## 8.8 Probability Density Function



A **random variable**, usually written *X*, is a variable whose possible values are numerical outcomes of a random phenomenon.

A **continuous random variable** is one which takes an infinite number of possible values (usually measurements)

A **probability density function** is a function *f* defined for all real *x* and having the following properties:

1. $f(x) \geq 0$ for all $x$
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

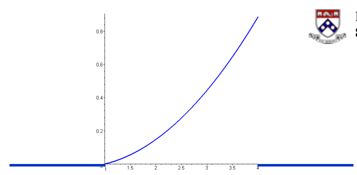
» Every continuous random variable, *X*, has a probability density function.

» Used to determine the probability that a continuous random variable lies between two values

$$\boxed{P(a \leq X \leq b) = \int_a^b f(x)\,dx}$$

Math 104 – Rimmer
8.8 Probability

$$f(x) = \begin{cases} \frac{2}{27}x(x-1) & 1 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$



a) Verify that *f* is a probability density function.

$f(x) \geq 0$ for all $x$

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{2}{27}\int_1^4 x(x-1)\,dx = \frac{2}{27}\int_1^4 (x^2 - x)\,dx = \frac{2}{27}\left(\frac{x^3}{3} - \frac{x^2}{2}\right)_1^4$$

$$= \frac{2}{27}\left(\left[\frac{64}{3} - 8\right] - \left[\frac{1}{3} - \frac{1}{2}\right]\right) = \frac{2}{27}\left(21 - 8 + \frac{1}{2}\right) = \frac{2}{27}\left(\frac{27}{2}\right) = \boxed{1}$$

b) Find $P(2 \leq x \leq 3)$.

$$P(2 \leq x \leq 3) = \frac{2}{27}\int_2^3 x(x-1)\,dx = \frac{2}{27}\left(\frac{x^3}{3} - \frac{x^2}{2}\right)_2^3$$

$$= \frac{2}{27}\left(\left[9 - \frac{9}{2}\right] - \left[\frac{8}{3} - 2\right]\right) = \frac{2}{27}\left(\frac{66 - 27 - 16}{6}\right)$$

$$= \boxed{\frac{23}{81} \approx 0.284}$$

**Mean** (average value) or expected value of a probability density function $f(x)$ is a measure of the center of a pdf.

$$\boxed{\mu = \int_{-\infty}^{\infty} x f(x)\,dx}$$

**Median** ( $m$ ) of a probability density function $f(x)$ is a number such that ½ the area under the graph of $f$ lies to the right of it (and half the area lies to the left of it).

The median $m$ solves the equation

$$\int_{m}^{\infty} f(x)\,dx = \frac{1}{2}$$

**Variance** $(\sigma^2)$ of a probability density function is a number that is used to measure the spread of a probability density function.

$$\boxed{\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx}$$

**Standard Deviation** $(\sigma)$ of a probability density function is a better measurement of spread because the units aren't squared.

$$\boxed{\sigma = \sqrt{\sigma^2}}$$

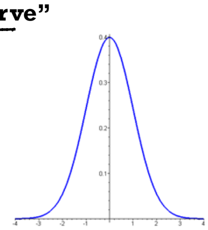


**Normal Density Function** **"bell curve"**

$$\boxed{f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}}$$

$\mu$ = mean

$\sigma$ = standard deviation



$\left.\begin{array}{l}\mu = 0 \\ \sigma = 1\end{array}\right\}$ standard normal

z-scores

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-(x)^2/2}$$

$$f(x)=\begin{cases} kx^2 & 2\le x\le 5 \\ 0 & \text{otherwise} \end{cases}$$

Find $k$ such that $f(x)$ is a probability density function.

$$f(x)\ge 0 \text{ for all } x \;\Rightarrow \boxed{k\ge 0}$$

$$\int_{-\infty}^{\infty} f(x)\,dx=1 \Rightarrow \int_2^5 kx^2\,dx=1$$

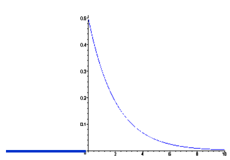
$$\Rightarrow k\left[\frac{x^3}{3}\right]_2^5=1 \Rightarrow k\left[\frac{125-8}{3}\right]=1$$

$$\Rightarrow k\left[\frac{117}{3}\right]=1 \Rightarrow 39k=1 \Rightarrow \boxed{k=\frac{1}{39}}$$



**Exponential Density Function**

$$f(x)=\begin{cases} 0 & x<0 \\ ke^{-kx} & x\ge 0 \end{cases},\; k>0$$



Examples of applications:

- » life span of electronic components
- » duration of telephone calls
- » waiting time in a doctor's office
- » time b/w successive flight arrivals and departures in an airport

$$\int_{-\infty}^{\infty} f(x)\,dx=\int_0^{\infty} ke^{-kx}\,dx=\lim_{b\to\infty}\int_0^b ke^{-kx}\,dx=\lim_{b\to\infty}\left(\cancel{k}\cdot\frac{-1}{\cancel{k}}e^{-kx}\right)_0^b$$

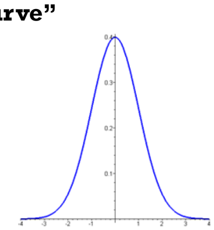
$$=\lim_{b\to\infty}\left(\frac{-1}{e^{kx}}\right)_0^b=\lim_{b\to\infty}\left(\frac{-1}{\underset{\nearrow 0}{e^{kb}}}-(-1)\right)=\boxed{1}$$

**Normal Density Function** **"bell curve"**

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/\left(2\sigma^2\right)}$$

$\mu$ = mean

$\sigma$ = standard deviation



**Mean** (average value) or expected value of a probability density function, a measure of the center of a pdf.

$$\boxed{\mu = \int_{-\infty}^{\infty} x f(x)\,dx}$$

$$f(x) = \begin{cases} \frac{2}{27} x(x-1) & 1 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

Find the mean.

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx = \frac{2}{27}\int_{1}^{4} x^2(x-1)\,dx = \frac{2}{27}\left(\tfrac{x^4}{4} - \tfrac{x^3}{3}\right)\Big|_1^4$$

$$= \frac{2}{27}\left[\left(64 - \tfrac{64}{3}\right) - \left(\tfrac{1}{4} - \tfrac{1}{3}\right)\right] = \frac{2}{27}\left[64 - 21 - \tfrac{1}{4}\right] = \tfrac{2}{27}\left[\tfrac{171}{4}\right] = \boxed{\frac{19}{6}}$$

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$$f(x) = \begin{cases} 0 & x < 0 \\ ke^{-kx} & x \ge 0 \end{cases}, \quad k > 0$$

Find the mean.

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{0}^{\infty} kxe^{-kx}\,dx = \lim_{b\to\infty}\int_{0}^{b} kxe^{-kx}\,dx$$

$$= \lim_{b\to\infty}\left[-xe^{-kx} - \tfrac{1}{k}e^{-kx}\right]_0^b$$

$$= \lim_{b\to\infty}\left[\left(\cancelto{0}{\frac{-b}{e^{kb}}} - \cancelto{0}{\frac{1}{ke^{kb}}}\right) - \left(0 - \frac{1}{k}\right)\right]$$

$$= \boxed{\frac{1}{k}}$$

$$\boxed{\begin{aligned} u &= kx \quad dv = e^{-kx} \\ du &= k\,dx \quad v = \tfrac{-1}{k}e^{-kx} \\ &= -xe^{-kx} + \int e^{-kx}\,dx \\ &= -xe^{-kx} - \tfrac{1}{k}e^{-kx} \end{aligned}}$$

$$\boxed{\begin{aligned} &\lim_{b\to\infty}\frac{-b}{e^{kb}} = \frac{\text{"}\infty\text{"}}{\infty} \\ &\overset{L'H}{=} \lim_{b\to\infty}\frac{-1}{ke^{kb}} = 0 \end{aligned}}$$



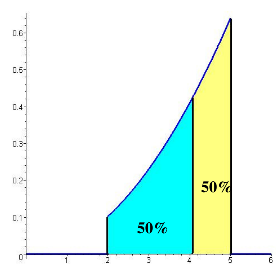
**Median** ( $m$ ) of a probability density function is a number such that ½ the area under the graph of $f$ lies to the right of it.

The median $m$ solves the equation

$$\int_{m}^{\infty} f(x)\,dx = \frac{1}{2}$$

$$f(x) = \begin{cases} \frac{1}{39}x^2 & 2 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$$

Find the median.




$$\int_{m}^{\infty}\frac{x^2}{39}\,dx = \frac{1}{2} \Rightarrow \int_{m}^{5}\frac{x^2}{39}\,dx = \frac{1}{2} \Rightarrow \left[\frac{x^3}{117}\right]_m^5 = \frac{1}{2} \Rightarrow \frac{125 - m^3}{117} = \frac{1}{2}$$

$$\Rightarrow 250 - 2m^3 = 117$$

$$\Rightarrow 2m^3 = 133$$

$$\Rightarrow \boxed{m = \sqrt[3]{\frac{133}{2}} \approx 4.05}$$

**Exponential Density Function**

$$f(x)=\begin{cases}0 & x<0\\ ke^{-kx} & x\geq 0\end{cases},\ k>0 \qquad \mu=\frac{1}{k}$$

Say $\mu$ is known

$$\Rightarrow k=\frac{1}{\mu}$$

We can now rewrite $f$ :

$$f(x)=\begin{cases}0 & x<0\\ \dfrac{1}{\mu}e^{-x/\mu} & x\geq 0\end{cases},\ \mu>0$$

The length of time spent waiting in line at a certain bank is modeled by an exponential density function with mean 8 minutes.
(a) What is the probability that a customer is served in the first 3 minutes?
(b) What is the probability that a customer has to wait more than 10 minutes?
(c) What is the median waiting time?

$$f(x)=\begin{cases}0 & x<0\\ \dfrac{1}{8}e^{-x/8} & x\geq 0\end{cases},$$

$$(a)\ P(X<3)=\int_{-\infty}^{3} f(x)\,dx=\frac{1}{8}\int_{0}^{3} e^{-x/8}dx=\left[\frac{1}{\cancel{8}}\cdot-\cancel{8}e^{-x/8}\right]_0^3=\left[-e^{-x/8}\right]_0^3=-e^{-3/8}+1$$

$$\approx 0.3127$$



The length of time spent waiting in line at a certain bank is modeled by an exponential density function with mean 8 minutes.
(a) What is the probability that a customer is served in the first 3 minutes?
(b) What is the probability that a customer has to wait more than 10 minutes?
(c) What is the median waiting time?

$$(b)\ P(X>10)=\int_{10}^{\infty} f(x)\,dx=\frac{1}{8}\lim_{b\to\infty}\int_{10}^{b} e^{-x/8}dx=\lim_{b\to\infty}\left[-e^{-x/8}\right]_{10}^{b}=\lim_{b\to\infty}\left[\frac{-1}{e^{x/8}}\right]_{10}^{b}$$

$$=\lim_{b\to\infty}\left[\frac{-1}{e^{b/8}}-\frac{-1}{e^{5/4}}\right]=\overset{0}{\frac{-1}{\lim_{b\to\infty} e^{b/8}}}+\frac{1}{e^{5/4}}=\frac{1}{e^{5/4}}\approx 0.2685$$

$$(c)\ \frac{1}{8}\int_{m}^{\infty} e^{-x/8}dx=\frac{1}{2}\ \Rightarrow\frac{1}{8}\lim_{b\to\infty}\int_{m}^{b} e^{-x/8}dx=\frac{1}{2}\Rightarrow\lim_{b\to\infty}\left[-e^{-x/8}\right]_m^b=\frac{1}{2}\Rightarrow\lim_{b\to\infty}\left[\frac{-1}{e^{x/8}}\right]_m^b=\frac{1}{2}$$

$$\Rightarrow\lim_{b\to\infty}\overset{0}{\frac{-1}{e^{b/8}}}-\frac{-1}{e^{m/8}}=\frac{1}{2}\ \Rightarrow\frac{1}{e^{m/8}}=\frac{1}{2}\ \Rightarrow e^{m/8}=2\Rightarrow\frac{m}{8}=\ln 2 \qquad m=8\ln 2$$

$$\approx 5.55\text{ min.}$$

The standard deviation for a random variable with probability density function $f$ and mean $\mu$ is defined by

$$\sigma = \left[\int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx\right]^{1/2}$$

Find the standard deviation for an exponential density function with mean $\mu$.

$$\int (x-\mu)^2 \frac{1}{\mu} e^{-x/\mu} dx = -(x-\mu)^2 e^{-x/\mu} - 2\mu(x-\mu)e^{-x/\mu} - 2\mu^2 e^{-x/\mu} + C$$

| *Diff* | | *Int* |
|---|---|---|
| $(x-\mu)^2$ | $+$ | $\frac{1}{\mu}e^{-x/\mu}$ |
| $2(x-\mu)$ | $-$ | $-e^{-x/\mu}$ |
| $2$ | $+$ | $\mu e^{-x/\mu}$ |
| $0$ | | $-\mu^2 e^{-x/\mu}$ |

$$\int (x-\mu)^2 \frac{1}{\mu} e^{-x/\mu} dx = \frac{-(x-\mu)^2}{e^{x/\mu}} - \frac{2\mu(x-\mu)}{e^{x/\mu}} - \frac{2\mu^2}{e^{x/\mu}} + C$$

Math 104 – Rimmer
8.8 Probability

$$\int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx = \lim_{b\to\infty}\int_0^b (x-\mu)^2 \frac{1}{\mu}e^{-x/\mu}dx = \lim_{b\to\infty}\left[\frac{-(x-\mu)^2}{e^{x/\mu}} - \frac{2\mu(x-\mu)}{e^{x/\mu}} - \frac{2\mu^2}{e^{x/\mu}}\right]_0^b$$

$$= \lim_{b\to\infty}\left[\frac{-(b-\mu)^2}{e^{b/\mu}} - \frac{2\mu(b-\mu)}{e^{b/\mu}} - \frac{2\mu^2}{e^{b/\mu}}\right] - \left[-(0-\mu)^2 - 2\mu(0-\mu) - 2\mu^2\right]$$

$$= \lim_{b\to\infty}\frac{-\left[(b-\mu)^2 + 2\mu(b-\mu) + 2\mu^2\right]}{e^{b/\mu}} - \left[-\mu^2 + 2\mu^2 - 2\mu^2\right]$$

$$= \lim_{b\to\infty}\frac{-\left[b^2 - \cancel{2b\mu} + \mu^2 + \cancel{2b\mu} - \cancel{2\mu^2} + \cancel{2\mu^2}\right]}{e^{b/\mu}} + \mu^2$$

$$= \lim_{b\to\infty}\frac{-\left[b^2+\mu^2\right]}{e^{b/\mu}} + \mu^2 \overset{L'H}{=} \lim_{b\to\infty}\frac{-2b}{\frac{1}{\mu}e^{b/\mu}} + \mu^2 \overset{L'H}{=} \lim_{b\to\infty}\cancelto{0}{\frac{-2}{\frac{1}{\mu^2}e^{b/\mu}}} + \mu^2 = \mu^2$$

$$\sigma = \sqrt{\int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx}$$

$$\boxed{\sigma = \mu}$$

**For the Exponential Density Function**

$$f(x)=\begin{cases}\frac{3}{32}\left(4-x^2\right) & -2\le x\le 2\\ 0 & \text{otherwise}\end{cases}$$

Find the standard deviation $\sigma$.

$$\mu=\int_{-2}^{2}\frac{3}{32}x(4-x^2)\,dx=\frac{3}{32}\int_{-2}^{2}\underbrace{(4x-x^3)}_{\text{this is an odd function}}dx=0$$

$$\sigma^2=\int_{-2}^{2}(x-0)^2\cdot\frac{3}{32}(4-x^2)\,dx=\frac{3}{32}\int_{-2}^{2}x^2(4-x^2)\,dx$$

$$\sigma^2=\frac{3}{32}\int_{-2}^{2}(\underbrace{4x^2-x^4}_{\text{even function}})\,dx=\frac{3}{32}\cdot 2\int_{0}^{2}(4x^2-x^4)\,dx=\frac{3}{16}\left(\frac{4x^3}{3}-\frac{x^5}{5}\right)\Bigg|_0^2$$

$$\sigma^2=\frac{3}{16}\left(\frac{32}{3}-\frac{32}{5}\right)=\frac{3}{16}\cdot 32\left(\frac{1}{3}-\frac{1}{5}\right)$$

$$6\cdot\left(\frac{5-3}{15}\right)=\cancel{6}^{\,2}\cdot\frac{2}{\cancel{15}_{\,5}}=\frac{4}{5}$$

$$\sigma^2=\frac{4}{5}\Rightarrow\boxed{\sigma=\frac{2}{\sqrt{5}}}$$