From Arithmetic to Geometric Langlands

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1 Preliminaries: Lie Groups and Representation Theory

1.1 The Haar Measure

Let G be a locally compact topological group. Then, up to multiplication by a positive real number, there is a unique measure μ which satisfies:

- $\forall x \in X, \exists$ a neighborhood U of x such that $\mu(U) < \infty$.
- $\mu(M) = \sup_{K<<M} \mu(K)$ for any M open.
- μ is right-invariant, i.e. $\forall f \in L^1(G, d\mu)^+, \forall g \in G$, we have $\int_G f(xg)d\mu(x) = \int_G f(x)d\mu(x)$.

The first two are criterion of Radon measure. Such μ is called a Haar measure. (Kowalski, Theorem 5.2.1)

1.2 The Language of Representation Theory

A representation is irreducible if it does not have a proper invariant subspace. Let $\pi_1 : G \to GL_1(V), \pi_2 : G \to GL_2(V)$ be two representations. Then $\pi_1 \oplus \pi_2 : G \to V \oplus W$ where G acts by $g(v, w) = (gv, gw)$ is the direct sum of two representations. A representation is semisimple if it is isomorphic to a direct sum of irreducible representations.

Representation-Function Correspondence Let $\pi : G \to GL_1(\mathbb{C}) = \mathbb{C}^\times$ be a character. Then we can define a corresponding right $\mathbb{C}[G]-$module $\mathbb{C}^G$, with G acting via $g \bullet f(x) = f(xg) = \pi(g)f(x), \forall g, x \in G$. Such right modules are in bijection with characters on G. (Refer to the definition of induced representations, Kowalski Page 44)

Let $\pi : G \to GL_n(\mathbb{C})$ be a representation. Define its $\mathbb{C}-$character to be $\chi : G \to \mathbb{C}, g \mapsto Tr(\pi(g))$. (Kowalski, Page 111). Over an algebraically closed field of characteristic 0, then two finite-dimensional semisimple representations are isomorphic if and only if they have the same character. (Kowalski, Page 112)

2 Review of Arithmetic Langlands

Let K be a function field over $\mathbb{F}_q$. Let X be the abstract algebraic curve defined by the places of K, and let $|X|$ denote the set of places (i.e. valuations) of K. Let $|X|$ denote the points of X, and for $x \in |X|$, let K_x be the completion of K at x, and $k_x = \mathcal{O}_x/\mathcal{P}_x$ the residue field at x. Then we can define the group of adèles $\mathbb{A} = \prod'_{x\in|X|} K_x$. Here, the restricted product means that for $(a_x)_{x\in|X|} \in \mathbb{A}$, all but finitely many a_x come from $\mathcal{O}_x$. We can define a topology[1] on $\mathbb{A}$ that makes it a locally compact group (so that we can perform harmonic analysis on it). Note that $K \hookrightarrow \mathbb{A}$, and $\mathbb{A}/K$ is compact.

Now define $\mathbb{O} = \prod_{x\in|X|} \mathcal{O}_x$, and we have $GL_n(\mathbb{O}) \leq GL_n(\mathbb{A})$ is a maximal compact subgroup.

Remark. *The reason behind considering a maximal compact subgroup is so that*

[1] The adèle topology is strictly finer than the product of the metric topology induced by valuations on each component.

we can define a Haar measure under which $GL_n(\mathbb{O})$ is finite. We can even scale it so that it restricts to a probability measure on $GL_n(\mathbb{O})$, i.e. $\mu(GL_n(\mathbb{O})) = 1$.

We know from class field theory and the representation-function correspondence that characters on $Gal(\overline{K}/K)$ corresponds bijectively to representations of $GL_1(\mathbb{A})$ on functions of $GL_1(K)\backslash GL_1(\mathbb{A})$.

Conjecture 2.1. *Langlands conjectures that there is such bijection for $n-$dimensional representations, called automorphic representations.*

Refer to Frenkel, Page 16 for a more detailed explanation.

2.1 Smoothness and Cuspidality

2.1.1 Smoothness

Representations Let π is a representation of a locally compact group G on a complex vector space E. Then π is called smooth if $\forall v \in E$, it is invariant under the action of an open compact subgroup of G. Equivalently, this means that the action of G on E is continuous under the discrete topology of E. (Refer to Florian Herzig's notes on smooth representations) Equivalently, this means that $E = \bigcup_{K \leq G \text{ compact}} E^K$

Functions Let F be a field of characteristic zero. Since all places of a function field are non-archimedean, we have $GL_n(K)\backslash GL_n(A) \cong \prod_{x\in|X|} GL_n(\mathcal{O}_x)$.

Definition 2.1. *A function $f : GL_n(K)\backslash GL_n(\mathbb{A}) \to F$ is smooth if it is locally constant on each of $GL_n(\mathcal{O}_x) \Leftrightarrow f$ is invariant with respect to an open compact subgroup. The latter is the general definition.*

Remark. *For intuitions of this definition, refer to the smoothness of representations and the function-representation correspondence. The real significance of this definition is that it is an analogue of the Schwartz Space (in Euclidean* *spaces, these are smooth functions with tame growth), which is closed under Fourier transform and is dense in L^p, for any finite $p > 1$. In the case of a locally compact group, and our definition of smoothness, the Schwartz class is simply the compactly supported smooth functions. For more context, refer to Section 3.1 of Bump (Starts on Page 254).*

Denote $\mathsf{Funct}(GL_n(K)\backslash GL_n(\mathbb{A}))$ the set of smooth functions. Then $GL_n(\mathbb{A})$ acts on $\mathsf{Funct}(GL_n(K)\backslash GL_n(\mathbb{A}))$ by right translation.

Remark. *1.*

$$\mathsf{Funct}(GL_n(K)\backslash GL_n(\mathbb{A}))^{GL_n(\mathbb{O})} \cong_{\mathsf{Set}} \mathsf{Funct}(GL_n(K)\backslash GL_n(\mathbb{A})/GL_n(\mathbb{O})).$$

2. $\mathsf{Funct}(GL_n(K)\backslash GL_n(\mathbb{A}))$ *is not stable under the action of* $GL_n(\mathbb{A})$.

2.1.2 Cuspidality

A matrix A is unipotent if $\exists n \in \mathbb{N}$ such that $(A-1)^n = 0$. A parabolic subgroup of GL_n is a proper subgroup of GL_n containing a conjugate of all upper-triangular matrices (Bump, Page 426). The unipotent radical of a parabolic subgroup be the subgroup of all unipotent matrices.

Functions

Definition 2.2. $f \in \mathsf{Funct}(GL_n(K)\backslash GL_n(\mathbb{A}))$ *is cuspidal if* $\forall g \in GL_n(\mathbb{A})$ *and* $\forall$ *unipotent radical* U *of any parabolic subgroup,* $\int_{U(\mathbb{A})/U(K)} f(u \bullet g) d\mu(u) = 0$, *where* μ *is a Haar measure on* $U(\mathbb{A})/U(K)$.

Denote the set of cuspidal functions $\mathsf{Funct}_{cusp}(GL_n(K)\backslash GL_n(\mathbb{A}))$. We have the following properties of cuspidal functions

1. $\mathsf{Funct}_{cusp}(GL_n(K)\backslash GL_n(\mathbb{A}))$ has a $GL_n(\mathbb{A})$ action.

2. Cuspidal functions are automatically compactly supported.

For a motivation of this definition, refer to Bump, Pages 421-422.

Representations Let π be an irreducible representation of $GL_n(\mathbb{A})$. Then π is cuspidal automorphic if $Hom_{GL_n(\mathbb{A})}(\pi, \mathsf{Funct}_{cusp}(GL_n(K)\backslash GL_n(\mathbb{A}))) \neq 0$. (Note that by Multiplicity-One Theorem on Page 52 of Bump, this module of homomorphisms of $GL_n(\mathbb{A})-$modules is either 0 or has dimension-1.) Here is an important structure theorem:

Proposition 2.1. $\mathsf{Funct}_{cusp}(GL_n(K)\backslash GL_n(\mathbb{A}))$ *decomposes into a Hilbert space direct sum of irreducible subrepresentations.*

Refer to Theorem 3.3.2 in Bump. By definition of cuspidal representation, these subrepresentations are apparently cuspidal. (Inclusion into the direct sum) Again, by the Multiplicity-One theorem, each direct summand has multiplicity one.

2.2 The Hecke Algebra

Although $\mathsf{Funct}(GL_n(K)\backslash GL_n(\mathbb{A})/GL_n(\mathbb{O}))$ does not admit a $GL_n(\mathbb{A})-$acion, it does admit an action by the Hecke Algebra.

2.2.1 Definition

Let $x \in |X|$, then $\mathcal{H}(GL_n(K_x), GL-n(\mathcal{O}_x))$ is the set of functions on $GL_n(K_x)$ that are compactly supported and $GL_n(\mathcal{O}_x)-$biinvariant. Mutiplication is given by convolution:

$$f_1 * f_2 = \int_{GL_n(K_x)} f_1(g_1 \bullet g_2) f_2(g_1^{-1}) \mu(g_1),$$

where μ is the probability Haar measure on $GL_n(\mathcal{O}_x)$. The multiplicative identity is the characteristic function of $GL_n(\mathcal{O}_x)$. Here are several basic properties:

1. Compactly supported $GL_n(\mathcal{O}_x)-$biinvariant functions on $GL_n(K_x)$ correspond to compactly supported functions on $GL_n(\mathcal{O}_x)\backslash GL_n(K_x)/GL_n(\mathcal{O}_x)$.

2. The Hecke algebra is commutative. Idea of proof is this: define the transform $f(-) \mapsto f((-^{-1})^{tr})$. This is an anti-homomorphism. Note that $GL_n(K_x) = \bigcup_{a_1 \leq \cdots \leq a_n} GL_n(\mathcal{O}_x) diag(\pi^{a_1}, \cdots, \pi^{a_n}) GL_n(\mathcal{O}_x)$, where π is a uniformizer. Since the transform preserves the diagonal matrices, this anti-homomorphism is the identity.[2]

3. The Hecke algebra is associative

The global Hecke algebra $\mathcal{H}(GL_n(\mathbb{A}), GL_n(\mathbb{O}))$ is defined as $\bigotimes'_{x\in|X|} \mathcal{H}(GL_n(K_x), Gl_n(\mathcal{O}_x))$. Here the restricted tensor product means that this algebra is spanned by tensors in which almost all components are units in their corresponding local Hecke algebras.

Note that by definition of the adèle, $GL_n(\mathbb{A}) = \prod'_{x\in|X|} GL_n(K_x)$, and also $GL_n(\mathbb{O}) = \prod_{x\in|X|} GL_n(\mathcal{O}_x)$, and by definition of the local Hecke algebra it is easy to check that elements in the Hecke algebra define class functions on $GL_n(K)\backslash GL_n(\mathbb{A})/GL_n(\mathbb{O})$.

Proposition 2.2. *If π is a (irreducible) representation of $GL_n(K_x)$, then $\pi^{GL_n(\mathcal{O}_x)}$ is a (irreducible) representation of the Hecke algebra.*

The idea of proof is as follows: for any $v \in \pi$ and compactly supported function f on $GL_n(K_x)$, define $f \bullet v = \int_{GL_n(K_x)} f(g)g(v)\mu(g)$. Apparently if f is in the Hecke algebra and $v \in \pi^{GL_n(\mathcal{O}_x)}$, then $f \bullet v \in \pi^{GL_n(\mathcal{O}_x)}$.

[2]https://math.stackexchange.com/questions/4425013/why-is-the-hecke-algebra-commutative. Alternatively, refer to Bump, Theorem 1.4.2.

Corollary 2.0.1. *We can strengthen the proposition a bit: There is a bijection between isomorphism classes of irreducible spherical representations of* $GL_n(K_x)$ *and irreducible representations of the Hecke algebra given by the above.*

Proof can be found at (https://bogdanzavyalov.com/refs/notes/Spherical_Representations.pdf, Theorem 1.18, proof right after Remark 1.19) [3] For the other direction, let π be an irreducible representation of the Hecke algebra into M, then consider the $GL_n(K_x)-$ module $\mathcal{H}_{GL_n(K_x)} \otimes_{\mathcal{H}(GL_n(K_x),GL_n(\mathcal{O}_x))} (M/X)$, where X is the maximal subspace for which $X^{GL_n(\mathcal{O}_x)} = 0$, and $\mathcal{H}_{GL_n(K_x)}$ the algebra of locally constant (i.e. smooth) functions on $GL_n(K_x)$ under $*$. Note that $GL_n(K_x)$ carries an obvious action on $\mathcal{H}_{GL_n(K_x)} \otimes_{\mathcal{H}(GL_n(K_x),GL_n(\mathcal{O}_x))} (M/X)$. For a more general result with stronger restrictions, refer to Bump, Proposition 4.2.7.

2.2.2 Structure Theorem

Hecke Operators For $x \in |X|$, and w_x a uniformizer in $\mathcal{O}_x$. For $1 \leq i \leq n$, define $T_x^i \in \mathcal{H}(GL_n(K_x), GL_n(\mathcal{O}_x))$ to be the characteristic function of the $GL_n(\mathcal{O}_x)-$double coset of $(w_x, \cdots, w_x, 1, \cdots, 1)$.

Theorem 2.1. $\mathcal{H}(GL_n(K_x), GL_n(\mathcal{O}_x)) \cong \mathbb{C}[T_x^1, \cdots, T_x^n, (T_x^n)^{-1}]$.

This is a corollary of Satake's Isomorphism Theorem. [4]

Consequences Firstly, since every irreducible module over a ring is cyclic, and by the structure theorem $\mathcal{H}(GL_n(K_x), GL_n(\mathcal{O}_x))$ is an integral domain, we know that it must be a $1-$dimensional free module. Since the annihilator of a $1-$dimensional simple free module is a prime ideal, the isomorphism classes of $\mathcal{H}(GL_n(K_x), GL_n(\mathcal{O}_x))-$modules correspond to $Spec(\overline{\mathbb{Q}_l}[T_x^1, \cdots, T_x^n, (T_x^n)^{-1}])$. On the other hand, since irreducible $\mathcal{H}(GL_n(K_x), GL_n(\mathcal{O}_x))-$modules correspond one-to-one to spherical irreducible representations of $GL_n(K_x)$, we know

[3] Actually can replace $\mathcal{O}_x$ with any compact open subgroup of K_x.

[4] Reference: http://sporadic.stanford.edu/bump/math263/hecke.pdf, Proposition 37

that they correspond to conjugacy classes of diagonalizable matrices in $GL_n(\overline{\mathbb{Q}_l})$. (Frenkel, Page 31 right after 2.8) This is because if A is an irreducible module over the Hecke algebra under the representation π_x, then A is $f\mathcal{H}(GL_n(K_x), GL_n(\mathcal{O}_x))$. Let the eigenvalues of f under $T_x^1, \cdots, T_x^n$ be respectively $z_1, \cdots, z_n$. Note that that $(z_1, \cdots, z_n)$. If π'_x is isomorphic to π_x, then the set of eigenvalues of f under π'_x is $(y_1, \cdots, y_n)$, equal to a conjugation of $(z_1, \cdots, z_n)$. Recall that each irreducible $GL_n(\mathbb{A})-$representation $\pi = \bigotimes'_{x\in|X|} \pi_x$, where π_x is a spherical representation of $GL_n(K_x)$. Then corresponding to each such π, there is a collection $\{\gamma_x\}_{x\in|X|}$, of conjugacy classes of diagonal matrices.

3 Transitioning to Geometric Langlands

3.1 Algebraic Geometry Preliminaries

3.1.1 Étale Morphism and Étale Fundamental Group

Let X, Y be schemes. A morphism $f : Y \to X$ is finite if for any affine open subset $U \subseteq X, \Gamma(f^{-1}(U), \mathcal{O}_Y)$ is a finite $\Gamma(U, \mathcal{O}_X)-$algebra. (Milne, Page 4). This should be thought of an analogy of finite cover in topology.

f is flat if for any affine open set $V \subseteq Y$ and $U \subseteq X$ such that $f(V) \subseteq U, \Gamma(V, \mathcal{O}_Y)$ is a flat $\Gamma(U, \mathcal{O}_X)$-module. (Milne, Page 8) When X and Y are varieties, this means that for all closed points $x \in X$ (i.e. residue field at the point is a finite extension of the base field) such that $f^{-1}(x) \neq \emptyset$, we have $dim(f^{-1}(x)) = dim(Y) - dim(X)$. (Milne, Page 10)

$f : Y \to X$ of finite type is unramified if $\forall y \in Y, \mathcal{O}_{Y,y}/m_{f(y)}\mathcal{O}_{Y,y}$ is an unramified extension of the residue field at x. Note that $\mathcal{O}_{Y,y}$ can be viewed as an $\mathcal{O}_{X,x}-$module. A morphism is étale if it is flat and unramified. (Milne, 21-22)

We can think of an étale morphism as an analogue of local homeomorphisms in topology. (Milne, Page 39)

Let $Y \xrightarrow{f} X, Z \xrightarrow{g} X$ be two étale morphisms. Then $Hom_X(Y,Z)$ are defined to be morphims $\phi : Y \to Z$ suh that $f = \phi(g)$. A family of morphisms $\{\phi_i : X_i \to X\}_{i \in I}$ is an étale cover of X if each ϕ_i is étale, and the images cover X. With this in mind we can define étale sheafs on X. (Stacks Project, 59.4)

Recall that the fundamental group of a topologcal space is isomorphic to the group of deck transformations of its universal cover. (Hatcher, Proposition 1.40) We aim to generalize this to schemes. Pick $x \in X$ a geometric point. We say X_i is Galois over X if $X_i \xrightarrow{f_i} X$ is a finite étale surjection and $Aut_X(X_i) \cong_{\mathsf{Set}} Hom_X(x, Y)$, which can be understood as the geometric fiber of Y over x.

There exists a projective system $\{X_i\}_{i \in I}$ of Galois covers of X with surjective finite Galois $X-$homomorphisms $\phi_{ij} : X_j \to X_i$ for $j \geq i$ that uniquely pro-represents the functor $F : \mathsf{FET}/X \to \mathsf{Set}, Y \mapsto Hom_X(x, Y)$.[5] This means that $\forall Z \in \mathsf{FET}/X, \varinjlim Hom_X(X_i, Z) \cong_{\mathsf{Set}} F(Z)$. Then define the profinite group $\pi_1(X, x) = \varprojlim Aut_X(X_i)$. (Milne, Pages 39-40) Note that the projective system plays the role of the universal cover.

3.1.2 $\ell-$adic Sheafs and Cohomology

An $\ell-$adic sheaf on X_{et} is a projective system $\{F_n\}_{n \in \mathbb{N}}$ of étale sheaves (i.e. contravariant functors from the category of étale morphisms of schemes to X to sets satisfying the usual gluing conditions of the sheaf definition) such that $F_{n+1} \to F_n$ induces an isomorphism $F_{n+1}/\ell^n F_{n+1} \xrightarrow{\cong} F_n$. Note that $F_0 = 0$ and F_n is a $\mathbb{Z}/(\ell^n)$-module. (Milne, Page 163-164)

[5] FeT/X is the category of finite étale $X-$schemes.

Define the cohomology to be $H^r(X, F) = \varprojlim_n H^r(X, F_n)$. Note that $\varprojlim \mathbb{Z}/(\ell^n) = \mathbb{Z}_\ell$ acts on $H^r(X, F)$. The cohomology of an $\ell-$adic sheaf is constructible if X can be written as a finite union of locally closed subschemes Y on which the sheaf of cohomology groups is locally constant. (Stacks project, 59.76)

3.1.3 Stacks and Algebraic Stacks

(Referece: John Voight's notes on Deligne-Mumford Stacks. Can also be found in Milne.) Stacks arise in two situations:

1. When the moduli space parametrizing certain objects does not distinguish between isomorphic objects.

2. When a group acts on a variety, the quotient might not be a scheme.

Category fibered in groupoids Let S be a scheme, $\mathcal{F}$ a category. Call $\mathcal{F}$ an $S-$category if it is equipped with a functor $p : \mathcal{F} \to \mathsf{Sch}/S$. Now $\mathcal{F}$ is fibered in groupoids if:

1. (Lifting of arrows) For all arrows $\phi : U \to V$ in Sch/S and $\forall y \in p^{-1}(V)$, there exists an arrow $f : x \to y$ in $\mathcal{F}$ such that $p(x) = U$ and $p(f) = \phi$.

2. (Lifting of diagrams) For all diagrams $x \xrightarrow{f} z \xleftarrow{g} y$ in $\mathcal{F}$, and for any $\phi : p(x) \to p(y)$ with $p(f) = p(g) \circ \phi$, there is a unique $\psi : x \to y$ in $\mathcal{F}$ such that $f = g \circ \psi$ and $p(\psi) = \phi$.

Note that (2) implies that the map in (1) is unique up to isomorphism. It also implies that f is an isomorphism in $\mathcal{F}$ if and only if $p(f)$ is an isomorphism in Sch/S. Let $U \in \mathsf{Sch}/S$, then $\mathcal{F}(U)$, which is the fiber of $\mathcal{F}$ over U, is the category whose objects are $p^{-1}(U)$ and whose arrows f are sent to the identity on U under p.

Here is one example of categories fibered in groupoids which provides strong topological intuitions: recall that in topology, if G is a topological group, then the category of principal $G-$bundles is a groupoid. In that case we can define the classifying space BG of principal G-bundles. For example, S^1 is the classifying space for $\mathbb{Z}, \mathbb{T}^n$ is the classifying space for $\mathbb{Z}^n, \wedge^n S^1$ is the classifying space for the free group with n generators, and a connected hyperbolic manifold is the classifying space for its fundamental group... Now let G be a group scheme over $Spec(K)$. Then we can define a category fibered in groupoids BG, whose objects are principal $G-$bundles (or $G-$torsors), and morphisms are morphisms of $G-$torsors.

Stacks Now we define stacks. Let $\mathcal{F}$ be a category fibered in groupoids over S such that the assignment $\mathsf{Sch}/S \to \mathsf{Set}$ is a sheaf of groupoids:

1. For U a scheme over S and $x, y \in \mathcal{F}(U)$, the functor $\mathsf{Isom}_U(x, y) : \mathsf{Sch}/U \to \mathsf{Set}, V \mapsto$ the set of isomorphisms of $\mathcal{F}(V)$ between $x|_V$ and $y|_V$, is a sheaf over U in the étale topology. Here, let $\phi : V \hookrightarrow U$ be the inclusion, define $x|_V = \phi^*(x)$.

2. The descent datum needs to be effective. i.e. For all étale covers by $S-$schemes $\{U_i \to U\}, \forall x_i \in \mathcal{F}(U_i)$, and all isomorphisms $a_{ij} : x_i|_{U_i \times_U U_j} \to x_j|_{U_i \times_U U_j}$ satisfying the cocycle condition, there exists $x \in \mathcal{F}(U)$ and isomorphisms $a_i : x|_{U_i} \to x_i$ in $\mathcal{F}(U_i)$ such that $a_{ij} = a_j|_{U_i \times_U U_j} \circ a_i|^{-1}_{U_i \times_U U_j}$.

We can try to understand this definition from the perspectives of $\mathsf{Bun}_n/S, S$ a scheme. This is the category of vector bundles of rank n over $U \in \mathsf{Sch}/S$, with morphisms pullbacks, i.e. there is a morphism $M/U \to M'/U'$ if and only if $\exists \phi : U \to U'$ an S-scheme morphism such that $M = \phi^*(M')$. Note that Bun_n/S is a category fibered in groupoids by mapping a bundle over U to $U \in \mathsf{Sch}/S$, and the lifting of arrows the pullback. Moreover, $\mathsf{Bun_n}/S$ is a stack: Condition (1) says that isomorphisms of bundles on the same scheme can be defined locally on an open cover and glued in a unique way. Condition (2) says that line bundles can be glued.

Remark. Bun_n *is actually an algebraic stack (Olsson, Definition 8.1.4 and Lemma 8.1.8), which are stacks with extra properties. The standard reference for algebraic stacks is Champs Algébriques.*

3.1.4 Grothendieck Correspondence between Functions and Sheaves

Let Y be a scheme and $D^b(Y)$ the derived category of $\ell-$adic sheaves with constructible cohomologies. Given $\phi : Y \to X$ a morphism of schemes, we have the pullback functor $\phi^* : D^b(X) \to D^b(Y)$.

Now suppose Y is a scheme over $\mathbb{F}_q$. Then note that $Mor_{\mathsf{Sch}/\mathbb{F}_q}(Spec(\mathbb{F}_q), Y) \cong_{\mathsf{Set}} Hom(I(Y), \mathbb{F}_q)$, which is a finite set. The claim is that every $\mathcal{F} \in D^b(Y)$ gives rise to a function on $Mor_{\mathsf{Sch}/\mathbb{F}_q}(Spec(\mathbb{F}_q), Y)$. If $Y = Spec(\mathbb{F}_q)$, then an object of $D^b(Y)$ is a complex of finite-dimensional $\overline{\mathbb{Q}_l}$-vector spaces, acted on by $Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q) \cong \pi_1^{et}(Spec(\mathbb{F}_q))$. Note that the action of the Frobenius element of $Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q) \cong \widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$. The Frobenius element is the element in $\widehat{\mathbb{Z}}$ that has image 1 under projection to every $\mathbb{Z}/n\mathbb{Z}$.[6] We define the function to be the alternating sum of the traces of the Frobenius element.

Remark. *Here is the action of* $Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q)$: *Firstly note that* $\overline{\mathbb{Q}_l}-$*vector spaces, when viewed as schemes over* $Spec(\mathbb{F}_q)$*, which is a point, are trivially étale Galois covers (note that locally free modules are projective, and projective modules are flat) of* $Spec(\mathbb{F}_q$*. Then refer to the definition of the étale fundamental group*

[6]Note that the Frobenius element is a generator of every finite quotient of $Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q)$. It is a topological generator of $\widehat{\mathbb{Z}}$ with the Krull topology, i.e. $Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ is the closure of the cyclic subgroup generated by the Frobenius.

as the inverse limit of automorphism groups of étale Galois covers.

Now let Y be any scheme, $\mathcal{F} \in D^b(Y)$, then $\forall y \in Y$, let $f(y)$ be the value of the function defined above for $\mathcal{F}_y \in D^b(Spec(\mathbb{F}_q))$, which is the pullback of $\mathcal{F}$ under inclusion.

Tensor products of complexes give rise to products of functions, and pullbacks/pushforwards of complexes under scheme morphisms give rise to pullbacks/pushforwards of functions. Actually many functions come from $\ell-$adic sheaves in this way. For example, given a character on an algebraic group over $\mathbb{F}_q$, we can construct its corresponding character sheaf.

3.2 Langlands Correspondence

More importantly, we know from the structure theorem that every irreducible direct summand of $\mathsf{Funct}_{cusp}(GL_n(K)\backslash GL_n(\mathbb{A}))$ must be a common eigenspace of $T_x^i, \forall i$. Such decomposition has spectrum $\{\gamma_x\}_{x\in|X|}$, where γ_x are conjugacy classes of diagonal matrices.

In the definition of the étale fundamental group, an implicit fact is that this definition is functorial. Therefore $f \in Mor_{\mathsf{Sch}}(X, Y)$ with $f(x) = y$ induces a group homomorphism $f^* : \pi_1^{et}(Y, y) \to \pi_1^{et}(X, x)$. (Milne, Lectures on Étale Cohomology, Page 29) Therefore let $x \in |X|$ then an irreducible representation $Gal(\overline{K}/K) \cong \pi_1(x) \to GL_n(\overline{\mathbb{Q}_l})$ induces an irreducible representation of $\pi_1(X, x)$.

Remark. *Note that an irreducible representation $\sigma : \pi_1^{et}(X, x) \to GL_n(\overline{\mathbb{Q}_l})$ can be viewed as an $\ell-$adic sheaf on X_{et}. This is because under the Krull Topology on X_{et}, we can consider the maximal compact subgroup of $\pi_1(X, x)$. Then the irreducible ϕ's can be regarded as lisse (each F_n is locally constant, Stacks Project*

Section 64.18) $\ell-$adic sheaves on X_{et}. (This correspondence can be found on Milne, Page 164, in a comment right after the definition of $\ell-$adic sheaves)

Now consider the Frobenius conjugacy class of x in $Gal(\overline{K}/K)$, and its image in $\pi_1(X, x)$. Then under σ, we have a conjugacy class A_x in $GL_n(\overline{\mathbb{Q}_l})$. Now take σ_x to be the conjugacy class of the semisimple part of any element in A_x. This exists and is well-defined by the Jordan-Chevalley decomposition theorem.

We say σ and π, a cuspidal automorphic representation of $GL_n(\mathbb{A})$ correspond in the sense of Langlands if $\forall x \in |X|, \sigma_x = \gamma_x$, where γ_x is as we defined before. The Langlands Conjecture (proven by Drinfeld for $n = 2$ and Lafforgue for $n = 3$) states that under the Langlands correspondence, there is a bijection between the set of σ's and π's.

3.3 Hecke Operators and Vector Bundles

3.3.1 $GL_n(K)\backslash GL_n(\mathbb{A})/GL_n(\mathbb{O})$

Let Bun_n denote the isomorphism classes of rank-n vector bundles on X.

We know that $GL_1(K)\backslash GL_1(\mathbb{A})/GL_1(\mathbb{O})$ is the divisor class group, which is in bijection with the group of line bundles. (i.e. For any line bundle $S, \exists D$ such that $S = \mathcal{O}_X(D + M)$, for any principal divisor M) In other words $GL_1(K)\backslash GL_1(\mathbb{A})/GL_1(\mathbb{O}) \cong_{\mathsf{Set}} Bun_1$. The same can be said for arbitrary n, since any rank-n vector bundle can be trivialized as K^n at the generic point of X and $\mathcal{O}_x^n$ at all other $x \in |X|$, and the quotienting are merely changes of trivializations.

3.3.2 Hecke Operators and Bun_n

For $x \in |X|$, denote H_x^i the set of isomorphism classes of triples (M, M', β) where $M, M' \in Bun_n$ and β an embedding of M into M' such that $M'/\beta(M)$ has length i and is (noncanonically) isomorphic to k_x^i. Here $k_x = \mathcal{O}_x/m_x$ is the residue field at x. Now we have functions $h^{\leftarrow} : H_x^i \to Bun_n, (M, M', \beta) \mapsto M$ and $h^{\rightarrow} : H_x^i \to Bun_n, (M, M', \beta) \mapsto M'$. Now we have the following

Proposition 3.1. *$\forall x \in |X|, (h^{\leftarrow})^{-1}(M)_x \cong_{\mathsf{Set}} Gr^i(M_x)$, where $Gr^i(M_x)$ is the i^{th} Grassmannian of M_x as a k_x-vector space. Similarly, $(h^{\rightarrow})^{-1}(M')_x \cong_{\mathsf{Set}} Gr^{n-i}(M'_x)$.*

To prove this, view β as an injection of coherent sheaves and consider the short exact sequence

$$0 \longrightarrow ker(\beta|_x) \longrightarrow M_x \longrightarrow M'_x \longrightarrow (M'/\beta(M))_x \longrightarrow 0$$

Note that since k_x are finite fields, the fibers have finite cardinality. Here is the main claim of this subsection:

Theorem 3.1. *Under the correspondence between Bun_n and $GL_n(K)\backslash GL_n(\mathbb{A})/GL_n(\mathbb{O})$, we have the following: if f is a function on Bun_n, then $(h^{\rightarrow})_!(h^{\leftarrow})^*(f) = T_x^i(f)$. Here $*$ means pull-back, and $!$ means summation along the fibers.*

Remark. *We have the diagram $Bun_n h^{\leftarrow} \leftarrow H_x^i \xrightarrow{h^{\rightarrow}} Bun_n$. Therefore $(h^{\leftarrow})^*(f)$ is a function on H_x^i.*

3.4 Cuspidality and Flag Manifolds

Consider the set of flags Fl_{n_1,n_2}^n, with $n_1+n_2 = n$, corresponding to isomorphism classes of short exact sequences $0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$. Then we have natural maps $Bun_n \xleftarrow{p} Fl_{n_1,n_2}^n \xrightarrow{p} Bun_{n_1} \times Bun_{n_2}$. Then f is cuspidal if and only if $q_! p^*(f) = kf$ for some k for any n_1, n_2. Observe the similarity to the definition of the Hecke operators via vector bundles, and remember the decomposition in Proposition 2.1, and that a function is cuspidal if and only if it is in the eigenspace of each T_x^i.

4 Formulation of Geometric Langlands using Stacks

4.1 Derived Category of $\ell-$adic sheaf

Firstly, let $\mathcal{F}$ be an $S-$stack. We first define a quasi-coherent sheaf on $\mathcal{F}$. Recalled that $\mathcal{F} \xrightarrow{p} S$ is a category fibered into groupoids. Then a quasi-coherent sheaf on Y consists of the following data:

1. For each object $f \in \mathcal{F}$, a quasi-coherent sheaf A_f on $p(f) \in \mathsf{Sch}/S$.
2. For each morphism $H : f \to g$ in $\mathcal{F}$, with $h = p(H)$, an isomorphism $\phi_H : h^*(A_g) \xrightarrow{\cong} A_f$ that satisfies the cocycle condition.

For a more technical definition using the method of descent (for a more detailed account, refer to Olsson, section 4.3) Now let Y be a category fibered into groupoids over $Spec(\mathbb{F}_q)$.

Remark. *We can similarly use the descent method to define $\ell-$adic sheaves on $\mathcal{F}$. Now it makes sense to define $D^b(\mathsf{Bun}_n)$ and then define the function on $\mathsf{Bun}_n(U), U \in \mathsf{Sch}/Spec(\mathbb{F}_q)$ corresponding to $\mathcal{F} \in D^b(\mathsf{Bun}_n)$, satisfying similar properties as in the case of schemes.*

4.2 "Local" and "Global" Hecke Stacks

Fixing $x \in |X|$, define the local Hecke stack $\mathcal{H}_x^i$ by the condition: for scheme $S/Spec(\mathbb{F}_q)$, we define $\mathcal{H}_x^i(S)$ to be the category of triples (M, M', β), with M and M' rank-n vector bundles on $X \times S$ and β an embedding $M \hookrightarrow M'$ of coherent sheaves, such that M'/M is a rank-i vector bundle on $\{x\} \times S$ and vanishes elsewhere. We can similarly define $h^{\rightarrow}, h^{\leftarrow} : \mathcal{H}_x^i \to \mathsf{Bun}_n$. The fiber of $h^{\leftarrow}$ is a scheme isomorphic to $Gr^i(M_x)$ and the fiber of $h^{\rightarrow}$ is a scheme isomorphic to $Gr^{n-i}(M_x^{'})$. Define the Hecke Functors: $D^b(\mathsf{Bun}_n) \to D^b(\mathsf{Bun}_n), \mathcal{F} \mapsto (h^{\rightarrow})_!(h^{\leftarrow})^*(\mathcal{F})[i(n-i)]$, where the bracket denotes the shift functor on the derived category.

Remark. *Note that H_x^i, as a set, is in bijection with the set of isomorphism classes of $\mathbb{F}_q$-points of $\mathcal{H}^i$.*

Now define the global Hecke stack $\mathcal{H}^i$ by $\mathcal{H}^i(S)$ (for $S \in \mathsf{Sch}/Spec(\mathbb{F}_q)$) being the category of (ϕ, M, M', β). Here, $\phi \in Hom_{\mathsf{Sch}/Spec(\mathbb{F}_q)}(S, X)$, and M'/M is a rank-i vector bundle supported on the graph of ϕ in $X \times S$. We can similarly define $h^{\leftarrow}$ and $h^{\rightarrow}$. Let $s : \mathcal{H}^i \to X, (\phi, M, M', \beta) \mapsto \phi$. Then define the Hecke functor $T_i : D^b(\mathsf{Bun}_n) \to D^b(X \times \mathsf{Bun}_n), \mathcal{F} \mapsto (s \times h^{\rightarrow})_!(h^{\leftarrow})^*(\mathcal{F})[i(n-i)]$.

4.3 Hecke Eigensheaves and Geometric Langlands

Let σ be a representation of the étale fundamental group in $\overline{\mathbb{Q}_l}$ and E_σ the corresponding $\ell-$adic sheaf on X, as defined before. An $\mathcal{F}_\sigma$ is a Hecke eigensheaf with respect to σ if for every $i, T^i(\mathcal{F}_\sigma) \cong \wedge^i(E_\sigma) \boxtimes \mathcal{F}_\sigma$, where $\wedge^i(E_\sigma) \boxtimes \mathcal{F}_\sigma$ is an element in $D^b(X \times Bun_n)$. Here $\boxtimes$ denotes the external tensor product: let X be an $A-$module, Y a $B-$module, p_1 and p_2 the two projections. Then $X \boxtimes Y$ is the $A \times B-$module $p_1^*(X) \otimes_{A \times B} p_2^*(Y)$.

Remark. *If $\mathcal{F}_\sigma$ is an eigensheaf with respect to σ, then the corresponding function on $GL_n(K) \backslash GL_n(\mathbb{A})/GL_n(\mathbb{O})$ is an eigen-function for the Hecke operators.*

The Geometric Langlands conjecture predicts that for each irreducible σ, there exists a cuspidal Hecke eigen-sheaf $\mathcal{F}_\sigma$. (Cuspidality again defined by conditions on $r^n_{n_1,n_2}$.