Math 600: Integration on Chains and Stoke's Theorem

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Outline

Integration on Chains in Euclidean Space

The subset $[0,1]^k \subset \mathbb{R}^k$ is the **standard unit cube** in $\mathbb{R}^k$.

Let U be an open subset of $\mathbb{R}^n$. A **singular** k**-cube** in U is a continuous map $c : [0,1]^k \to U$.

A singular 0-cube in U is, in effect, just a point of U, and a singular 1-cube in U is a parametrized curve in U.

The **standard (singular) k-cube** $I^k : [0,1]^k \to \mathbb{R}^k$ is the inclusion map of the standard unit cube.

A **(singular) k-chain** in U is a formal finite sum of singular k-cubes in U with integer coefficients, such as

$$2c_1 + 3c_2 - 4c_3.$$

It is clear how k-chains in U can be added and multiplied by integers.

For each singular k-chain c in U we will define a singular $k-1$ chain in U called the **boundary** of c and denoted by $\partial(c)$.

We begin by defining the boundary of the standard k-cube $I^k : [0,1]^k \to \mathbb{R}^k$.

For each i with $1 \leq i \leq k$ we define two singular $k-1$ cubes, $I^k_{(i,0)} : [0,1]^{k-1} \to [0,1]^k \subset \mathbb{R}^k$ $I^k_{(i,1)} : [0,1]^{k-1} \to [0,1]^k \subset \mathbb{R}^k$, as follows.

$$I^k_{(i,0)}(x^1, ..., x^{k-1}) = (x^1, ..., x^{i-1}, 0, x^i, ..., x^{k-1})$$

$$I^k_{(i,1)}(x^1, ..., x^{k-1}) = (x^1, ..., x^{i-1}, 1, x^i, ..., x^{k-1})$$

We call $I^k_{(i,0)}$ the $(i,0)$-face of I^k and $I^k_{(i,1)}$ the $(i,1)$-face. of I^k. Then we define

$$\partial(I^k) = \sum_{i=1}^{k} \sum_{\alpha=0,1} (-1)^{i+\alpha} I^k_{(i,\alpha)}.$$

If $c : [0,1]^k \to U$ is a singular k-cube in U, we define its (i,α)-**face** by $c(i,\alpha) = c \circ I^k_{(i,\alpha)}$, and then define

$$\partial(c) = \sum_{i=1}^{k} \sum_{\alpha=0,1} (-1)^{i+\alpha} c(i,\alpha).$$

We extend the definition of boundary to k-chains by linearity: $\partial(\Sigma a_i c_i) = \Sigma a_i \partial(c_i)$.

Fact: If c is a k-chain in U, show that $\partial(\partial c) = 0$. Briefly, $\partial^2 = 0$.

Now suppose that U is an open set in $\mathbb{R}^n$, that c is a k-chain in U, and that ω is a differential k-form on U. We want to define the integral $\int_c \omega$ of ω over c, and do this in several steps.

First suppose that ω is a differential k-form on the unit k-cube $[0,1]^k$ in $\mathbb{R}^k$. Then

$$\omega = f(x^1, ..., x^k) dx^1 \wedge ... \wedge dx^k.$$

In that case we define

$$\int_{[0,1]^k} \omega = \int_{[0,1]^k} f = \int_{[0,1]^k} f(x^1, ..., x^k) dx^1 ... dx^k.$$

If ω is a differential k-form on the open set U in $\mathbb{R}^n$ and $c : [0,1]^k \longrightarrow U$ is a singular k-cube in U, we define

$$\int_c \omega = \int_{[0,1]^k} c^* \omega.$$

In other words, integration of a k-form over a singular k-cube is defined by pulling the k-form back to the unit k-cube in $\mathbb{R}^k$ and then doing ordinary integration.

In the special case that $k = 0$, a 0-form ω on U is a real-valued function on U, and a singular 0-cube is a map $c : \{0\} \rightarrow U$ of a point into U. So we define

$$\int_c \omega = \omega(c(0)).$$

Finally, the integral of a k-form ω on U over a singular k-chain $c = \Sigma a_i c_i$ is defined by

$$\int_c \omega = \Sigma a_i \int_{c_i} \omega.$$

Theorem

Stokes' Theorem. *Let U be an open set in $\mathbb{R}^n$, ω a differential $k-1$ form on U, and c a singular k-chain on U. Then*

$$\int_c d\omega = \int_{\partial c} \omega.$$

Theorem

Green's Theorem. *Let U be a compact region in $\mathbb{R}^2$ bounded by finitely many smooth, simple closed curves.*

Let $u(x,y)$ and $v(x,y)$ be smooth functions on U.

Then

$$\int_{\partial(U)} u(x,y)\ dx\ +\ v(x,y)\ dy = \int_U (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\ dx\ dy.$$

Proof. Let c be a singular 2-chain which covers the region U, so that $\partial(c)$ covers $\partial(U)$. There is some subtlety in proving the existence of c, but we will deal with this at a later time.

Let $\omega = u(x,y)\ dx\ +\ v(x,y)\ dy$. Then

$$d\omega = (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})\ dx \wedge dy.$$

So Green's Theorem states that

$$\int_{\partial c} \omega = \int_c d\omega,$$

which is just a special case of Stokes' Theorem.

Theorem

Divergence Theorem. *Let U be a compact region in $\mathbb{R}^3$ bounded by finitely many smooth surfaces. Let n be the outward pointing unit normal vector field along $\partial(U)$. Let V be a differentiable vector field on U. Then*

$$\int_U \nabla \bullet V \ \ d(vol) = \int_{\partial(U)} V \bullet \mathbf{n} \ \ d(area).$$

Proof. In words, the integral of the divergence of V over the region U equals the flux of V through its boundary. Let

$$V = u(x,y,z)\mathbf{i} + v(x,y,z)\mathbf{j} + w(x,y,z)\mathbf{k}$$

and

$$\mathbf{n} = n_x(x,y,z)\mathbf{i} + n_y(x,y,z)\mathbf{j} + n_z(x,y,z)\mathbf{k}.$$

Then $\int_U \nabla \bullet V d(vol) = \int_U \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} dxdydz$
$\int_{\partial(U)} V \bullet \mathbf{n} d(area) = \int_{\partial(U)} (un_x + vn_y + wn_z) d(area)$.
Now define a 2-form ω on U by

$$\omega = u(x,y,z)dy \wedge dz + v(x,y,z)dz \wedge dx + w(x,y,z)dx \wedge dy.$$

Then $d\omega = (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z})\ dx \wedge dy \wedge dz$.

Thus

$$\int_U \nabla \bullet V d(vol) = \int_c d\omega,$$

where c is a singular 3-chain that covers the region U so that ∂c covers ∂U, as in Green's Theorem.

Fact:

$$(un_x + vn_y + wn_z)d(area) = udy \wedge dz + vdz \wedge dx + wdx \wedge dy.$$

Using the result of the above problem, we have that

$$\int_{\partial U} V \bullet \mathbf{n} d(area) = \int_{\partial U} (un_x + vn_y + wn_z) d(area)$$

$$= \int_{\partial c} (udy \wedge dz + vdz \wedge dx + wdx \wedge dy)$$

$$= \int_{\partial c} \omega.$$

So the Divergence Theorem,

$$\int_U \nabla \bullet V d(vol) = \int_{\partial U} V \bullet \mathbf{n} d(area),$$

is a special case of Stokes' Theorem,

$$\int_c d\omega = \int_{\partial c} \omega.$$

Theorem

Classical Stokes' Theorem. *Let S be a compact, smooth oriented surface in $\mathbb{R}^3$ with finitely many smooth boundary curves.*

Let $\mathbf{n}$ be the unit "outward" normal vector field along S, and T the unit tangent vector field along ∂S.

Let V be a smooth vector field defined on an open set in $\mathbb{R}^3$ which contains S.

Then

$$\int_S (\nabla \times V) \bullet \mathbf{n} d(area) = \int_{\partial S} V \bullet T d(length).$$