C \times [2]) \\
\downarrow \xi^* & & \downarrow \xi^{-1*} & & \downarrow \xi^* & & \downarrow \xi^* & & \downarrow \xi^* \\
\underline{\mathcal{V}}(B) & \xrightarrow{r} & \underline{\mathcal{V}}(Y) & \xrightarrow{t} & \underline{\mathcal{V}}(C) & & & & \\
\downarrow \nabla_* & & \downarrow \nabla_* & & \downarrow \nabla_* & & & & \\
\underline{\mathcal{V}}(B) & \xrightarrow{r} & \underline{\mathcal{V}}(Y) & \xrightarrow{t} & \underline{\mathcal{V}}(C) & & & &
\end{array}$$

To show they agree we make the following large diagram. There are four 3-cells in it. The trapezoidal prism on the left commutes as all of its side strictly commute (only trivial 2-cells). Checking the triangular prism commutes comes down to writing out the definition of Λ from 11.1.8 and checking the appropriate pasting diagrams agree. The top cube commutes by

Lemma 13.0.15, and the bottom cube commutes by Lemma 13.0.13.



□

Definition 14.1.2. Given a permutative SGA \mathcal{V} , we define its equivariant K -theory as the K theory of $M_{\mathcal{V}}$.

Proposition 14.1.3. For $X \in \mathbf{Fin}^G$, $\Sigma_G \wr X$ is sent to the PMF $S_X := \mathbf{GE}_{ord}^{op}(X, -) = \mathbf{GE}_{ord}(-, X)$

This is by a direct check of the definitions.

Proposition 14.1.4. (Equivariant Barratt-Priddy-Quillen)

$$K(\mathbf{Fin}^G) = \mathbb{S}_G$$

Proof. We recall that $\mathbf{Fin}^G = \Sigma_G \wr *$ and we apply Theorem 9.4 of [BO] that $\Phi K_{\bullet}(S_X)$ corresponds to $\Sigma_G^{\infty}(X_+)$. Plugging in $X = *$ we have that $K(\mathbf{Fin}^G) = \Sigma_G^{\infty}(*_+) = \mathbb{S}_G$. □

14.2 PFMs to SGAs

In this section we let $\underline{M} : \mathbf{GE}_{ord}^{op} \rightarrow \mathbf{Perm}$ be a PMF.

Notation 14.2.1. Let $f : A \rightarrow B, g : C \rightarrow D$ in \mathbf{Fin}^G . We call $(B \xleftarrow{f} A = A)$ *restriction* and denote it by f^* . We call $(C = C \xrightarrow{g} D)$ *transfer* and denote it by g_* . Note that \underline{M} is implicit in these definitions.

Remark 14.2.2. We observe that using restrictions \underline{M} defines a functor $M^* : (\mathbf{Fin}^G)^{op} \rightarrow \mathbf{Cat}$ sending $A \mapsto \underline{M}(A), f : A \rightarrow B \mapsto f^* : \underline{M}(B) \rightarrow \underline{M}(A)$. Similarly using transfers we can define a functor $M_* : \mathbf{Fin}^G \rightarrow \mathbf{Cat}$ sending $A \mapsto \underline{M}(A), f : A \rightarrow B \mapsto f_* : \underline{M}(A) \rightarrow \underline{M}(B)$.

These follow from the associativity and one-sided unitality of lex-pullbacks in \mathbf{Fin}^G as well as \underline{M} being strictly functorial.

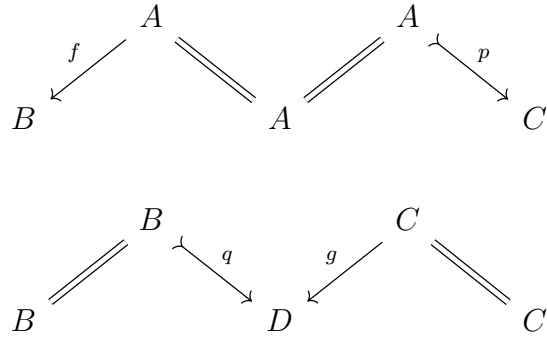
Lemma 14.2.3. *Restrictions and transfers created by \underline{M} satisfy a version of the double coset formula.*

Given a pullback square in \mathbf{Fin}^G :

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ p \downarrow & & \downarrow q \\ C & \xrightarrow{g} & D \end{array}$$

We have that $p_ f^* \cong g^* q_*$ with equality if the square is a lex-pullback. This is also natural similarly to the case with SGAs.*

Proof. We first prove this in the case of a lex-pullback. As \underline{M} functorial, this follows from the two pairs of spans in \mathbf{GE}_{ord}^{op} having the same composite:



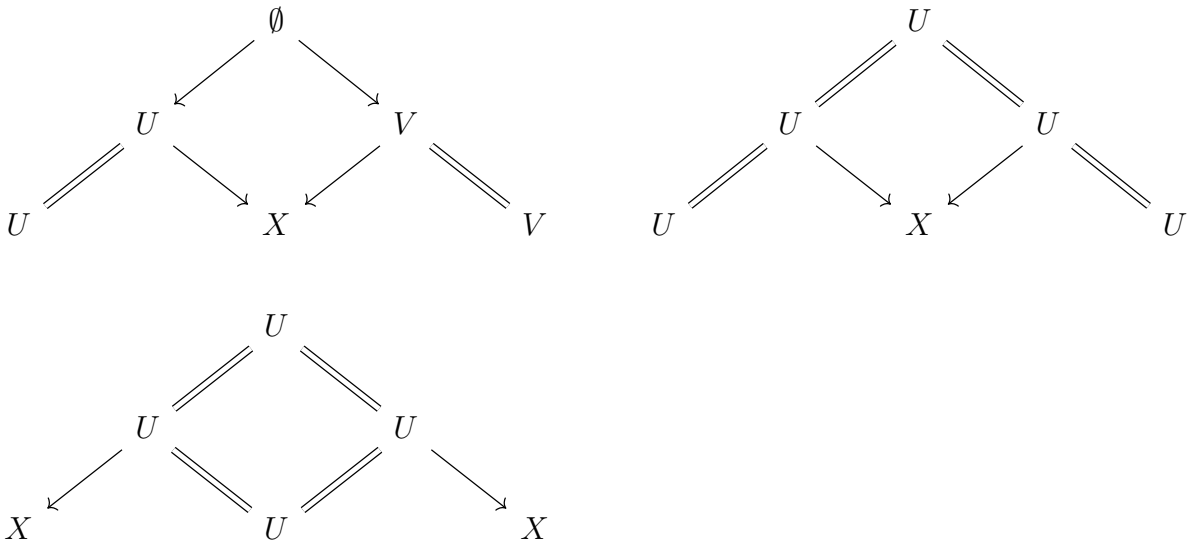
In the case that A is not the lex-pullback here, then the composite of the top span is $(B \xleftarrow{f} A \xrightarrow{p} C)$ which is isomorphic as a span to the composite of the lower two spans. Thus \underline{M} sends it to a natural isomorphism. Showing the naturality comes down to similar diagram chasing arguments. \square

Lemma 14.2.4. *Let $\iota_i : U_i \rightarrow X := \amalg_i U_i$ be the inclusion. Then we have the following identities:*

$$\iota_i^* \iota_{i*} = Id, \iota_j^* \iota_{i*} = 0 \text{ for } i \neq j, \text{ and } \sum_i \iota_{i*} \iota_i^* = Id_X.$$

We use this notation through this section.

Proof. These we verify directly with the help of these diagrams showing composition in \mathbf{GE}_{ord}^{op}



And direct computations show that the sums of the $(X \leftarrow U \rightarrow X)$ over U is $(X = X = X)$ in $\text{GE}_{ord}^{op}(X, X)$.

□

Lemma 14.2.5. *Let $i_{\underline{M}(U)} : \underline{M}(U) \rightarrow \prod_i \underline{M}(U)$ denote the inclusion which is the identity on the $\underline{M}(U)$ factor and the zero functor on all others.*

Then for $\mathcal{C} \in \text{Perm}$, $f_i : \mathcal{C} \rightarrow \underline{M}(U_i)$, for each i , as functors $\mathcal{C} \rightarrow \prod_i \underline{M}(U_i)$,

$$(f_i)_i = \sum_i i_{\underline{M}(U_i)} \circ f_i.$$

This follows from direct computation.

Proposition 14.2.6. *PMFs are weakly additive in that $\underline{M}(X) \simeq \prod_i \underline{M}(U_i)$.*

Proof. In one direction we have $\iota^* : \underline{M}(X) \rightarrow \prod_i \underline{M}(U_i)$ defined as $(\underline{M}(\iota_i^*))_i$.

By the lemma this is equal to $\sum_i i_{\underline{M}(U_i)} \circ \underline{M}(\iota_i^*)$.

In the other direction we construct $\lambda : \prod_i \underline{M}(U_i) \rightarrow \underline{M}(X)$ as $\sum_i \underline{M}(\iota_{i*}) \pi_{\underline{M}(U_i)}$. We observe that λ here is associative and symmetric in the sense of 11.1.4 and ???. We now construct the unit and counit of this equivalence.

$$\eta : Id_{\prod \underline{M}(U)} = \sum_i i_{\underline{M}(U_i)} \circ \pi_{\underline{M}(U_i)} = \sum_{i,j} i_{\underline{M}(U_i)} \circ \underline{M}(\iota_i^* \circ \iota_{j*}) \circ \pi_{\underline{M}(U_j)} =$$

$$\sum_{i,j} i_{\underline{M}(U_i)} \circ \underline{M}(\iota_i^*) \circ \underline{M}(\iota_{j*}) \circ \pi_{\underline{M}(U_j)} \xrightarrow[\text{Perm}]{\delta_{\sigma,=} } \left(\sum_i i_{\underline{M}(U_i)} \circ \underline{M}(\iota_i^*) \right) \circ \left(\sum_j \underline{M}(\iota_{j*}) \circ \pi_{\underline{M}(U_j)} \right) = \iota^* \circ \lambda$$

$$\epsilon : \lambda \circ \iota^* = \left(\sum_i \underline{M}(\iota_{i*}) \circ \pi_{\underline{M}(U_i)} \right) \circ (\underline{M}(\iota_j^*))_j \xleftarrow[\text{Perm}]{=} \sum_{i=j} \underline{M}(\iota_{i*}) \circ \underline{M}(\iota_j^*) =$$

$$\sum_{i=j} \underline{M}(\iota_{i*} \circ \iota_j^*) \xrightarrow[\delta_{\underline{M}}]{\cong} \underline{M} \sum_i \iota_{i*} \circ \iota_{U_i}^* = \underline{M}(Id_X) = Id_{\underline{M}(X)}$$

The first morphism (*Perm*) is an equality because the left distributivity morphism for composition is equality (see [BlO15], top of page 10) and there is no addition on the right here. δ_M denotes the distributivity morphisms of M . □

Remark 14.2.7. This is where we needed PMFs and thus PC-categories to be strong monoidal instead of lax.

Lemma 14.2.8. ϵ is natural along the restrictions of isomorphisms in \mathbf{Fin}^G in the sense of Diagram 11.1.3.

Notation 14.2.9. In this proof we let $X = U \amalg V, X' = U' \amalg V'$ and $r : U \rightarrow U', s : V \rightarrow V'$ isomorphisms

Written out we require that the two pasting diagrams (of monoidal functors) agree:

$$\begin{array}{ccccc} & & \underline{M}(U') \times \underline{M}(V') & & \\ & \nearrow & \parallel \epsilon \downarrow & \searrow & \\ \underline{M}(X') & \xrightarrow{\quad} & & \xrightarrow{\quad} & \underline{M}(X') \xrightarrow{\underline{M}((r \amalg s)^*)} \underline{M}(X) \\ & \xlongequal{\quad} & & \xlongequal{\quad} & \end{array}$$

$$\begin{array}{ccccc} & & \underline{M}(U) \times \underline{M}(V) & & \\ & \nearrow & \parallel \epsilon \downarrow & \searrow & \\ \underline{M}(X') & \xrightarrow{\underline{M}((r \amalg s)^*)} \underline{M}(X) & \xrightarrow{\quad} & \xrightarrow{\quad} & \underline{M}(X) \\ & \xlongequal{\quad} & & \xlongequal{\quad} & \end{array}$$

Proof. By the definition of ϵ , this is the same as these two pasting diagrams agreeing

$$\begin{array}{ccc}
\underline{M}(X') & \xrightarrow{\Sigma \underline{M}(\iota_* \iota^*)} & \underline{M}(X') \xrightarrow{\underline{M}((r \amalg s)^*)} \underline{M}(X) \\
& \searrow \delta_M \downarrow \swarrow & \\
\underline{M}(X') & \xrightarrow{\underline{M}(\Sigma(\iota_* \iota^*))} & \underline{M}(X)
\end{array}$$

$$\begin{array}{ccc}
\underline{M}(X') \xrightarrow{\underline{M}((r \amalg s)^*)} \underline{M}(X) & \xrightarrow{\Sigma \underline{M}(\iota_* \iota^*)} & \underline{M}(X) \\
& \searrow \delta_M \downarrow \swarrow & \\
\underline{M}(X') & \xrightarrow{\underline{M}(\Sigma(\iota_* \iota^*))} & \underline{M}(X)
\end{array}$$

As $(r \amalg s)^*$ is invertible with inverse $((r \amalg s)^{-1})^*$ it suffices to show these agree:

$$\begin{array}{ccc}
\underline{M}(X') \xrightarrow{\underline{M}(((r \amalg s)^{-1})^*)} \underline{M}(X') & \xrightarrow{\Sigma \underline{M}(\iota_* \iota^*)} & \underline{M}(X') \xrightarrow{\underline{M}((r \amalg s)^*)} \underline{M}(X) \\
& \searrow \delta_M \downarrow \swarrow & \\
\underline{M}(X') & \xrightarrow{\underline{M}(\Sigma(\iota_* \iota^*))} & \underline{M}(X)
\end{array}$$

$$\begin{array}{ccc}
\underline{M}(X) & \xrightarrow{\Sigma \underline{M}(\iota_* \iota^*)} & \underline{M}(X) \\
& \searrow \delta_M \downarrow \swarrow & \\
\underline{M}(X) & \xrightarrow{\underline{M}(\Sigma(\iota_* \iota^*))} & \underline{M}(X)
\end{array}$$

This follows from the diagram commuting as monoidal functors - the distributivity morphisms (in this case the middle one) are suitably preserved.

$$\begin{array}{ccc}
\mathbf{GE}_{ord}^{op}(X, X') \times \mathbf{GE}_{ord}^{op}(X', X') \times \mathbf{GE}_{ord}^{op}(X', X) & \xrightarrow{comp} & \mathbf{GE}_{ord}^{op}(X, X) \\
\downarrow \underline{M}, \underline{M}, \underline{M} & & \downarrow \underline{M} \\
\mathbf{Perm}(\underline{M}(X), \underline{M}(X')) \times \mathbf{Perm}(\underline{M}(X'), \underline{M}(X')) \times \mathbf{Perm}(\underline{M}(X'), \underline{M}(X)) & \xrightarrow{comp} & \mathbf{Perm}(\underline{M}(X), \underline{M}(X))
\end{array}$$

We apply this to $(r \amalg s)^{-1)*}$, $\iota_* \iota^*$, $(r \amalg s)^*$ in the top left, where $\iota = \iota_U, \iota_V$. And using the fact that $((r \amalg s)^{-1})^* \circ \iota_{U*} \circ \iota_U^* \circ (s \amalg r)^* = \iota_{U'*} \circ \iota_{U'}^*$ and similarly for V in place of U .

□

Theorem 14.2.10. *Given a PMF \underline{M} we can construct a strict, split SGA, furthermore it is additive when \underline{M} satisfies a certain technical condition.*

This technical condition is stated in Equation 14.2.1. It can be summarized as saying that either $M_{A,B}$ is strict monoidal its image is strict monoidal in a particular circumstance.

Proof. We construct $\underline{\mathcal{V}}$ as follows.

- As a \mathbf{Fin}^G category, $\underline{\mathcal{V}}$ is given by applying the Grothendieck construction to $M^* : (\mathbf{Fin}^G)^{op} \rightarrow \mathbf{Cat}$. So $\underline{\mathcal{V}}(A) = \underline{M}(A)$ and the chosen lift of $f : A \rightarrow B$ is $\underline{M}(f^*) : \underline{\mathcal{V}}(B) \rightarrow \underline{\mathcal{V}}(A)$. This is clearly a split fibration as M^* is a functor.
- Additivity comes from that of M , which we showed in Proposition 14.2.6.

By construction λ is associative and commutative.

ϵ is preserved by restriction along isomorphism as shown in Lemma 14.2.8. On the other hand, we do not know the original η of Lemma 14.2.6 is adjoint to ϵ so we replace it with η' which is. Then η' is preserved by restriction along isomorphisms.

- \otimes comes from transfers. Showing this is a map of split fibrations will be the tricky part. We need to show that it is actually a functor, might follow from Grothendieck construction but we need to check.

$$\otimes : \hat{\Sigma}_G \wr \underline{\mathcal{V}}(A) \rightarrow \underline{\mathcal{V}}(A)$$

is given by

$$A' \xrightarrow{p} A, x \in \underline{\mathcal{V}}(A') \mapsto \underline{M}(f_*)(x)$$

A morphism in $\hat{\Sigma}_G \wr \underline{\mathcal{V}}$ over $f : B \rightarrow A$

Consists of a pullback square in \mathbf{Fin}^G

$$\begin{array}{ccc}
B' & \xrightarrow{g} & A' \\
q \downarrow & & \downarrow p \\
B & \xrightarrow{f} & A
\end{array}$$

Along with a morphism $\alpha : \underline{M}(g^*)(x) \rightarrow x'$ in $\underline{M}(B')$. It is cartesian if the morphism α is an isomorphism, and the chosen lift if the pullback square is a lex-pullback, and α is an equality.

\otimes applied to such a square is defined as

$$f^* p_*(x) \cong q_* g^*(x) \xrightarrow{q_*(\alpha)} q_*(x')$$

where the first isomorphism comes from the double coset formula of Lemma ??; this is a morphism in $\underline{\mathcal{V}} = \int M^*$ as hoped.

If the original morphism is cartesian, this is as well. If it is the chosen cartesian lift of f , then the square is a lex-pullback, so the double coset isomorphism is an identity and $g_*(\alpha)$ is an equality as well. This is the chosen lift in the Grothendieck construction. This shows that \otimes is a map of split fibrations.

- Next we show that \otimes is associative and unital, in other words $\underline{\mathcal{V}}$ is an algebra. These essentially come down to the fact that transfers are functorial. For this we simply check that the diagrams 12.0.1 and 12.0.2 commute. For unitality we claim that $\underline{\mathcal{V}} \xrightarrow{\delta_{-1}} \hat{\Sigma}_G \wr \underline{\mathcal{V}} \xrightarrow{\otimes} \underline{\mathcal{V}}$ is the identity. This composite is given by $(A, x \in \underline{M}(A)) \mapsto (A, Id_A : A \rightarrow A, x \in \underline{M}(A)) \mapsto (A, Id_{A^*} x)$ which is the identity as $Id_{A^*} = Id_{\underline{M}(A)}$. For associativity we consider

$(A \xrightarrow{f} B \xrightarrow{g} C, x \in \underline{M}(A))$ Applying $\hat{\Sigma}_G \wr \otimes$ to this gets $(B \xrightarrow{g} C, f_*(x) \in \underline{M}(B))$ applying \otimes to this gives $g_* f_*(x)$.

On the other side we first apply σ_0 to get $(A \xrightarrow{gf} C, x)$; applying \otimes to this gives $(gf)_*(x)$.

This is equal as transfers are functorial.

- We now discuss the additivity of \otimes recalling that it means the following diagram strictly

commutes:

$$\begin{array}{ccc} \hat{\Sigma}_G \wr \underline{\mathcal{V}}(U) \times \hat{\Sigma}_G \wr \underline{\mathcal{V}}(V) & \xrightarrow{\lambda} & \hat{\Sigma}_G \wr \underline{\mathcal{V}}(X) \\ \otimes \downarrow & & \downarrow \otimes \\ \underline{\mathcal{V}}(U) \times \underline{\mathcal{V}}(V) & \xrightarrow{\lambda} & \underline{\mathcal{V}}(X) \end{array}$$

We start with $(f, x)(g, y) \in \hat{\Sigma}_G \wr \underline{\mathcal{V}}(U) \times \hat{\Sigma}_G \wr \underline{\mathcal{V}}(V)$, where $f : U' \rightarrow U, g : V' \rightarrow V$ and let $X' = U' \amalg V'$.

Going across then down gives $\underline{M}((f \amalg g)_*)(\underline{M}(\iota_*)(x) + \underline{M}(\iota_*)(y))$. Going down then across gives $\underline{M}(\iota_*)\underline{M}(f_*)(x) + \underline{M}(\iota_*)\underline{M}(g_*)(y)$.

Thus additivity is exactly having the equality:

$$\underline{M}(\iota_*)\underline{M}(f_*)(x) + \underline{M}(\iota_*)\underline{M}(g_*)(y) = \underline{M}((f \amalg g)_*)(\underline{M}(\iota_*)(x) + \underline{M}(\iota_*)(y)). \quad (14.2.1)$$

This holds if $\underline{M}((f \amalg g)_*)$ is strict monoidal when applied here. This is also satisfied if

$\underline{M}((f \amalg g)_*) = \underline{M}(f_*) + \underline{M}(g_*) = \underline{M}(X' \xleftarrow{\iota} U' \xrightarrow{\iota f} X) + \underline{M}(X' \xleftarrow{\iota} V' \xrightarrow{\iota g} X)$, which holds if \underline{M} is strict monoidal on hom-categories.

We note that 14.2.1 is satisfied for PMFs originating from permutative SGAs.

□

Theorem 14.2.11. *On the underlying categories, restrictions, transfers, and λ s, these two constructions are inverses.*

By construction the categories, restrictions, and transfers correspond. In the case of permutative SGAs we recall that λ is defined as a sum of transfers so this follows as well.

We note that this does not imply these are inverse constructions as it does not take into account any of the relevant 2-cells.

Conjecture 14.2.12. We believe that these form equivalent 2-categories.

Chapter 15

Comparison with Equivariant Symmetric Monoidal Structures of Hill Hopkins and G -Symmetric Monoidal ∞ -categories

15.1 Hill Hopkins

In this chapter we review the symmetric monoidal Mackey functors and equivariant symmetric monoidal structures of [HH16] and compare them with SGAs. All numbers refer to that paper.

Definition 15.1.1. A *symmetric monoidal coefficient system* is a pseudofunctor $(\mathcal{O}_G)^{op} \rightarrow \mathit{Sym}$, the category of symmetric monoidal categories and strong monoidal functors.

We observe that this is equivalent to a strictly additive pseudo- Fin^G -category.

Definition 15.1.2. A *symmetric monoidal Mackey functor* $\underline{M} = (M^*, M_*)$ consists of two pseudofunctors $M^* : (\mathcal{O}_G)^{op} \rightarrow Sym, M_* : \mathcal{O}_G \rightarrow Sym$ called *restriction* and *transfer* which agree on objects and satisfy a double coset formula up to isomorphism.

Remark 15.1.3. We note that this definition of [HH16] is rather vague, and one of the goal of developing SGAs was to formalize the details of this definition and the coherences involved.

Remark 15.1.4. We can realize these as pseudo-SGAs, strictly additive as a \mathbf{Fin}^G category.

Definition 15.1.5. \underline{Set} is the symmetric monoidal coefficient system given by $G/H \mapsto \mathbf{Fin}^H \simeq \mathbf{Fin}_{/(G/H)}^G$. Restriction is given by pullback along $G/K \rightarrow G/H$, transfer by postcomposition.

We observe that this is equivalent to the SGA $\underline{\mathbf{Fin}}^G$.

Definition 15.1.6. \underline{Set}^{Iso} is the symmetric monoidal coefficient system given by taking the objectwise maximal subgroupoid of \underline{Set} .

Proposition 15.1.7 ([HH16] 3.1). *The category of symmetric monoidal coefficient systems has a product given by objectwise cartesian products.*

Definition 15.1.8. A *bilinear* functor of symmetric monoidal coefficient systems $\underline{\mathcal{C}}_1 \times \underline{\mathcal{C}}_2 \rightarrow \underline{\mathcal{D}}$ which is bilinear on objects and suitably coherent.

Definition 15.1.9. A *genuine G -symmetric monoidal structure* on a symmetric monoidal coefficient system $\underline{\mathcal{C}}$ is a bilinear map $\square : \underline{Set}^{Iso} \times \underline{\mathcal{C}} \rightarrow \underline{\mathcal{C}}$

such that when restricting to trivial H -sets $X \in \underline{Set}^{Iso}(G/H), \square(G/H) : X \times \underline{\mathcal{C}}(G/H) \rightarrow \underline{\mathcal{C}}(G/H)$ is simply the exponentiation functor $A \mapsto A^{\otimes |X|}$ and such that the following commutative diagram commutes up to natural isomorphism:

$$\begin{array}{ccc}
\underline{Set}^{Iso} \times \underline{Set}^{Iso} \times \underline{\mathcal{C}} & \xrightarrow{Id \times \square} & \underline{Set}^{Iso} \times \underline{\mathcal{C}} \\
(-\times -) \times Id \downarrow & & \downarrow \square \\
\underline{Set}^{Iso} \times \underline{\mathcal{C}} & \xrightarrow{\square} & \underline{\mathcal{C}}
\end{array}$$

Proposition 15.1.10 ([HH16]). *Let \underline{M} be a symmetric monoidal Mackey functor, we a a genuine G -symmetric monoidal structure on it given by $\square(H/K, -) : \underline{M}(G/K) \rightarrow \underline{M}(G/K)$ by $Tr_K^H Res_K^H$.*

15.2 G -Symmetric Monoidal ∞ -categories

We briefly discuss the connections between SGAs and the G -symmetric monoidal ∞ -categories of [BDG⁺16, NS22].

Definition 15.2.1. A G -symmetric monoidal ∞ -category is a product preserving functor $Span_2(Fin^G) \rightarrow Cat_\infty$.

These can also be defined in terms of spans of fibrations over a version of Fin_*^G which satisfy a Segal condition. This is best explained in [Hor19] §2.4 and App B.

Remark 15.2.2. Given a SGA we can make a (pseudofunctorial) PMF, then take its nerve we get a G -symmetric monoidal ∞ -category.

Chapter 16

Genuine Commutative Monoids

Definition 16.0.1. A G -commutative monoid in a SGA $\underline{\mathcal{V}}$ is a map of SGAs $\mu : \underline{\mathbf{Fin}}^G \rightarrow \underline{\mathcal{V}}$.

Remark 16.0.2. The non-equivariant intuition here is that in a symmetric monoidal category \mathcal{C} , a commutative monoid is equivalent to a symmetric monoidal functor $\mathbf{Fin} \rightarrow \mathcal{C}$, where the monoid object is the image of $\underline{1}$.

Expanding this, for each H we have a symmetric monoidal functor $\mathbf{Fin}^H \rightarrow \underline{\mathcal{V}}(G/H)$.

Definition 16.0.3. A *morphism of G -commutative monoids* is a natural transformation of fibrations that commutes with the 2-cell.

Lemma 16.0.4. *Given a morphism of SGAs $\underline{\mathcal{V}} \rightarrow \underline{\mathcal{W}}$ and a G -commutative monoid in $\underline{\mathcal{V}}$, its image is a G -commutative monoid in $\underline{\mathcal{W}}$.*

This explains why homotopy groups of a ring spectrum are Tambara functors.

Remark 16.0.5. We do not know of any version of monoids in a PMF or in the G -symmetric monoidal ∞ -categories of [GMMO19]. We believe that in we can similarly view monoids as maps out of the analogue of $\underline{\mathbf{Fin}}^G$.

Definition 16.0.6. In a genuine symmetric monoidal structure of [HH16], a G -commutative monoid is an object $M \in \underline{\mathcal{C}}(G/G)$ and an extension:

$$\begin{array}{ccc} \underline{Set}^{iso} & \xrightarrow{-\square M} & \underline{\mathcal{C}} \\ & \searrow & \nearrow N \\ & \underline{Set} & \end{array}$$

In the case that $\underline{\mathcal{C}}$ comes from the symmetric monoidal Mackey functor \underline{M} we also require that N is a map of symmetric monoidal Mackey functors.

The condition that N is a map of symmetric monoidal Mackey functors can be explicitly described as in [Hor19] 3.2.2.

In that case a G -commutative monoid is a commutative monoid $M \in \underline{M}(G/G)$, with commutative monoid maps $N_H^G \square M \rightarrow * \square M \cong M$, which are pseudonatural in that $N_H^G N_K^H \cong N_K^G$.

Remark 16.0.7. A map of symmetric monoidal Mackey functors $F : \underline{Set} \rightarrow \underline{M}$ nearly determines a G -commutative monoid in \underline{M} . Indeed we calculate:

$$H/K \square Res_H^G M = Tr_K^H Res_K^H Res_H^G M \cong Tr_K^H Res_K^G M = Tr_K^H Res_K^G F(*) F(Tr_K^H Res_K^G *) = F(H/K)$$

This isomorphism arises from symmetric monoidal Mackey functors being defined as pseudo-functors from \mathcal{O}_G and $(\mathcal{O}_G)^{op}$. The extension property simply requires this to be an equality instead essentially adding a particular sort of strictness on \underline{Set}^{iso} which is not present in general.

Example 16.0.8. [Hor19] Tambara functors are the G -commutative monoids in \underline{Mack}_G , viewed as a symmetric monoidal Mackey functor.

We believe that this result also hold for monoids in the SGA \underline{Mack}_G , however we do not have a full proof of this at the moment.

Conjecture 16.0.9. A monoid in \underline{Sp}_G is a genuine G -ring spectrum.

Conjecture 16.0.10. We expect that monoids in \underline{Fin}_G are equivalent to semi-Mackey functors. This is essentially Thm 5.6 of [HH16].

Remark 16.0.11. We can also attempt to define G -commutative monoids in PMFs. For this we can consider \underline{Fin}^G as the PMF S_* defined in [BIO15] Defn . 9.1; given by $S_*(A) := \mathbf{GE}_{ord}(*, A)$; equivalently $A \mapsto \mathbf{Fin}_{/A, ord}^G$. Restrictions are lex-pullbacks, transfers are by post-composition.

Then we can define a genuine G -commutative monoid in a PMF \underline{M} is a map of PMFs $\underline{M} \rightarrow \underline{Fin}^G$. This is philosophically the same as our approach for monoids in SGAs and for G -commutative monoids in [HH16].

However we anticipate that the extreme strictness of PMFs relative to both SGAs and symmetric monoidal Mackey functors could prevent these from including useful examples. There might be workarounds such as finding strict versions using ordered G -sets similar to much of the work in this paper. One could also attempt to define a suitable weak morphism of PMFs and define a monoid using these instead.

Chapter 17

Musings and Further Directions

There are several choices we made in our general approach to SGAs as a way of unifying the different versions of genuine equivariant symmetric monoidal categories, and it is worthwhile considering other ways we could have done it.

- We could have just as well defined Fin^G categories and SGAs in terms of functors $(\text{Fin}^G)^{op} \rightarrow \text{Cat}$ instead of as fibrations over Fin^G . Of course we know that these should yield (2-categorically) equivalent theories. But the comparisons with PMFs and the symmetric monoidal Mackey functors of [HH16] might be cleaner working entirely with functors as opposed to fibrations. In particular we might have a more tractable relationship between the 2-categories of SGAs and PMFs.
- We could have chosen to work primarily in bicategories as opposed to strict 2-categories.
- We could have chosen to work internally to PMFs. Originally a major motivation for the development of SGAs was a way to bridge the K -theoretic machinery of PMFs with the monoids of [HH16]. Only at the very end of the process did we realize that we can

interpret monoids as maps of PMFs from S_* .

On its own this approach this would not account for the symmetric monoidal Mackey functors of [HH16] as these use pseudofunctors. But combining this with bicategories could work well. We still might not be able to fine-tune the different types of strictness as we can with SGAs.

- A major innovation of this paper was to use \mathbf{GE}_{ord} instead of \mathbf{GE}' or \mathbf{GE} . This required the lengthy technical results on pullbacks in \mathbf{Fin}^G .

Further directions

- Flesh out the 2-category of SGAs. In particular what do morphisms and natural transformations look like on the fibers. We expect them to be symmetric monoid functors and monoidal natural transformations, however this has not been proven.
- Compare the various Mackey functor constructions to Shimakawa's Γ_G categories. These have their own K -theory construction, which we hope would be equivalent to ours. These also are more closely connected to [GMMO].
- Develop a notion of multiplicative equivariant K -theory for SGAs and PMFs. This has already been done in [GMMO23] to prove a multiplicative equivariant Barratt-Priddy-Quillen theorem.
- For this we would define a (possibly genuine equivariant) symmetric monoidal structure on the category of SGAs, whose monoids are *bipermutative SGAs*. Then K theory of SGAs should extend to a multiplicative functor and the K -theory of bipermutative SGAs should be genuine G -ring spectra.

- In many equivariant situations, it is necessary to work with indexing systems - roughly a subset of maps along which we can define transfers or norms. This is necessary as equivariant algebraic structures are not well behaved under localization. We expect that most of the constructions of this paper could be generalized to indexing systems. In the case of G -commutative monoids, this is already done by [HH16] §4.
- A major desiderata of a good theory of genuine equivariant symmetric monoidal categories is a version of Thomason's theorem and ideally something even stronger such as an inverse equivariant K -theory functor. A version of Thomason's theorem has been proven in [Len22]. Another active area of research in that direction is to use multiplicative inverse K -theory the Guillou-May theorem to give an inverse of the K -theory of PMFs.

I am also not aware of any version of K -theory directly using [HH16], however it is relatively straightforward to strictify those into PMFs.

- Hitherto there has been minimal writing on the theory of monoids in a general equivariant symmetric category. I believe this would be a fertile ground for new research.

One avenue is applying genuine symmetric monoidal structures to the homological algebra of Mackey functors.

- As mentioned in the desiderata, we expect that genuine equivariant colored operads, such as those of [BP21], specialize to them such as those of [BP21]; analogous to how symmetric colored operads (multicategories) specialize to permutative categories. In a sense the version in [NS22] is simply defined as equivariant operads satisfying a suitable condition (analous to how symmetric monoidal ∞ -categories are a type of ∞ -operads as

in [Lur17]). However we do not expect this to carry over neatly to the lower-categorical constructions, so would require independent work.

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