Math 600 Day 9: Lee Derivatives

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Outline

Lee Derivatives

Rate of change of a smooth function w.r.t. a tangent vector.

Let M be a smooth manifold, $x \epsilon M$ a point of M, and $V \epsilon T_x M$ a tangent vector to M at x.

Let $f : M \rightarrow \mathbb{R}$ be a smooth function.

We want to measure the rate of change of f at x with respect to V.

Here are two sensible ways to do this:

(1) Let $\alpha : \mathbb{R} \to M$ be a smooth function such that

$$\alpha(0) = x \text{ and } \alpha'(0) = V.$$

Then the real number $(f \circ \alpha)'(0)$ measures this rate of change.

(2) Consider the differential of f at x,

$$df_x = (f_*)_x : T_x M \to T_{f(x)}\mathbb{R} \cong \mathbb{R}.$$

Then the real number $df_x(V(x))$ also measures this rate of change.

Problem 1. (a) Check that these two definitions coincide.

(b) Suppose that $(x_1, x_2, ..., x_n)$ are local coordinates in a neighborhood of the point $x \epsilon M$, and that the tangent vector V at x is given in these coordinates by

$$V = v^1 \frac{\partial}{\partial x^1} + ... + v^n \frac{\partial}{\partial x^n}.$$

Show that the rate of change of f at x with respect to V is given by

$$\Sigma_i v^i (\frac{\partial f}{\partial x^i}(x)).$$

Now suppose that we still have the smooth function $f : M \rightarrow \mathbb{R}$, but that instead of a tangent vector to M at a single point, we have a smooth vector field V on M.

Let $\{\phi_t\}$ be the corresponding local one-parameter group of diffeomorphisms of M generated by the vector field V. For brevity, we'll call $\{\phi_t\}$ a **local flow**.

Then the rate of change of f at the point $x \epsilon M$ is also given by the formula

$$df_x(V(x)) = lim_{t \rightarrow 0} \frac{(f(\phi_t(x)) - f(x))}{t}.$$

If we regard this rate of change $df_x(V(x))$ as a function of x, then its expression $\Sigma_i v^i(x)(\frac{\partial f}{\partial x^i})(x)$ in local coordinates shows that it is a smooth function of x, which for brevity we write as Vf.

Note that V is serving here as a differential operator, taking functions to functions.

When we want to emphasize this operator character of V even further, we write the operator as L_V and call it the Lie derivative with respect to V. Thus

$$L_V(f) =_{defn} Vf.$$

The Lie derivative of one vector field with respect to another.

We continue to work in the smooth manifold M, but this time with two smooth vector fields V and W.

We want to measure the rate of change of W with respect to V.

Again let $\{\phi_t\}$ be the local flow generated by V.

We define the Lie derivative of W with respect to V at the point $x \epsilon M$ to be the vector

$$(L_V W)(x) = lim_{t \to 0} \frac{(\phi_{-t})_* W \phi_t(x) - Wx}{t}.$$

lying in the tangent space $T_x M$.

Comments. (1) It's not clear yet that this limit exists.

(2) Note that $(\phi_{-t})_* W \phi_t(x) = ((\phi_{-t})_* W)_x$.

(3) Note that $W_{\phi_t}(x)$ and W_x lie in tangent spaces to M at different points, and so can not be subtracted from one another without first moving one of these vectors to the tangent space containing the other.

Problem 2. (a) Show that

$$L_v(W_1 + W_2) = L_v(W_1) + L_v(W_2).$$

(b) Show that

$$L_v(fW) = (L_V f)W + f(L_V W) = (Vf)W + f(L_V W).$$

Hint: Remember that the Leibniz Rule in Freshman Calculus was proved by adding and subtracting a convenient middle term.

Theorem

Suppose that in local coordinates $(x_1, ..., x_n)$, the smooth vector fields V and W are given by

$$V = \Sigma_i v^i(x^1, ..., x^n)\frac{\partial}{\partial x^i}$$

and

$$W = \Sigma_i w^i(x^1, ..., x^n)\frac{\partial}{\partial x^i}.$$

Then $L_V W$ exists and in these local coordinates is given by

$$L_V W = \Sigma_j(\Sigma_i(v^i \frac{\partial w^j}{\partial x^i} - w^i \frac{\partial v^j}{\partial x^i}))\frac{\partial}{\partial x^j}$$

$$= (v^i \frac{\partial w^j}{\partial x^i} - w^i \frac{\partial v^j}{\partial x^i})\frac{\partial}{\partial x^j},$$

using the Einstein convention of summing over repeated indices.

Some preliminaries before giving the proof.

The simplest case. Suppose we are on the line $\mathbb{R}^1$, that $V = \frac{\partial}{\partial x}$ and that $W = w(x)\frac{\partial}{\partial x}$. Then the flow $\{\varphi_t\}$ of V is given by $\varphi_t(x) = x + t$. Hence

$$(L_V W)(x) = lim_{t \to 0} \frac{(\varphi_{-t})_* W(\varphi_t(x)) - W(x)}{t}$$

$$= lim_{t \to 0} \frac{w(x+t)\frac{\partial}{\partial x} - w(x)\frac{\partial}{\partial x}}{t}$$

$$= \frac{dw}{dx}\frac{\partial}{\partial x},$$

which agrees with the proposed formula.

Lemma 1 Suppose we are in $\mathbb{R}^n$, that $V = \frac{\partial}{\partial x^1}$ and that $W = w^j \frac{\partial}{\partial x^j}$ (we continue to use the summation convention). Compute, as above, that

$$L_V W = \left(\frac{\partial w^j}{\partial x^1}\right)\frac{\partial}{\partial x^j},$$

which again agrees with the proposed formula.

Lemma 2 Let V be a smooth vector field on the smooth manifold M, and let x be a point of M at which $V(x) \neq 0$. Show how to find local coordinates $(x^1, x^2, ..., x^n)$ about x, in terms of which $V = \frac{\partial}{\partial x^1}$.

Lemma 3 Let $V = v^i \frac{\partial}{\partial x^i}$ be a smooth vector field given in local coordinates $x^1, ..., x^n$. Suppose that h is a diffeomorphism carrying this coordinate neighborhood to an open set on which we have local coordinates $y^1, ..., y^n$. Show that $h_* V$ is given by

$$h_* V = (v^i \frac{\partial y^j}{\partial x^i}) \frac{\partial}{\partial y^j},$$

where $(\frac{\partial y^j}{\partial x^i})$ is the Jacobian matrix for h_*.

Note that this same formula applies for transforming a vector field V given in one set of local coordinates to the same vector field given in a different set of local coordinates.

Consistency check on the proposed formula

$$L_V W = (v^i \frac{\partial w^j}{\partial x^i} - w^i \frac{\partial v^j}{\partial x^i}) \frac{\partial}{\partial x^j}.$$

Suppose we use the result of Lemma 3 to transform the vector fields V and W from one set $(x^1, ..., x^n)$ of local coordinates to another set $(y^1, ..., y^n)$. We must check that the proposed formula for $L_V W$ transforms in the same way, for otherwise it could not possibly be correct.

We will write $V = v^i \frac{\partial}{\partial x^i}$ in the first set of coordinates, and $V = \underline{v}^r \frac{\partial}{\partial y^r}$ in the second set, where $\underline{v}^r = v^i \frac{\partial y^r}{\partial x^i}$, and likewise for W.

We now write the formula for $L_V W$ in the y-coordinates, transform the appearances of V and W in that formula back to x-coordinates, and see (thanks to some convenient cancellations) that the result is the appropriate transform of $L_V W$:

$$L_V W = (\underline{v}^r \frac{\underline{w}^s}{\partial y^r} - \underline{w}^r \frac{\partial \underline{v}^s}{\partial y^r}) \frac{\partial}{\partial y^s}$$

$$= (v^i \frac{\partial y^r}{\partial x^i} \frac{\partial}{\partial y^r}(w^j \frac{\partial y^s}{\partial x^j}) - w^j \frac{\partial y^r}{\partial x^j} \frac{\partial}{\partial y^r}(v^i \frac{\partial y^s}{\partial x^i})) \frac{\partial}{\partial y^s}.$$

This expands from two to four terms, and then the second derivative terms cancel, thanks to compressions such as $(\frac{\partial y^r}{\partial x^i})(\frac{\partial x^k}{\partial y^r}) = \delta_i^k$; the first derivative terms compress to

$$(v^i \frac{w^j}{\partial x^i} - w^i \frac{\partial v^j}{\partial x^i}) \frac{\partial}{\partial y^s},$$

which is the transform of $L_V W$ from x- to y-coordinates.

Proof of the Theorem. We must show that

$$L_V W = (v^i \frac{\partial w^j}{\partial x^i} - w^i \frac{\partial v^j}{\partial x^i}) \frac{\partial}{\partial x^j}.$$

To show this in a neighborhood of the point x, suppose first that $V(x) \neq 0$. Then by Lemma 2, we can choose local coordinates about x in terms of which $V = \frac{\partial}{\partial x^1}$, in which case the proposed formula is correct according to Lemma 1. By our previous consistency check, if the formula is correct in one coordinate system about x, it is correct in all coordinate systems about x.

Now, by continuity, the formula is correct in a neighborhood of x if x is in the closure of the set of points where V is nonzero.

All that remains is an open set of points where $V \equiv 0$, where by inspection the formula is correct. This completes the proof.