Spring 2012

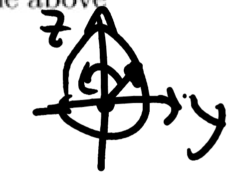
$$\iint_S (F\cdot n)\,ds \overset{\text{Div Thm}}{=} \iiint_D \operatorname{div} F \, dV$$

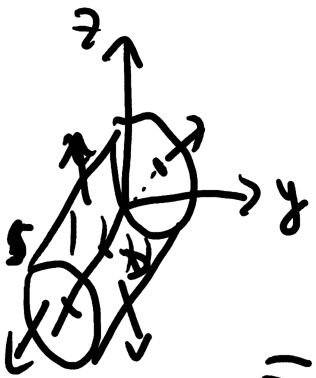
$\left.\begin{matrix} x \\ y \end{matrix}\right\} \rightarrow \left.\begin{matrix} y \\ z \end{matrix}\right\}$ $y = \cos t$, $z = \sin t$

$z \qquad X$

3. Calculate the outward flux of $\vec{F}$ across S if $\vec{F}(x,y,z) = 3xy^2\vec{i} + xe^z\vec{j} + z^3\vec{k}$ and S is the surface of the solid bounded by the cylinder $y^2+z^2=1$ and the planes $x=-1$ and $x=2$.

a) 0 b) $-\frac{\pi}{4}$ c) $\frac{11\pi}{8}$ d) 3π e) $\frac{9\pi}{5}$ f) $\frac{9\pi}{2}$ g) none of the above





$$F = \langle \underbrace{3xy^2}_{P}, \underbrace{xe^z}_{Q}, \underbrace{z^3}_{R} \rangle$$

$$\operatorname{div} F = P_x + Q_y + R_z = 3y^2 + 0 + 3z^2 = 3(y^2+z^2)$$

$$= \iiint 3\underbrace{(y^2+z^2)}_{r^2}\, dx\, \boxed{dz\,dy}_{r\,dr\,d\theta}$$

$$= \int_0^{2\pi}\int_0^1\int_{-1}^{2} 3r^3\,dx\,dr\,d\theta = 9\int_0^{2\pi} \underbrace{\left[\frac{r^4}{4}\right]\Big|_0^1}_{\frac{1}{4}} = \frac{9}{4}\int_0^{2\pi} d\theta$$

$$3r^3 [x]\Big|_{-1}^{2} \quad 2-(-1) = 3$$

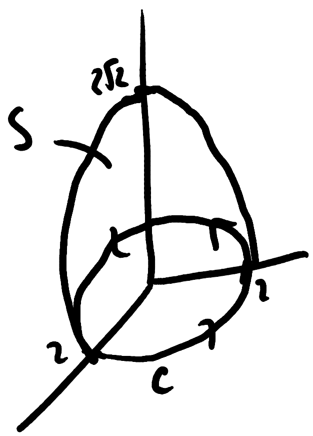
$$= \frac{9}{4}\cdot 2\pi = \boxed{\frac{9\pi}{2}}$$

Spring 2012

$$\iint_S \underbrace{(\nabla \times F)}_{\text{curl } F} \cdot n \, dS \overset{\text{Stokes' Thm}}{=} \oint_C F \cdot dr$$

4. Compute the outward flux of $\nabla \times \vec{F}$ through the surface of the ellipsoid $2x^2 + 2y^2 + z^2 = 8$ lying above the plane $z = 0$, where
$$\vec{F} = (3x - y)\vec{i} + (x + 3y)\vec{j} + (1 + x^2 + y^2 + z^2)\vec{k}.$$

a) 0 b) 2π c) 3π d) 8π e) 12π f) 16π g) none of the above



$F_{\text{on } C} = \langle 3\cdot 2\cos t - 2\sin t, \; 2\cos t + 3\cdot 2\sin t, \; 1 + 4 + 0 \rangle$

$F_{\text{on } C} = \langle 6\cos t - 2\sin t, \; 2\cos t + 6\sin t, \; 5 \rangle$

$F \cdot dr = \cancel{-12\sin t\cos t} + \underbrace{4\sin^2 t + 4\cos^2 t}_{4} + \cancel{12\sin t\cos t} + 0$

C: $x = 2\cos t$
$y = 2\sin t \quad 0 \le t \le 2\pi$
$z = 0$

$r = \langle 2\cos t, 2\sin t, 0 \rangle$

$dr = \langle -2\sin t, 2\cos t, 0 \rangle dt$

$x^2 + y^2 = 4$

$$\oint_C F \cdot dr = \int_0^{2\pi} 4 \, dt = 4 \cdot 2\pi = \boxed{8\pi}$$

Fall 2011

Problem 5. Find the value of the line integral

$$I = \int_C (3\pi x^2 y + ye^x)\,dx + (\pi x + \pi x^3 + e^x)\,dy$$

where C is the curve parametrized by $x = \sin t$, $y = t$ for $0 \le t \le \pi$, and oriented in the direction of increasing t.

$\int F \cdot dr$ $\quad F = \langle \underbrace{3\pi x^2 y + ye^x}_{P}, \underbrace{\pi x + \pi x^3 + e^x}_{Q} \rangle$

$Q_x = \pi + 3x^2\pi + e^x$

$P_y = 3x^2\pi + e^x$

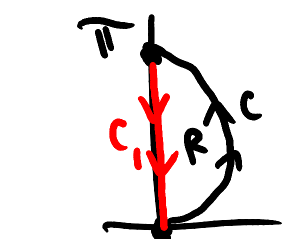
$Q_x \neq P_y$ $\quad Q_x - P_y = \pi$

Not path independent

$t=0$ $\quad x=0 \quad y=0$
$t=\pi$ $\quad x=0 \quad y=\pi$
Not Closed

① Use given Parametrization

② Close the Path & use Green's Thm. ✓



C_1: $x=0$
$y=t$
t st. at π
ends at 0

$$\int_C F\cdot dr + \int_{C_1} F\cdot dr = \iint_R (Q_x - P_y)\,dA$$

$$\int_C F\cdot dr = \underbrace{\iint_R (Q_x - P_y)\,dA}_{2\pi} - \underbrace{\int_{C_1} F\cdot dr}_{-\pi}$$

$$= 2\pi - (-\pi)$$

$$= \boxed{3\pi}$$

$$\iint_R \underbrace{(Q_x - P_y)}_{\pi}\,dA = \pi \underbrace{\boxed{\iint_R dA}}_{\text{area of } R}$$

$$= \boxed{2\pi}$$

$\int_0^\pi \sin x\,dx = 2$

$r = \langle 0, t \rangle$

$dr = \langle 0, 1 \rangle dt$

$F_{\text{on } C_1} = \langle t, 1 \rangle$

$F \cdot dr = 1 \cdot dt$

$$\int_{C_1} F\cdot dr = \int_\pi^0 1\,dt = -\int_0^\pi 1\,dt = -\pi$$

Fall 2011

Problem 6. Let S be the square with vertices

$(1, 0, 0)$, $(0, 1, 0)$, $(0, 1, \sqrt{2})$, $(1, 0, \sqrt{2})$,

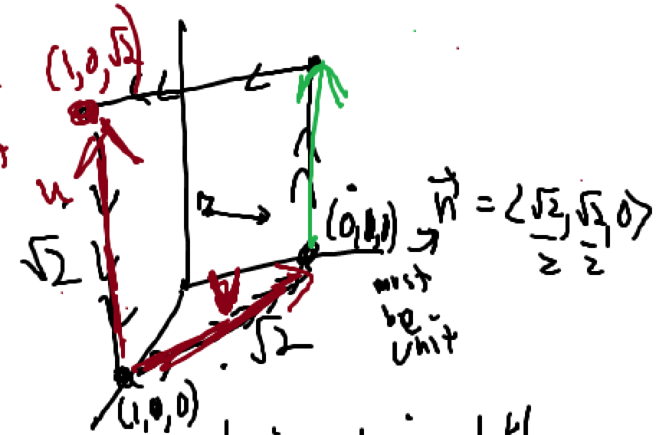
and let C be the boundary of S , traversed in this order of vertices.

Let W be the vector field $W = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$.

Find the value of the integral $I = \int_C W \cdot dr = \int_C z\,dx + x\,dy + y\,dz$.

$\vec{v} \times \vec{u} = n$

might not unit



$$\oint_C F\cdot dr = \iint_S \underbrace{(\nabla\times F)\cdot n}_{\sqrt{2}}\, dS$$

$$= \sqrt{2}\,\boxed{\iint_S dS}^{\,2}$$

surface Area

$$= \boxed{2\sqrt{2}}$$

$$\nabla\times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$$

$$\nabla\times F = \langle 1, 1, 1 \rangle$$

Fall 2011

Problem 7. Evaluate

$$I = \int_C \; y\,(x^2 + y^2)^{-1}\,dx \;-\; x\,(x^2 + y^2)^{-1}\,dy$$

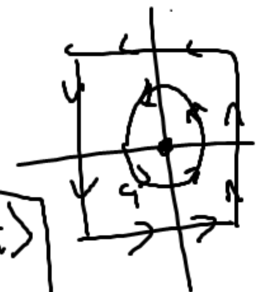
$Q_x = P_y$

where C is the boundary of the square with vertices at $(2, -2)$, $(2, 2)$, $(-2, 2)$ and $(-2, -2)$, traversed counterclockwise.

closed

$$F = \left\langle \frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2} \right\rangle$$

$F \cdot dr$



$r = \langle \cos t, \sin t \rangle$

$dr = \langle -\sin t, \cos t \rangle$

$$\begin{cases} x = \cos t \\ y = \sin t \quad 0 \le t \le 2\pi \\ x^2 + y^2 = 1 \end{cases}$$

$$\boxed{F_{\text{on } C} = \langle \sin t, -\cos t \rangle}$$

$dr = \langle -\sin t, \cos t \rangle$

$F \cdot dr = -(\sin^2 t + \cos^2 t) = -1$

$$\int_{C_1} F \cdot dr = \int_0^{2\pi} -1\,dt = -2\pi$$

Fall 2011

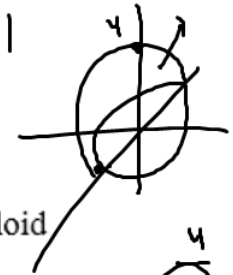
$\boxed{D = S \cup S_1}$

Problem 8. Consider the vector field

$\mathrm{div} = \cancel{3x^2y^2} + \cancel{-3x^2y^2} + 1$

$$W = x^3y^2\,\mathbf{i} - x^2y^3\,\mathbf{j} + (1+z)\,\mathbf{k}.$$

Find the outward flux of W through the portion S of the paraboloid $z = 4 - x^2 - y^2$ which lies above the xy-plane.



$$\underbrace{\text{Find this } \iint_S (w\cdot n)\,ds}_{\text{Surface integral}} + \iint_{S_1} (w\cdot n)\,ds_{\text{ clockwise}} \overset{\text{Div. Thm}}{=} \iiint_D (\mathrm{div}\,W)\,dV$$

$$= 1\cdot \boxed{\iiint_D dV}_{\text{volume}}$$

$$\iint_S w\cdot n\,ds = \iiint_D dV - \iint_{S_1} (w\cdot n)\,ds_1 \quad \text{clockwise}$$

$$\iint_S (w\cdot n)\,ds = \iiint_D dV + \iint_{S_1} (w\cdot n)\,ds_1$$

S_1: $1+z$, Counter Clockwise

$\vec{n} = \langle 0,0,1\rangle$

$\vec{n} = k$

$w\cdot n = 1+z$

but $z = 0$ on S_1

$w\cdot n = 1$

$$\iint_S (w\cdot n)\,ds = \iiint_D dV + \boxed{\iint_{S_1} 1\,dS_1} = 8\pi + 4\pi = \boxed{12\pi}$$

Planar surface $\Rightarrow$ area of shape

$A = 4\pi$

$$\iiint_D dV = \int_0^{2\pi}\int_0^2\int_0^{4-r^2} dz\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 (4-r^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 (4r - r^3)\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \left[2r^2 - \frac{r^4}{4}\right]_0^2$$

$$= 2\pi \cdot (\underbrace{8-4}_{4}) = 8\pi$$

Spring 2011

5. (10 pts) Let C_1 and C_2 be the closed curves

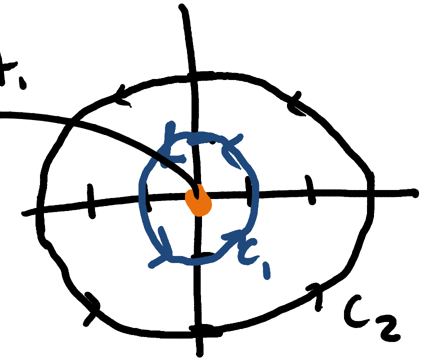
$$\frac{x^2}{9}+\frac{y^2}{4}=1$$

$$C_1=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=1\},\quad C_2=\{(x,y)\in\mathbb{R}^2 \mid 4x^2+9y^2=36\}$$

on the (x,y)-plane, oriented counterclockwise. Consider the line integrals

$$\oint_{C_i}\frac{(x-y)\,dx+(x+y)\,dy}{x^2+y^2},\qquad i=1,2.$$

discont. at (0,0)



(a) Are the two integrals $\oint_{C_1}\frac{(x-y)\,dx+(x+y)\,dy}{x^2+y^2}$ and $\oint_{C_2}\frac{(x-y)\,dx+(x+y)\,dy}{x^2+y^2}$ equal? Why? (Justify your answer.)

Yes

$P=\frac{x-y}{x^2+y^2}\qquad Q=\frac{x+y}{x^2+y^2}\qquad Q_x=P_y$

$$Q_x=\frac{(x^2+y^2)\cdot 1-(x+y)\cdot 2x}{(x^2+y^2)^2}=\frac{\overset{y^2-x^2-2xy}{x^2+y^2-2x^2-2xy}}{(x^2+y^2)^2}$$

$$P_y=\frac{(x^2+y^2)(-1)-(x-y)(2y)}{(x^2+y^2)^2}=\frac{\overset{y^2-x^2-2xy}{-x^2-y^2-2xy+2y^2}}{(x^2+y^2)^2}$$

The two integrals are the same

$$\int_{C_2}F\cdot dr+\int_{-C_1}F\cdot dr=\underbrace{\iint_R (Q_x-P_y)\,dA}_{0}$$

$$\int_{C_2}F\cdot dr=-\int_{-C_1}F\cdot dr=\int_{C_1}F\cdot dr$$

(b) Evaluate these two line integrals.

$$\oint_{C_1}\frac{(x-y)\,dx+(x+y)\,dy}{x^2+y^2}=\underline{\quad 2\pi\quad}$$

$$\oint_{C_2}\frac{(x-y)\,dx+(x+y)\,dy}{x^2+y^2}=\underline{\quad 2\pi\quad}$$

$C_1:\begin{cases}x=\cos t\\ y=\sin t\end{cases}\quad 0\le t\le 2\pi$

$r=\langle\cos t,\sin t\rangle$

$dr=\langle-\sin t,\cos t\rangle\,dt$

See next slide

For C_1: $\left\langle\frac{\cos t-\sin t}{1},\ \frac{\cos t+\sin t}{1}\right\rangle$

$$C_1: \begin{cases} x = \cos t \\ y = \sin t \end{cases} \quad 0 \le t \le 2\pi$$

$$r = \langle \cos t, \sin t \rangle$$

$$dr = \langle -\sin t, \cos t \rangle dt$$

$$F_{\text{on } C_1} = \left\langle \frac{\cos t - \sin t}{1}, \frac{\cos t + \sin t}{1} \right\rangle$$

$$F \cdot dr = -\cancel{\cos t \sin t} + \underbrace{\sin^2 t + \cos^2 t}_{1} + \cancel{\cos t \sin t}$$

$$F \cdot dr = 1$$

$$\int_{C_1} F \cdot dr = \int_0^{2\pi} 1 \cdot dt = \boxed{2\pi}$$

Spring 2011

6. (10 pts) Let $S = \partial D$ be the boundary of the solid region D contained in the cylinder $x^2 + y^2 = 4$ between $z = x$ and $z = 8$, i.e.

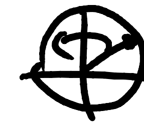
$$D = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \le 4,\ x \le z \le 8\}.$$

Let $\mathbf{n}$ be the unit normal vector field on S pointing outward relative to D. Calculate the flux

$$\iint_S \mathbf{F} \cdot \mathbf{n}\ \mathrm{d}S \overset{\text{Div. Thm.}}{=} \iiint_D \operatorname{div} F\, dV \rightarrow \text{cylindrical}$$

of the vector field

$$\mathbf{F} = \langle x, y^2, z + y \rangle = x\,\vec{i} + y^2\,\vec{j} + (z + y)\,\vec{k}.$$



$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$x^2 + y^2 \le 4$

$x \le z \le 8$

$r\cos\theta \le z \le 8$

$$\operatorname{div} F = 1 + 2y + 1 = 2 + 2y = 2(1 + y) = 2(1 + r\sin\theta)$$

$$\iint_S F\cdot n\, ds = \int_0^{2\pi}\int_0^2\int_{r\cos\theta}^{8} 2(1 + r\sin\theta)\, r\, dz\, dr\, d\theta$$

$$= 2\int_0^{2\pi}\int_0^2 (r + r^2\sin\theta)\,[z]_{r\cos\theta}^{8}\, dr\, d\theta$$

$$\boxed{\begin{array}{l}(r + r^2\sin\theta)(8 - r\cos\theta)\\ 8r - r^2\cos\theta + 8r^2\sin\theta\\ \qquad - r^3\sin\theta\cos\theta\end{array}}$$

$$= 2\int_0^{2\pi}\int_0^2 (8r - r^2\cos\theta + 8r^2\sin\theta - r^3\sin\theta\cos\theta)\, dr\, d\theta$$

$$= 2\int_0^{2\pi}\left[4r^2 - \frac{r^3}{3}\cos\theta + \frac{8r^3}{3}\sin\theta - \frac{r^4}{4}\sin\theta\cos\theta\right]_0^2 d\theta$$

$$\iint_S \vec{F}\cdot\vec{n}\,dS = 2\int_0^{2\pi}\left[4r^2 - \frac{r^3}{3}\cos\theta + \frac{8r^3}{3}\sin\theta - \frac{r^4}{4}\sin\theta\cos\theta\right]_0^2 d\theta$$

$$= 2\int_0^{2\pi}\left(16 - \frac{8}{3}\underbrace{\cancelto{0}{\cos\theta}} + \frac{64}{3}\underbrace{\cancelto{0}{\sin\theta}} - 4\underbrace{\cancelto{0}{\sin\theta\cos\theta}}_{\text{2π periodic}}\right)d\theta$$

Full period of $\cos\theta$ and $\sin\theta$

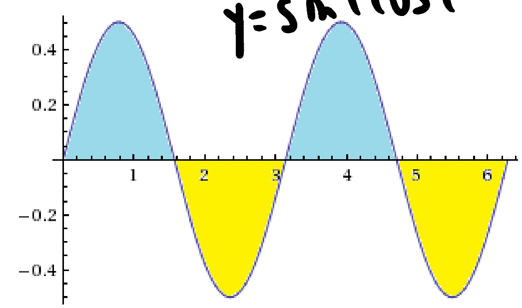
2π periodic (product of 2π periodic functions is 2π periodic (at least))

$$= 2\left[16\theta\right]_0^{2\pi}$$

$$= 2\cdot 16\cdot 2\pi$$

$$= \boxed{64\pi}$$

$y = \sin t\cos t$

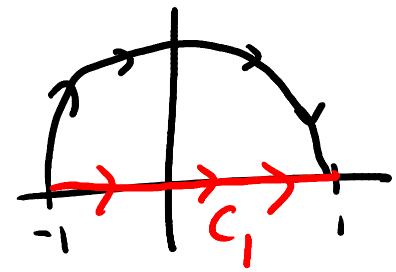


Fall 2010

11. Determine the value of the line integral

$$\int_C \overbrace{6x^2e^{2x^3-2y^3}}^{P}\,dx - \overbrace{6y^2e^{2x^3-2y^3}}^{Q}\,dy$$

where C is the semicircle $x^2+y^2=1,\ y \geq 0$, traversed from $(-1,0)$ to $(1,0)$.



$$\left.\begin{array}{l} Q_x = -6y^2 e^{2x^3-2y^3}\cdot(6x^2) \\ P_y = 6x^2 e^{2x^3-2y^3}\cdot(-6y^2) \end{array}\right\}\ Q_x = P_y \Rightarrow \text{Ind. of Path}$$

Not closed → Pick a conv. path ✓

→ Find ϕ and use FTLI

C_1: $x = t$, $y = 0$, $-1 \leq t \leq 1$

$r = \langle t, 0\rangle$

$dr = \langle 1, 0\rangle\, dt$

$F_{\text{on } C_1} = \langle 6t^2e^{2t^3}, 0\rangle$

$F\cdot dr = 6t^2e^{2t^3}\,dt$

$$\int_C F\cdot dr = \int_{C_1} F\cdot dr = \int_{-1}^{1} 6t^2e^{2t^3}\,dt = \left[e^{2t^3}\right]_{-1}^{1} = \boxed{e^2 - e^{-2}}$$

$u = 2t^3$

$du = 6t^2\,dt$ $\quad \int e^u\,du = e^u$

Fall 2010

12. Find the value of a so that the line integral

$$\int_C \underbrace{ay^3z}_{P}\,dx + \underbrace{xy^2z}_{Q}\,dy + \underbrace{\frac{1}{3}xy^3}_{R}\,dz$$

is independent of the path, C, taken between any two given points.

$$\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ay^3z & xy^2z & \frac{1}{3}xy^3 \end{vmatrix} = \left\langle \underbrace{xy^2 - xy^2}_{0},\ -\underbrace{\left(\tfrac{1}{3}y^3 - ay^3\right)}_{\substack{=0 \\ \text{if } a=1/3}},\ \underbrace{y^2z - 3ay^2z}_{\substack{=0 \\ \text{if} \\ a=1/3}} \right\rangle$$

Fall 2010

13. Find the outward flux

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS \overset{\text{Div. Thm}}{=} \iiint_D \operatorname{div}\mathbf{F}\,dV$$

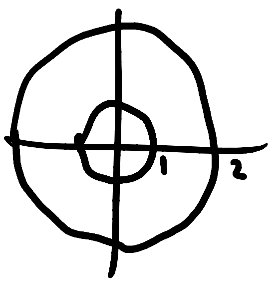
of the vector field

$$\mathbf{F} = 4xy^2\mathbf{i} + 3y\mathbf{j} + 4zx^2\mathbf{k}$$

where the surface S is the boundary of the region $1 \le x^2+y^2 \le 4$, $\boxed{0 \le z \le 1.}$ $\quad 1 \le r \le 2 \quad 0 \le \theta \le 2\pi$

$$\operatorname{div}\mathbf{F} = 4y^2 + 3 + 4x^2$$

$$\operatorname{div}\mathbf{F} = 4(\underbrace{x^2+y^2}_{r^2}) + 3$$



$$\iint_S \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^{2\pi}\int_1^2\int_0^1 (4r^2+3)\,dz\; r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^2 (4r^3+3r)\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[r^4 + \frac{3r^2}{2}\right]_1^2 d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \underbrace{\left[\underbrace{(16+6)}_{22} - \underbrace{\left(1+\frac{3}{2}\right)}_{2.5}\right]}_{19.5}$$

$$= 2\pi \cdot 19.5$$

$$= \boxed{39\pi}$$

Fall 2010

14. Compute

$$\iint_S \text{curl}\mathbf{F}\cdot\mathbf{n}\ dS \overset{\text{Stokes' thm.}}{=} \oint_C F\cdot dr$$

where

$$\mathbf{F} = xz\mathbf{i} + (zy - 2y)\mathbf{j} + y^2 z\mathbf{k}$$

and S is the cone $z^2 = x^2 + y^2$ with $0 \le z \le 2$ and $\mathbf{n}$ the outward (i.e. downward pointing) normal.

$z=2 \Rightarrow x^2+y^2=4$

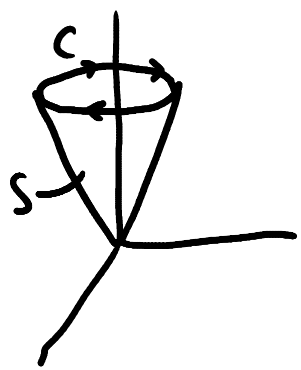
C: $x = 2\cos t$ t st @ 2π
$y = 2\sin t$ t ends @ 0
$z = 2$

$r = \langle 2\cos t, 2\sin t, 2\rangle$

$dr = \langle -2\sin t, 2\cos t, 0\rangle$

$F_{\text{on } C} = \langle 4\cos t, \underbrace{4\sin t - 4\sin t}_{0}, 8\sin^2 t\rangle$

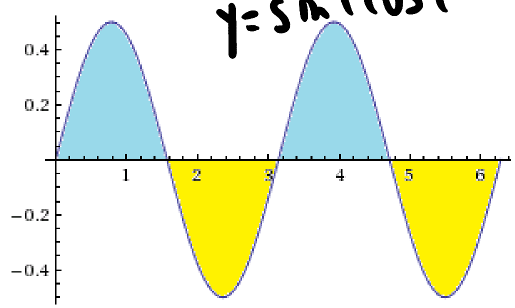
$F\cdot dr = -8\sin t\cos t$



$$\oint_C F\cdot dr = \int_{2\pi}^{0} -8\sin t\cos t\, dt = 8\int_0^{2\pi} \underbrace{\sin t\cos t\, dt}_{0} = \boxed{0}$$

Product of 2π periodic functions is 2π periodic (at least)

$y = \sin t\cos t$



Fall 2010

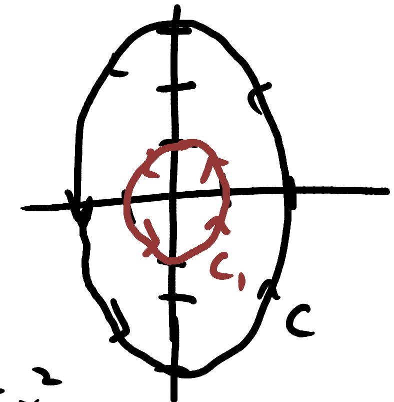
15. Consider the curve C, traced counter clockwise, which is the ellipse $\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$. (C lies between a circle of radius 1 and a circle of radius 5.)

Compute the line integral

$$\int_C \mathbf{F} \cdot \mathbf{dr}$$

where

$$\mathbf{F}(\mathrm{x}, \mathrm{y}) = \underbrace{\frac{-y}{x^2+y^2}}_{P}\mathbf{i} + \underbrace{\frac{x}{x^2+y^2}}_{Q}\mathbf{j}$$



$$\int_C F \cdot dr = \int_{C_1} F \cdot dr$$

$$Q_x = \frac{(x^2+y^2)\cdot 1 - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$Q_x = P_y$ ✓

$$P_y = \frac{(x^2+y^2)(-1) - (-y)\cdot 2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$x^2+y^2=1$

C_1: $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$

$r = \langle \cos t, \sin t \rangle$

$dr = \langle -\sin t, \cos t \rangle dt$

$F_{\text{on } C_1} = \langle -\sin t, \cos t \rangle$

$F \cdot dr = \sin^2 t + \cos^2 t = 1$

$$= \int_0^{2\pi} 1 \, dt = \boxed{2\pi}$$

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Closed? No

Ind. of Path? Yes $\rightarrow$ pick conv. path / Find ϕ use FTLI

$$\text{Curl}\,\mathbf{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^y & xe^y + e^z & ye^z \end{vmatrix} = \langle 0,0,0\rangle$$

11. Find the work done by the force field

$$\mathbf{F}(x,y,z) = e^y\,\mathbf{i} + (xe^y + e^z)\,\mathbf{j} + ye^z\,\mathbf{k}$$

in moving a particle from $(1,0,0)$ to $(0,1,\pi)$ along the helix $x = \cos(t)$, $y = \sin(t)$, $z = t$.

Answer:

(a) e^π (b) $e^\pi - 2$ (c) $e^\pi + 1$ (d) $e^\pi - 1$ (e) $2e^\pi - 1$ (f) $2e^\pi - 3$

$$\left.\begin{aligned} \phi &= \int e^y\,dx = xe^y + G(y,z) \\ \phi &= \int (xe^y + e^z)\,dy = xe^y + ye^z + H(x,z) \\ \phi &= \int (ye^z)\,dz = ye^z + K(x,y) \end{aligned}\right\} \phi = xe^y + ye^z + C$$

$$\phi(0,1,\pi) = 0 + e^\pi$$

$$-\ \phi(1,0,0) = 1 + 0$$

$$= \boxed{e^\pi - 1}$$

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S: plane $z = 1 - x - y$

$z_x = -1 \quad (z_x)^2 = 1$
$z_y = -1 \quad (z_y)^2 = 1$

12. Let C be the curve that is the intersection of the plane $x+y+z=1$ and the cylinder $x^2+y^2=9$ oriented counter-clockwise as viewed from above. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F}(x, y, z) = -yx^2\,\mathbf{i} + y^2z\,\mathbf{j} + z^2\,\mathbf{k}.$$

Answer:

(a) 0 (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) -2 (e) $-\frac{1}{4}$

$$\int_C F \cdot dr \overset{\text{Stokes' Thm}}{=} \iint_S (\text{curl}F \cdot n)\, ds$$

$$= \iint_R (x^2 - y^2)\, dA$$

$$= \int_0^{2\pi} \int_0^3 r^2(\cos^2\theta - \sin^2\theta)\, r\, dr\, d\theta$$

$$= \underbrace{\int_0^{2\pi} \cos 2\theta\, d\theta}_{0 \text{ since its period is } \pi} \cdot \int_0^3 r^3\, dr = \boxed{0}$$

$$\text{curl}F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -yx^2 & y^2z & z^2 \end{vmatrix}$$

$$\text{curl}F = \langle 0 - y^2, 0, 0 + x^2 \rangle$$

n to plane $= \langle 1, 1, 1 \rangle \quad |n| = \sqrt{3}$

$$\boxed{\text{curl}F \cdot n = (-y^2 + x^2) \cdot \frac{1}{\sqrt{3}}}$$

$$ds = \sqrt{1 + (z_x)^2 + (z_y)^2}\, dA$$

$$\boxed{ds = \sqrt{3}\, dA}$$

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$Q_x = 3y \quad P_y = 2y$ Closed? Yes; Ind. of Path? No $\}$ → Parametrize; → Green's Thm.

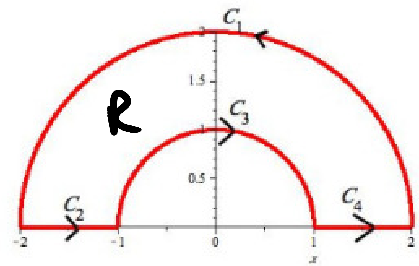
P Q

$Q_x - P_y = y$

13. Let $\mathbf{F}(x, y) = \langle y^2, 3xy \rangle$ be a vector field in the plane and let C be the closed curve shown in the following picture with a counter-clockwise orientation [the curve C_1 and C_3 travel along a circle of radius 2 and 1, respectively]. Evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

Answer:

(a) -10 (b) $\frac{14}{3}$ (c) $\frac{10}{3}$ (d) $-\frac{10}{5}$ (e) $\frac{3}{2}$



polar

$1 \le r \le 2$

$0 \le \theta \le \pi$

$$\oint_C F \cdot dr = \iint_R (Q_x - P_y)\, dA$$

$$= \int_0^{\pi} \int_1^2 r \sin\theta \; r\, dr\, d\theta$$

$$= \int_0^{\pi} \left[\frac{r^3}{3}\right]_1^2 \sin\theta\, d\theta = \frac{7}{3} \int_0^{\pi} \sin\theta\, d\theta = \boxed{\frac{14}{3}}$$

$$\left[-\cos\theta\right]_0^{\pi}$$

$$\underbrace{-(-1) - (-1)}_{2}$$

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$$\text{div}\,F = 3y^2+3z^2+3x^2 = 3(\underbrace{x^2+y^2+z^2}_{\rho^2})$$

14. Find the outward flux $\iint_S \mathbf{F}\cdot\mathbf{n}\;d\mathbf{S}$ of the vector field $\mathbf{F} = 3xy^2\mathbf{i} + 3yz^2\mathbf{j} + 3zx^2\mathbf{k}$ where the surface S is the boundary of the region $1 \le x^2+y^2+z^2 \le 4$.

$1 \le \rho \le 2$

$0 \le \phi \le \pi$

$0 \le \theta \le 2\pi$

$$\iint_S (F\cdot n)\,ds \overset{\text{Divergence Thm}}{=} \iiint_D \text{div}\,F\,dv$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_1^2 3\rho^2\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= 3\int_0^{2\pi}\int_0^{\pi}\underbrace{\left[\frac{\rho^5}{5}\right]_1^2}_{\frac{32-1}{5}}\sin\phi\;d\phi\,d\theta$$

$$= \frac{3\cdot 31}{5}\int_0^{2\pi}\underbrace{\int_0^{\pi}\sin\phi\,d\phi}_{\underbrace{\left[-\cos\phi\right]_0^{\pi}}_{2}}\,d\theta = \frac{3\cdot 31}{5}\cdot 2\int_0^{2\pi}d\theta$$

$$= \frac{3\cdot 31\cdot 2\cdot 2\pi}{5} = \boxed{\frac{372\pi}{5}}$$

$$\begin{array}{r} 31 \\ \times 12 \\ \hline 62 \\ 31 \\ \hline 372 \end{array}$$