# Math 600 Day 4: Differentiable Manifolds

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# Outline

1. Differentiable Manifolds
   - k-Dimensional Smooth Manifolds

# k-Dimensional Smooth Manifolds

## Definition

A subset $M$ of $\mathbb{R}^n$ is called a $k$-dimensional differentiable (or smooth) manifold if for every point $x \epsilon M$, there is an open set $U$ of $\mathbb{R}^n$ which contains $x$, another open set $V$ in $\mathbb{R}^n$, and a diffeomorphism $h : U \to V$, such that

$$h(U \cap M) = V \cap \mathbb{R}^k.$$

### Remark

(1) We regard

$$\mathbb{R}^k = \{x \epsilon \mathbb{R}^n : x_{k+1} = \ldots = x_m = 0\} \subset \mathbb{R}^n.$$

Thus $\mathbb{R}^1 \subset \mathbb{R}^2 \ldots \mathbb{R}^k \subset \ldots \mathbb{R}^m \subset \ldots$.

### Remark

(2) The above definition says that each point of $M$ has a neighborhood in $M$ which sits inside $\mathbb{R}^n$ the same way, up to diffeomorphism, that $\mathbb{R}^k$ does.

### Remark

(3) If the diffeomorphisms are of class $C^r$, we say that the manifold $M$ is of class $C^r$. Most of the time, we will be dealing with differentiable manifolds of class $C^\infty$.

**Example** $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ is a 2-manifold

### Theorem

*Let $A \subset \mathbb{R}^n$ be open and let $f : A \to \mathbb{R}^p$ be a differentiable function whose derivative $f'(x)$ has maximal rank $p$ whenever $f(x) = 0$. Then $f^{-1}(0)$ is a $(n-p)$-dimensional manifold in $\mathbb{R}^n$.*

**Proof:**

Fix $a$ such that $f(a) = 0$. By hypothesis, the $p \times n$ matrix $f'(a)$ has rank $p$. Consider $f$ as a function $f : \mathbb{R}^{n-p} \times \mathbb{R}^p \to \mathbb{R}^p$.

Since $(D_j f^i(a))$ has rank $p$, we can reorder the coordinates so that the $p \times p$ matrix $M = (D_{n-p+j} f^i(a_1, a_2))$, where $1 \leq i, j \leq m$ and $(a_1, a_2) = a$, has the property that $det(M) \neq 0$.

Define $F : \mathbb{R}^{n-p} \times \mathbb{R}^p \to \mathbb{R}^{n-p} \times \mathbb{R}^p$ by $F(x, y) = (x, f(x, y))$. Then $det(F'(a_1, a_2)) = det(M) \neq 0$.

By the inverse function theorem, there is an open set $W \subset \mathbb{R}^{n-p} \times \mathbb{R}^p$ containing $F(a_1, a_2) = (a_1, 0)$ and an open set in $\mathbb{R}^{n-p} \times \mathbb{R}^p$ containing $(a_1, a_2)$, which may be taken to be of the form $A \times B$ such that $F : A \times B \to W$ has a differentiable inverse $h : W \to A \times B$.

$h$ is of the form $h(x, y) = (x, k(x, y))$ since $F$ is of this form.

Let $\pi : \mathbb{R}^{n-p} \times \mathbb{R}^p \to \mathbb{R}^p$ be the projective map. Then $\pi \circ F = f$. Therefore,

$$f(x, k(x, y)) = f \circ h(x, y) = (\pi \circ F) \circ h(x, y) = \pi \circ (F \circ h)(x, y) = \pi(x, y) = y$$

Hence we have found an $h$ such that $f \circ h(x_1, ..., x_n) = (x_{n-p+1}, ..., x_n)$ in a nbh of $a$ where $f(a) = 0$.

It follows that $f^{-1}(0)$ is a differentiable manifold.

**Example** $S^{n-1} \subset \mathbb{R}^n$ are all $n-1$ manifolds.

**Example** A subset $M$ of $\mathbb{R}^n$ has condition $C$ if for every $x \epsilon M$:

There is an open set $U$ of $\mathbb{R}^n$ containing $x$, an open subset $W \subset \mathbb{R}^k$ and a one-to-one differentiable function $f : W \to \mathbb{R}^n$ such that
(1) $f(W) = M \cap U$
(2) $f'(y)$ has maximal rank $k$ for each point $y \epsilon W$.
(3) $f^{-1} : f(W) \to W$ is continuous.

The function $f$ is called a **coordinate system** around $x \epsilon M$.

## Theorem

*M having condition (C) is equivalent to M being a k-manifold.*

**Proof:** Assume $M$ is a manifold. For every $x \epsilon M$ there exists an open set $U \subset \mathbb{R}^n$ containing $x$ and an open set $V \subset \mathbb{R}^n$ and a diffeomorphism $h : U \to V$ such that

$$h(U \cap M) = V \cap \{y \epsilon V : y^{k+1} = ... = y^n = 0\}.$$

Let $W = \{a \epsilon \mathbb{R}^k : (a, 0) \epsilon h(M)\}$ and define

$$f : W \to \mathbb{R}^n \ \ by \ \ f(a) = h^{-1}(a, 0).$$

$f(W) = M \cap U$

$f^{-1}$ is continuous since it is the restriction of a continuous map.

If $H : U \to \mathbb{R}^k$ is given by $H(z) = (h^1(z), h^2(z), ..., h^k(z))$, then $H(f(y)) = y$ for all $y \epsilon W$.
Therefore, $H'(f(x))f'(y) = Id$ and $f'(y)$ has rank $k$. Hence, $M$ has condition (C).

Assume $M$ has condition $(C)$ and show $M$ is a manifold.

There is an open set $U$ of $\mathbb{R}^n$ containing $x$, an open subset $W \subset \mathbb{R}^k$ and a one-to-one differentiable function $f : W \to \mathbb{R}^n$ such that
(1) $f(W) = M \cap U$
(2) $f'(y)$ has maximal rank $k$ for each point $y \epsilon W$.
(3) $f^{-1} : f(W) \to W$ is continuous.

Let $x = f(y) \epsilon M$. Assume that $(D_j f^i(y))$, $1 \leq i, j \leq k$ has non-zero determinant.

Define $g : W \times \mathbb{R}^{n-k} \to \mathbb{R}^n$ by

$$g(a, b) = f(a) + (0, b) = (f^1(a), ..., f^k(a), b_1, ..., b_{n-k})$$

Then $det(g'(a, b)) = det(D_j f^i(a)) \neq 0$

In particular, $g'(y,0) \neq 0$. By the inverse function theorem, there is an open set $V_1'$ containing $(y,0)$ and an open set $V_2'$ containing $g(y,0)$ such that $g : V_1' \to V_2'$ has a differentiable inverse $h : V_2' \to V_1'$.

Since, $f^{-1}$ is continuous, $\{f(a) : (a,0) \epsilon V_1'\} = U \cap f(W)$ for some open set $U$.

Let $V_2 = V_2' \cap U$ and $V_1 = g^{-1}(V_2)$. Then $V_2 \cap M$ is exactly $\{f(a) : (a,0)\epsilon V_1\} = \{g(a,0) : (a,0)\epsilon V_1\}$. So,

$$h(V_2 \cap M) = g^{-1}(V_2 \cap M) = g^{-1}(\{g(a,0) : (a,0)\epsilon V_1\}) = V_1 \cap \mathbb{R}^k$$

The **closed half-space** $H^k \subset \mathbb{R}^k$ is defined to be

$$H^k = x\epsilon\mathbb{R}^k : x_k \leq 0.$$

A subset $M$ of $\mathbb{R}^n$ is called a *k-dimensional differentiable manifold with boundary* if for every point $x\epsilon M$, there is an open set $U$ of $\mathbb{R}^n$ which contains $x$, another open set $V$ in $R^n$,and a diffeomorphism $h : U \to V$, such that either

$$h(U \cap M) = V \cap \mathbb{R}^k$$

or

$$h(U \cap M) = V \cap H^k.$$

The set of points in $M$ which satisfy the second condition is called the boundary of $M$ and denoted by $\partial M$.

Let $M$ be a k-dimensional differentiable manifold in $\mathbb{R}^n$, and let $(f_1, U_1, u_1)$ be a coordinate system around $x \epsilon M$. Since $f_1'(u_1)$ has rank $k$, the linear transformation $(f_1)_* : \mathbb{R}^k_{u_1} \to \mathbb{R}^n_x$ is one-to-one, and hence $(f_1)_*(\mathbb{R}^k_{u_1})$ is a k-dimensional subspace of $R^n_x$. If $(f_2, U_2, u_2)$ is another coordinate system around $x \epsilon M$, then

$$(f_2)_*(\mathbb{R}^k_{u_2}) = (f_1)_*(f_1^{-1} \circ f_2)_*(\mathbb{R}^k_{u_2}) = (f_1)_*(\mathbb{R}^k_{u_1}).$$

Thus the k-dimensional subspace $(f_1)_*(\mathbb{R}^k_{u1})$ does not depend on the choice of coordinate system around $x$. This subspace is denoted by $M_x$ or $TM_x$ or $T_xM$, and is called the **tangent space** to $M$ at $x$.