Class of 1880 Exam

(Math competition ONLY for Univ. of Pennsylvania freshmen and sophomores.)

April 9, 2021

Solve the problems on blank pages. Show your work and give justification for your answers as completely as possible. You will scan or take pictures of your solutions and submit them after the exam ends. Write clearly on each page which question you are solving (and part 1, 2 etc) if you need more then one page for a question.

Your submission file should bear your lastname_firstname. For example, "Merling_Mona." You should ideally combine all your pages into one scan, but if you do not have the means to do that and are submitting several pages, each file should have your name as above, followed by what question is on that page. For example "Mona_Merling_Q3" or "Mona_Merling_Q3_part1".

Your information (name, Penn id, year and email) will be recorded separately via a google form (this is what usually would be the cover of the exam).

The google form and the link for where to upload your work will be shared in the beginning of the exam, and after the exam, we will add 15 extra minutes to submit these.

Time available: 2 hours!

1. (10 points) Find all the functions $f, g\colon \mathbb{Q} \to \mathbb{Q}$ which satisfy the conditions

$$f(g(x) + g(y)) = f(g(x)) + y,$$
$$g(f(x) + f(y)) = g(f(x)) + y,$$

for all $x, y \in \mathbb{Q}$.

2. (10 points) There are 10 teams participating in a soccer league, where each pair is scheduled to have a match. In every match, the winner gets 3 points, the loser gets 0 point, while each of the two teams gets 1 point if there is a draw. Suppose that a team aims to be placed in top 5 (more precisely, there are no more than 4 other teams who have at least as many points as they do), how many points does this team need to earn?

3. (10 points) Let S be a subset of the real numbers. Consider the sets

$$A = \{x + y \ \mid x, y \in S\}, \qquad B = \{x - y \ \mid x, y \in S\}.$$

Show that

$$\operatorname{card}(S) \cdot \operatorname{card}(B) \leq (\operatorname{card}(A))^2,$$

where $\operatorname{card}(-)$ denotes set cardinality.

4. (10 points) Let n be a nonzero natural number, and let $f\colon \mathbb{N} \to \mathbb{N}$ be the function defined by

$$f(x) = \begin{cases} \dfrac{x}{2}, & \text{if } x \text{ is even,} \\[2ex] \dfrac{x-1}{2} + 2^{n-1}, & \text{if } x \text{ is odd.} \end{cases}$$

Find all $x \in \mathbb{N}$ such that $\underbrace{(f \circ f \circ \cdots \circ f)}_{n \text{ times}}(x) = x$.

5. (10 points) Let

$$M = \{(x, y) \in \mathbb{Z}^2 : 1 \leq x, y \leq 3\}.$$

Find the total number of possible functions f defined on M satisfying the following properties:

   1. $f(x, y)$ is a nonnegative integer, for all $(x, y) \in M$;
   2. $\sum_{y=1}^{3} f(x, y) = 2$, for all $1 \leq x \leq 3$;
   3. $f(x_1, y_1) f(x_2, y_2) > 0 \implies (x_1 - x_2)(y_1 - y_2) \geq 0$.