9.2 Linear Differential Equation

$$\frac{dy}{dx}+P(x)\,y=Q(x)$$ ← standard form of a linear first order diff. eq.

1) Find the integrating factor $\mu=e^{\int P(x)dx}$

2) Multiply the entire equation by it.

$$e^{\int P(x)dx}\frac{dy}{dx}+P(x)e^{\int P(x)dx}y=e^{\int P(x)dx}Q(x)$$

3) Identify the left hand side as the product rule with the two functions being the integrating factor and the unknown function y(x).

$$\underbrace{e^{\int P(x)dx}\frac{dy}{dx}+P(x)e^{\int P(x)dx}y}=e^{\int P(x)dx}Q(x)$$

$$\frac{d}{dx}\left(e^{\int P(x)dx}\cdot y\right)=e^{\int P(x)dx}Q(x)$$

4) Integrate both sides of the equation.

$$\cancel{\int}\left[\cancel{\frac{d}{dx}}\left(e^{\int P(x)dx}\cdot y\right)\right]\cancel{dx}=\int\left[e^{\int P(x)dx}Q(x)\right]dx$$

5) Use the FTC on the left hand side.

$$e^{\int P(x)dx}\cdot y=\int\left[e^{\int P(x)dx}Q(x)\right]dx$$

6) Divide both sides by μ.

$$y=\frac{\int\left[e^{\int P(x)dx}Q(x)\right]dx}{e^{\int P(x)dx}}$$

$$\frac{x^2y'}{x^2}+\frac{2xy}{x^2}=\frac{\cos^2 x}{x^2}$$

$$y'+\left(\frac{2x}{x^2}\right)\cdot y=\frac{(\cos x)^2}{x^2}$$

$$\boxed{\underbrace{y'+\left(\frac{2}{x}\right)}_{P(x)}y=\underbrace{\frac{(\cos x)^2}{x^2}}_{Q(x)}\quad\boxed{\text{std. form}}}$$

$$\int P(x)dx=\int\frac{2}{x}dx=2\cdot\ln x$$

$$\mu=e^{\int P(x)dx}=e^{2\cdot\ln x}=\cancel{e}^{\cancel{\ln}(x^2)}=x^2$$

① $\boxed{\mu=x^2}$

$$e^{2\ln x}\neq 2x$$

② mult. by x^2

$$x^2y'+\left(\frac{2}{x}\cdot x^2\right)y=\frac{(\cos x)^2}{x^2}\cdot x^2$$

$$\boxed{x^2y'+2xy=(\cos x)^2}$$

③ $$\frac{d}{dx}(\overset{\mu\cdot y}{x^2\cdot y})=(\cos x)^2$$

④ $$\cancel{\int}\cancel{\frac{d}{dx}}(x^2y)=\underbrace{\int(\cos x)^2dx}$$

$$x^2y=\int\frac{1}{2}(1+\cos 2x)\,dx$$

$$x^2 y = \int \frac{1}{2}(1+\cos(2x))\,dx$$

$$= \frac{1}{2}\int (1 + \cos(2x))\,dx$$

$$x^2 y = \frac{1}{2}\left(x + \frac{1}{2}\sin(2x)\right) + C$$

$$y = \frac{\frac{1}{2}x + \frac{1}{4}\sin(2x) + C}{x^2}$$

Ⓐ $\boxed{y = \frac{1}{2x} + \frac{\sin(2x)}{4x^2} + \frac{C}{x^2}}$

Ⓠ orig eq.
$x^2 y' + 2xy = (\cos x)^2$

$$x \ln x \frac{dy}{dx} + y = xe^x$$

divide by $x \ln x$

$$\boxed{\frac{dy}{dx} + \underbrace{\left(\frac{1}{x\ln x}\right)}_{P(x)} y = \underbrace{\frac{xe^x}{x\ln x}}_{Q(x)}}$$

Std. Form

$\int P(x)dx = \int \frac{1}{x\ln x}dx$

$u = \ln x$
$du = \frac{1}{x}dx$

$\int \frac{1}{u}du = \ln u = \boxed{\ln(\ln x)}$

$\int P(x)dx = \cancel{\ln}(\ln x)$

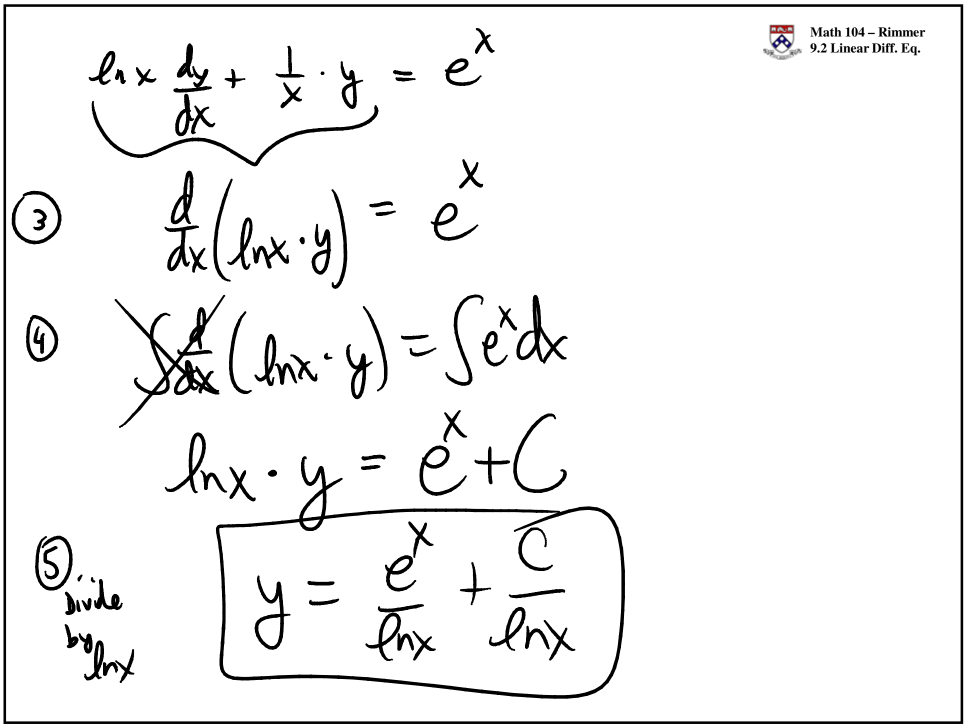
$\mu = e \qquad \cancel{e}$

① $\boxed{\mu = \ln x}$

② mult. by μ

$\ln x \frac{dy}{dx} + \frac{1}{x}y = \frac{\cancel{x}e^x \cancel{\ln x}}{\cancel{x}\cancel{\ln x}}$

$$\boxed{\ln x \frac{dy}{dx} + \frac{1}{x}y = e^x}$$



Math 104 – Rimmer
9.2 Linear Diff. Eq.

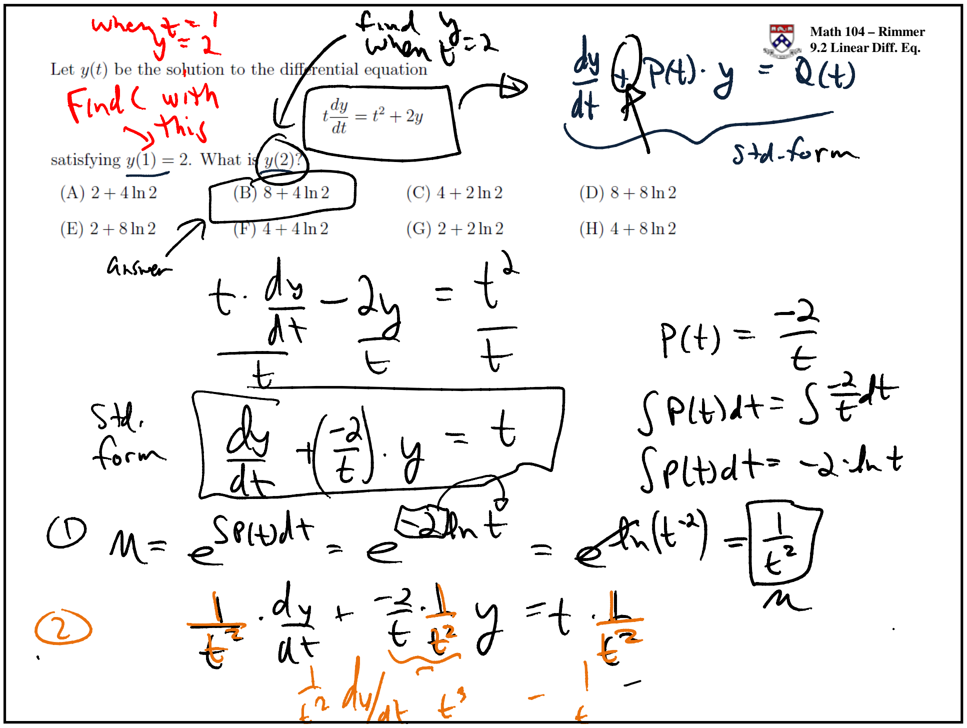
$$\underbrace{\ln x \frac{dy}{dx} + \frac{1}{x}\cdot y}_{} = e^x$$

③ $$\frac{d}{dx}\left(\ln x \cdot y\right) = e^x$$

④ $$\cancel{\int \frac{d}{dx}}\left(\ln x \cdot y\right) = \int e^x dx$$

$$\ln x \cdot y = e^x + C$$

⑤ Divide by $\ln x$ $$\boxed{y = \frac{e^x}{\ln x} + \frac{C}{\ln x}}$$



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9.2 Linear Diff. Eq.

when $t=1$, $y=2$

find y when $t=2$

Let $y(t)$ be the solution to the differential equation

Find C with this

$$\boxed{t\frac{dy}{dt} = t^2 + 2y}$$

$$\underbrace{\frac{dy}{dt} + P(t)\cdot y = Q(t)}_{\text{Std. form}}$$

satisfying $y(1) = 2$. What is $y(2)$?

(A) $2 + 4\ln 2$ (B) $\boxed{8 + 4\ln 2}$ (C) $4 + 2\ln 2$ (D) $8 + 8\ln 2$

(E) $2 + 8\ln 2$ (F) $4 + 4\ln 2$ (G) $2 + 2\ln 2$ (H) $4 + 8\ln 2$

Answer

$$\frac{t\cdot\frac{dy}{dt}}{t} - \frac{2y}{t} = \frac{t^2}{t}$$

$$P(t) = \frac{-2}{t}$$

$$\int P(t)dt = \int \frac{-2}{t}dt$$

$$\int P(t)dt = -2\cdot\ln t$$

Std. form $$\boxed{\frac{dy}{dt} + \left(\frac{-2}{t}\right)\cdot y = t}$$

① $$\mu = e^{\int P(t)dt} = e^{\boxed{-2}\ln t} = e^{\ln(t^{-2})} = \underbrace{\boxed{\frac{1}{t^2}}}_{\mu}$$

② $$\frac{1}{t^2}\cdot\frac{dy}{dt} + \underbrace{\frac{-2}{t}\cdot\frac{1}{t^2}}\, y = t\cdot\frac{1}{t^2}$$

$$\frac{1}{t^2}\, dy/dt \quad \frac{-}{t^3} \quad - \quad \frac{1}{t} \quad -$$

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9.2 Linear Diff. Eq.

$$\underbrace{\frac{1}{t^2}\frac{dy}{dt} + \frac{-2}{t^3}y} = \frac{1}{t}$$

③ $$\frac{d}{dt}\left(\frac{1}{t^2}\cdot y\right) = \frac{1}{t}$$

④ $$\cancel{\int \frac{d}{dt}}\left(\frac{1}{t^2}\cdot y\right)\cancel{dt} = \underbrace{\int \frac{1}{t}\,dt}$$

$(t=1)$ $(y=2)$

⑤ FTC $\nearrow$

$$\frac{1}{t^2}\cdot y = \ln(t) + C$$

$$\frac{1}{1^2}\cdot 2 = \underbrace{\ln 1}_{0} + c \Rightarrow c = 2$$

$$\boxed{\frac{1}{t^2}\cdot y = \ln(t) + 2}$$

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9.2 Linear Diff. Eq.

$$\frac{1}{t^2}\cdot y = \ln(t) + 2$$

Find y
when $t=2$
$y(2) = ?$
6

$$4\cdot\left[\cancel{\frac{1}{4}}\cdot y = \ln(2) + 2\right]$$

$$\boxed{y = 4\ln(2) + 8}$$

$\frac{dy}{dt} = -k(y+1)$ with $y(0)=2$ and $y(1)=1$. Find $y(2)$.

$$\frac{dy}{dt} = -ky - k$$

$$\boxed{\frac{dy}{dt} + \underbrace{k}_{P(x)} y = -k}$$

$$\int P(t)\,dt = \int k\,dt = kt$$

$$e^{\int P(t)\,dt} = e^{kt} = \mu \quad ①$$

$u = kt$
$-du = -k\,dt$
[illegible] $-\int e^u\,du$

mult. by $\mu = e^{kt}$

② $e^{kt}\frac{dy}{dt} + ke^{kt}y = -ke^{kt}$

③ $\frac{d}{dt}\left(e^{kt}\cdot y\right) = -ke^{kt}$

④ $\int \cancel{\frac{d}{dt}}\left(e^{kt}y\right) = \int -ke^{kt}\,dt$

$$e^{kt}\cdot y = -e^{kt} + C$$

solve for C

$$e^{kt}\cdot y = -e^{kt} + C$$

$\begin{cases} t=0 \\ y=2 \end{cases}$ $y(0)=2$

$$\underbrace{e^0}_{1}\cdot 2 = -\underbrace{e^0}_{1} + C$$

$$\begin{aligned} +2 &= -1 + C \\ +1 &\quad\; +1 \\ \hline 3 &= C \end{aligned}$$

$$e^{kt}\cdot y = -e^{kt} + 3$$

$y(1)=1$
$t=1$
$y=1$

$$e^{k\cdot 1}\cdot 1 = -e^{k\cdot 1} + 3$$

$$\begin{aligned} e^k &= -e^k + 3 \\ +e^k &\quad +e^k \end{aligned}$$

$$\frac{\cancel{2}e^k}{\cancel{2}} = \frac{3}{2}$$

$$\cancel{\ln}\left(e^k\right) = \ln\left(\frac{3}{2}\right)$$

$$\boxed{k = \ln\left(\frac{3}{2}\right)}$$

$$e^{\ln(3/2)\cdot t}\cdot y = -e^{\ln(3/2)t} + 3$$

$\begin{cases} \text{plug in } t=2 \\ \text{Find } y \end{cases}$

$$e^{\ln(3/2)2}\cdot y = -e^{\ln(3/2)2} + 3$$

Find y

$$\left(\cancel{e}^{\cancel{\ln}(3/2)}\right)^2$$

$$\left(\frac{3}{2}\right)^2$$

$$e^{\ln(3/2)\cdot 2} \neq 3$$

$$\frac{9}{4}\cdot y = \boxed{-\frac{9}{4}+3} \rightarrow \frac{12}{4}$$

$$\cancel{\frac{4}{9}}\,\cancel{\frac{9}{4}}\, y = \frac{3}{4}\cdot\cancel{\frac{4}{9}}_{3}$$

Answer

$$y = 1/3$$

$$e^{k\cdot t} \neq e^{k^t}$$

$$=(e^k)^t$$

In Section **7.2** we looked at mixing problems in which the volume of fluid remained constant and saw that such problems give rise to separable equations.

If the rates of flow into and out of the system are different, then the volume is not constant and the resulting differential equation is linear but not separable.

pure NO SALT

A tank contains 100 L of water. A solution with a salt concentration of 0.4 kg/L is added at a rate of 5 L/min. The solution is kept mixed and is drained from the tank at a rate of 3 L/min. If $y(t)$ is the amount of salt (in kilograms) after t minutes, find $y(t)$

$y(0)=0$

t=0
y=0

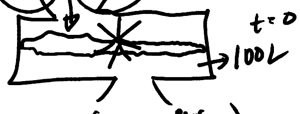
$$0.4 = \frac{4}{10} = \frac{2}{5}\ \frac{\text{kg}}{\text{L}}$$

$$\frac{dy}{dt} = \text{rate salt in} - \text{rate salt out}$$

$$\boxed{\frac{dy}{dt} = 2 - \frac{3y}{100+2t}}$$

$$\boxed{y(t) = \text{amount of salt (kg) in the tank @ time } t}$$

$\frac{2\text{ kg}}{5\text{ L}}$ rate 5 L/min



How much water in tank

t=0	100 L
t=1	102 L
t=2	104 L
⋮	
t	(100 +2t)

(conc.)(pour rate) (conc.)(pour rate)

$$\frac{dy}{dt} = \underbrace{\frac{2\text{ kg}}{\cancel{5}\cancel{\text{L}}}\cdot\frac{\cancel{5}\cancel{\text{L}}}{\text{min}}}_{\text{rate at which salt in}} - \underbrace{\left(\frac{y\ \text{kg}}{100+2t}\right)\frac{3\text{ L}}{\text{min}}}_{\text{rate at which salt is coming out}}$$

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9.2 Linear Diff. Eq.

$$\frac{dy}{dt} = 2 - \frac{3y}{100+2t}$$

~~#2~~

$$\frac{dy}{dt} + \underbrace{\left(\frac{3}{100+2t}\right)}_{P(t)} \cdot y = 2$$

$$\mu = e^{\int P(t)dt} = e^{\textcircled{\frac{3}{2}}\ln(100+2t)}$$

$$= e^{\cancel{\ln}(100+2t)^{3/2}}$$

① $\boxed{\mu = (100+2t)^{3/2}}$

$$\int \frac{3}{100+2t}\,dt$$

$u = 100+2t$

$du = 2dt$

$\frac{1}{2}du = dt$

$\frac{1}{2}\int \frac{3}{u}du$

$\frac{3}{2}\ln u$

$$\left(e^{\ln(100+2t)}\right)^3$$

$$e^{\ln(100+2t)^{3/2}}$$

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9.2 Linear Diff. Eq.

② mult. by $\mu = (100+2t)^{3/2}$

$$(100+2t)^{3/2}\left[\frac{dy}{dt} + \left(\frac{3}{100+2t}\right)y = 2\right]$$

$$(100+2t)^{3/2} \cdot \frac{3}{(100+2t)^1}$$

$$\underbrace{\overbrace{(100+2t)^{3/2}}^{1st} \cdot \overset{(2nd)'\,+}{\frac{dy}{dt}} + \overbrace{3(100+2t)^{1/2}}^{(1st)'} \overset{\cdot\, 2nd}{y}} = 2(100+2t)^{3/2}$$

③ $$\cancel{\int \frac{d}{dt}}\left(\underbrace{(100+2t)^{3/2}}_{1st} \cdot \underbrace{y}_{2nd}\right)dt = \int \underline{2}(100+2t)^{3/2}\underline{dt}$$

$$(100+2t)^{3/2} \cdot y = \frac{2}{5}(100+2t)^{5/2} + C$$

$u = 100+2t$

$du = \underline{2dt}$

$\int u^{3/2}\,\textcircled{du}$

$\frac{2u^{5/2}}{5}$

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9.2 Linear Diff. Eq.

$$(100+2t)^{3/2} \cdot y = \frac{2}{5}(100+2t)^{5/2} + C$$

$$(100)^{3/2} \cdot 0 = \frac{2}{5}(100)^{5/2} + C$$

$$0 = \frac{2}{5}(100)^{5/2} + C$$

$$C = -\frac{2}{5}(100)^{5/2}$$

$$= -\frac{2}{\cancel{5}}\left(\overset{20,000}{\cancel{100,000}}\right)$$

$$C = \boxed{-40,000}$$

$$\frac{\cancel{(100+2t)^{3/2}} \cdot y}{\cancel{(100+2t)^{3/2}}} = \frac{\frac{2}{5}(100+2t)^{5/2}}{(100+2t)^{3/2}} - \frac{40,000}{(100+2t)^{3/2}}$$

$$\boxed{\begin{array}{l} t=0 \\ y=0 \\ \text{Find } C \end{array}}$$

$$\left(100^{1/2}\right)^5$$

$$10^5$$

$$100,000$$

$$\boxed{y(t) = \frac{2}{5}(100+2t) - \frac{40,000}{(100+2t)^{3/2}}}$$