Strategies (which test to use on which series)

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1) Check at a glance to see if $\lim_{n\to\infty} a_n \neq 0$.

If this is true, then the series diverges by the **Test for Divergence**.

2) Series that we can find whether or not they converge rather quickly:

$a)$ $\boxed{p-\text{series}}$ $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$ and divergent if $p \le 1$.

$b)$ $\boxed{\text{geometric series}}$ $\sum_{n=1}^{\infty} ar^{n-1}$ or $\sum_{n=1}^{\infty} ar^{n}$ is convergent for $|r| < 1$ and divergent if $|r| \ge 1$.

3) Use the Comparison Test / Limit Comparison Test on series

$a)$ that have the form of $p-$series or geometric series

if a_n is a fraction involving polynomials only or polynomials under radicals compare this series with a $p-$series

$b)$ Note: make sure that the series has only positive terms to use the comp. tests

12.7 Strategies (which test to use on which series)

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4) The Alternating Series Test might work on series of the form $\sum_{n=1}^{\infty} (-1)^n b_n$

5) The Ratio Test works well on series involving factorials or constants raised to powers involving n

6) The Root Test works well if a_n is of the form $(b_n)^n$

7) The Integral Test works well if $\int_k^{\infty} f(x)\,dx$ is not difficult to evaluate,

where $\sum_{n=k}^{\infty} a_n$ with $a_n = f(n)$ and f is continuous, decreasing, and positive on $[k,\infty)$

$i)\ \sum_{n=1}^{\infty} \frac{n}{n^3+2}$ $\quad iii)\ \sum_{n=1}^{\infty} \frac{n^2}{e^n}$ $\quad v)\ \sum_{n=1}^{\infty} \pi^{1/n}$ $\quad vii)\ \sum_{n=2}^{\infty} \frac{\ln n}{n^2}$

$ii)\ \sum_{n=1}^{\infty} \frac{n^2+2}{n^3+3}$ $\quad iv)\ \sum_{n=1}^{\infty} \frac{\cos(5n)}{1+(1.4)^n}$ $\quad vi)\ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sqrt{n}}{n+1}$ $\quad viii)\ \sum_{n=2}^{\infty} \frac{n \ln n}{(n-1)^3}$

$i)\ \sum_{n=1}^{\infty} \frac{n}{n^3+2}$

$b_n = \frac{n}{n^3} = \frac{1}{n^2}$

$\frac{n}{n^3+2} < \frac{1}{n^2}$ since $n^3 < n^3+2$

$\sum_{n=1}^{\infty} b_n$ converges (p-series w/ $p=2$) so $\sum_{n=1}^{\infty} a_n$ converges by the Comp. Test.

$ii)\ \sum_{n=1}^{\infty} \frac{n^2+2}{n^3+3}$

$b_n = \frac{n^2}{n^3} = \frac{1}{n}$

$\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{n^2+2}{n^3+3}}{\frac{1}{n}}\right|$

$= \lim_{n\to\infty} \left|\frac{n^3+2n}{n^3+3}\right| = 1$

$\sum_{n=1}^{\infty} b_n$ diverges (Harmonic series) so $\sum_{n=1}^{\infty} a_n$ diverges by the Limit Comp. Test.

$iii)\ \sum_{n=1}^{\infty} \frac{n^2}{e^n}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^2}{n^2} \cdot \frac{e^n}{e^{n+1}}\right|$

$= \lim_{n\to\infty} \left|\frac{(n+1)^2}{n^2} \cdot \frac{e^n}{e^n e}\right| = \frac{1}{e} < 1$

so $\sum_{n=1}^{\infty} a_n$ converges by the Ratio Test.

$i)\ \sum_{n=1}^{\infty} \frac{n}{n^3+2}$ $\quad iii)\ \sum_{n=1}^{\infty} \frac{n^2}{e^n}$ $\quad v)\ \sum_{n=1}^{\infty} \pi^{1/n}$ $\quad vii)\ \sum_{n=2}^{\infty} \frac{\ln n}{n^2}$

$ii)\ \sum_{n=1}^{\infty} \frac{n^2+2}{n^3+3}$ $\quad iv)\ \sum_{n=1}^{\infty} \frac{\cos(5n)}{1+(1.4)^n}$ $\quad vi)\ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sqrt{n}}{n+1}$ $\quad viii)\ \sum_{n=2}^{\infty} \frac{n \ln n}{(n-1)^3}$

$iv)\ \sum_{n=1}^{\infty} \frac{\cos(5n)}{1+(1.4)^n}$ terms aren't all positive

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{|\cos(5n)|}{1+(1.4)^n}$ now they are

$b_n = \frac{1}{(1.4)^n} = \frac{1}{\left(\frac{7}{5}\right)^n} = \left(\frac{5}{7}\right)^n$

$\sum_{n=1}^{\infty} b_n$ converges (geom. series w/ $r=5/7$)

$\frac{|\cos(5n)|}{1+(1.4)^n} < \frac{1}{(1.4)^n}$

so $\sum_{n=1}^{\infty} a_n$ converges absolutely by the Comp. Test.

$v)\ \sum_{n=1}^{\infty} \pi^{1/n}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \pi^{1/n} = \pi^0 = 1 \neq 0$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges by the Test For Divergence

$vi)\ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sqrt{n}}{n+1}$

$b_n = \frac{\sqrt{n}}{n+1} > 0$

b_n is decreasing

$\lim_{n\to\infty} b_n = 0$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ converges by the Alternating Series Test

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$i)\ \sum_{n=1}^{\infty} \frac{n}{n^3+2}$ $\quad iii)\ \sum_{n=1}^{\infty} \frac{n^2}{e^n}$ $\quad v)\ \sum_{n=1}^{\infty} \pi^{1/n}$ $\quad vii)\ \sum_{n=2}^{\infty} \frac{\ln n}{n^2}$

$ii)\ \sum_{n=1}^{\infty} \frac{n^2+2}{n^3+3}$ $\quad iv)\ \sum_{n=1}^{\infty} \frac{\cos(5n)}{1+(1.4)^n}$ $\quad vi)\ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sqrt{n}}{n+1}$ $\quad viii)\ \sum_{n=2}^{\infty} \frac{n \ln n}{(n-1)^3}$

$vii)\ \sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ $\quad f(x) = \frac{\ln x}{x^2} > 0,$ continuous, decreasing

$$\int_2^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \left.\frac{-\ln x}{x}\right|_2^b + \int_2^b \frac{1}{x^2}\,dx$$

$u = \ln x \quad dv = \frac{1}{x^2}dx$

$du = \frac{1}{x}dx \quad v = -\frac{1}{x}$

$$= \lim_{b\to\infty} \frac{-\ln b}{b} + \frac{\ln 2}{2} + \frac{-1}{b} + \frac{1}{2}$$

$= \frac{1+\ln 2}{2}$ since $\lim_{b\to\infty} \frac{-\ln b}{b} \overset{L'H}{=} \lim_{b\to\infty} \frac{-\frac{1}{b}}{1} = 0$

and $\lim_{b\to\infty} \frac{-1}{b} = 0$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ converges by the Integral Test

$viii)\ \sum_{n=2}^{\infty} \frac{n \ln n}{(n-1)^3}$ $\quad b_n = \frac{n \ln n}{n^3} = \frac{\ln n}{n^2}$

$$\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{n\ln n}{(n-1)^3}}{\frac{\ln n}{n^2}}\right|$$

$$= \lim_{n\to\infty} \left|\frac{n \ln n}{(n-1)^3} \cdot \frac{n^2}{\ln n}\right| = \lim_{n\to\infty} \left|\frac{n^3}{(n-1)^3}\right| = 1$$

$\sum_{n=1}^{\infty} b_n$ converges so $\sum_{n=1}^{\infty} a_n$ converges by the Limit Comp. Test.

10.2

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Test for Divergence $\left(n^{th}\text{ term test}\right)$:

If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ or $\sum_{n=1}^{\infty} ar^n$ is convergent for $|r| < 1$ and divergent if $|r| \geq 1$.

10.3 The Integral Test

If $f(x)$ is: $a)$ continuous, on the interval $\underset{\text{constant } k>0}{[k,\infty)}$

$b)$ positive,

$c)$ and decreasing

,then the series $\sum_{n=k}^{\infty} a_n$ (with $a_n = f(n)$)

$i)$ is convergent when $\int_k^{\infty} f(x)\,dx$ is convergent.

$ii)$ is divergent when $\int_k^{\infty} f(x)\,dx$ is divergent.

$\boxed{p-\text{series}}$ $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$ and divergent if $p \le 1$.

10.4

The Comparison Test:

Given the series $\sum_{n=1}^{\infty} a_n$, $(a_n \ge 0)$

(i) if the terms a_n are **smaller** than the terms b_n of a known **convergent** series $\sum_{n=1}^{\infty} b_n$ $(b_n \ge 0)$, then our series $\sum_{n=1}^{\infty} a_n$ is also **convergent**.

(ii) if the terms a_n are **larger** than the terms b_n of a known **divergent** series $\sum_{n=1}^{\infty} b_n$ $(b_n \ge 0)$, then our series $\sum_{n=1}^{\infty} a_n$ is also **divergent**.

10.4

The Limit Comparison Test:

Given the series $\sum_{n=1}^{\infty} a_n,\ (a_n > 0)$ and a known convergent or divergent series $\sum_{n=1}^{\infty} b_n,\ (b_n > 0)$

If the $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is a finite positive number, then the series will behave alike, i.e. either both converge or both diverge.

10.5 The Ratio Test

Let $\{a_n\}$ be a sequence and assume that the following limit exists: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$

$i)$ If $L < 1,$ then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

$ii)$ If $L > 1$ or if the limit is infinite, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

$iii)$ If $L = 1,$ the Ratio Test is inconclusive.

(the series could be absolutely convergent, conditionally convergent, or divergent)

10.5 The Root Test

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Let $\{a_n\}$ be a sequence and assume that the following limit exists: $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L$

$i)$ If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

$ii)$ If $L > 1$ or if the limit is infinite, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

$iii)$ If $L = 1$, the Root Test is inconclusive.

(the series could be absolutely convergent, conditionally convergent, or divergent)

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10.6 The Alternating Series Test

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1} b_n$ (where $b_n > 0$) satisfies:

$i)$ $\lim_{n\to\infty} b_n = 0$

$ii)$ $\{b_n\}$ is a decreasing sequence, and

,then the series is **convergent**.