9.1 Introduction to Differential Eq.

A **differential equation** is an equation that involves a derivative

As soon as you learned how to find the anti-derivative, you were solving your first differential equation.

$$\int \left(x^2 - 3x + 4\right) dx$$

This is asking the question :

What function has $x^2 - 3x + 4$ as its derivative?

$$\textbf{Diff. eq.}: \ \frac{dy}{dx} = x^2 - 3x + 4. \ \text{Find } y(x).$$

$$\textbf{Answer}: y(x) = \frac{x^3}{3} - \frac{3x^2}{2} + 4x + C$$

This is called the **general solution**.

When given an **initial condition**: $y(0) = 7$

You can find the **particular solution**.

$$y(x) = \frac{x^3}{3} - \frac{3x^2}{2} + 4x + 7$$

So you have been solving differential equations for quite some time now.

These have all been of the form:

$$\frac{dy}{dx} = f(x)$$

A more difficult differential equation is:

$$\frac{dy}{dx} = f(x, y)$$

This is called a **first order differential equation**

This is asking you to find the function $y(x)$
that has a derivative that is the right hand side
(a function of the input variable x and the function itself)

$\frac{dy}{dx} = y^2 x$	$\frac{dy}{dx} = \frac{3x^2 y^3 - 6x^2}{y^2}$	$\frac{dy}{dx} = y - x$	$\frac{dy}{dx} = \frac{y + 2x\ln x}{x}$
Separable	Separable	Linear	Linear

These are examples where we can actually find an explicit formula for the solution.

In Math 104, we learn 2 techniques to solve first order differential equations:

- **Separable** Differential Equations — Covered in section 7.2
- **Linear** Differential Equations — Covered in section 9.2

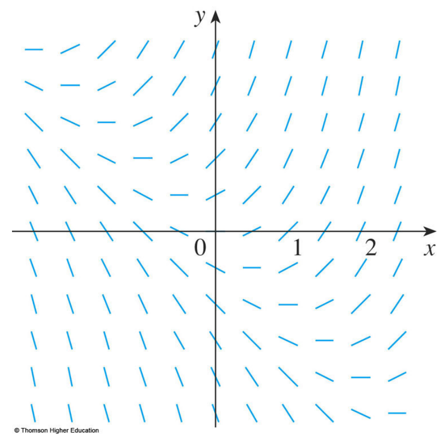
Geometric view of a differential equation

Math 104 – Rimmer
9.1 Intro to Diff. Eq.

$\frac{dy}{dx} = f(x, y)$ at each point (x, y), the solution curve should have a slope of $f(x, y)$

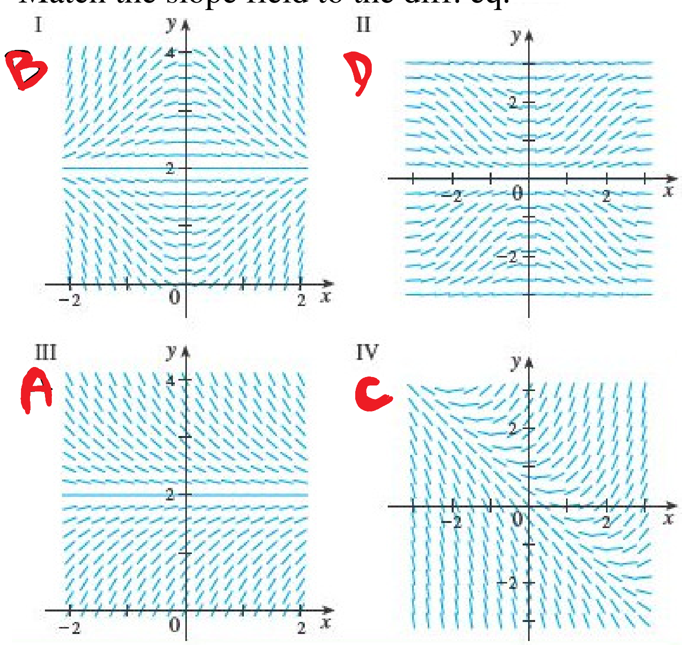
If we were to draw in lines that represent lines with the same slope as the value $f(x, y)$ for each point (x, y) we have a **slope field**.

Example: $\frac{dy}{dx} = x + y$



Match the slope field to the diff. eq.

Math 104 – Rimmer
9.1 Intro to Diff. Eq.



A. $y' = 2 - y$

no x ⇒ y' same for all x for a fixed y

B. $y' = x(2 - y)$

y' = 0 @ origin
y' = 0 for y = 2

C. $y' = x + y - 1$

y = -x ⇒ y' = -1

D. $y' = \sin x \sin y$

Periodic in x and y

When you can't find the solution to a differential equation, you can still study it using a numerical approximation called **Euler's Method**.

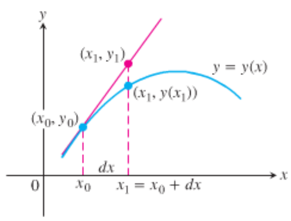


FIGURE 9.5 The first Euler step approximates $y(x_1)$ with $y_1 = L(x_1)$.

$$y_1 = y_0 + f\left(x_0, y_0\right)dx$$

$$y_2 = y_1 + f\left(x_1, y_1\right)dx$$

$$y_3 = y_2 + f\left(x_2, y_2\right)dx$$

continue...

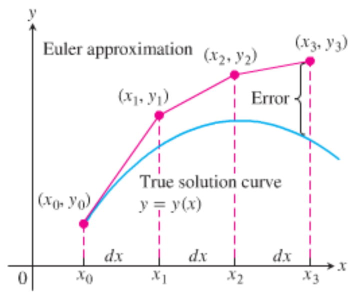


FIGURE 9.6 Three steps in the Euler approximation to the solution of the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$. As we take more steps, the errors involved usually accumulate, but not in the exaggerated way shown here.

TABLE 9.1 Euler solution of $y' = 1 + y$, $y(0) = 1$, step size $dx = 0.1$

x	y (Euler)	y (exact)	Error
0	1	1	0
0.1	1.2	1.2103	0.0103
0.2	1.42	1.4428	0.0228
0.3	1.662	1.6997	0.0377
0.4	1.9282	1.9836	0.0554
0.5	2.2210	2.2974	0.0764
0.6	2.5431	2.6442	0.1011
0.7	2.8974	3.0275	0.1301
0.8	3.2872	3.4511	0.1639
0.9	3.7159	3.9192	0.2033
1.0	4.1875	4.4366	0.2491

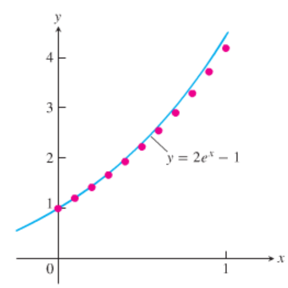


FIGURE 9.7 The graph of $y = 2e^x - 1$ superimposed on a scatterplot of the Euler approximations shown in Table 9.1