8.7 Improper Integrals

An integral can be called "improper" with one or any combination of the following:

Examples:

- **Infinite upper limit** $\displaystyle\int_1^{\infty} e^{-2x}dx = \lim_{t\to\infty}\int_1^{t} e^{-2x}dx$
- **Infinite lower limit** $\displaystyle\int_{-\infty}^{1} xe^{x}dx = \lim_{t\to-\infty}\int_t^{1} xe^{x}dx$
- **Infinite discontinuity at:**
  - **upper limit** $\displaystyle\int_0^{8}\frac{dx}{\sqrt[3]{8-x}} = \lim_{t\to 8^-}\int_0^{t}\frac{dx}{\sqrt[3]{8-x}}$
  - **lower limit** $\displaystyle\int_0^{9}\frac{dx}{\sqrt{x}} = \lim_{t\to 0^+}\int_t^{9}\frac{dx}{\sqrt{x}}$
  - **some value between the upper and lower limit** $\displaystyle\int_{-2}^{3}\frac{dx}{x^4} = \lim_{t\to 0^-}\int_{-2}^{t}\frac{dx}{x^4} + \lim_{t\to 0^+}\int_t^{3}\frac{dx}{x^4}$

If the limit exists, we say the integral converges and if it fails to exist (this includes infinite limits), we say the integral diverges.

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8.7 Improper Integrals

8.7 Improper Integrals

The skill for evaluating improper integrals relies on the skills of integration and evaluating limits.

2.6 Limits at Infinity

If $r > 0$ is a rational number, then $\displaystyle\lim_{x\to\infty}\frac{1}{x^r} = 0$

If $r > 0$ is a rational number such that x^r is defined for all x, then $\displaystyle\lim_{x\to-\infty}\frac{1}{x^r} = 0$

$f(x)$ is a rational function, with **deg. num. = deg. denom.** $\Rightarrow \displaystyle\lim_{x\to\pm\infty} f(x) = \frac{\textbf{coeff. of leading term in num.}}{\textbf{coeff. of leading term in denom.}}$

$f(x)$ is a rational function, with **deg. num. < deg. denom.** $\Rightarrow \displaystyle\lim_{x\to\pm\infty} f(x) = 0$

$f(x)$ is a rational function, with **deg. num. > deg. denom.** $\Rightarrow \displaystyle\lim_{x\to\pm\infty} f(x)$ does not exist (could be ∞ or $-\infty$)

8.7 Improper Integrals

The skill for evaluating improper integrals relies on the skills of integration and evaluating limits.

4.5 L'Hopital's Rule

$\lim_{x\to a}\frac{f(x)}{g(x)}$ when $f(x)\to 0$ and $g(x)\to 0$ as $x\to a$

$\lim_{x\to a}\frac{f(x)}{g(x)} = \text{"}\frac{0}{0}\text{"}$ ← this is called an indeterminate form

$\lim_{x\to a}\frac{f(x)}{g(x)}$ when $f(x)\to\pm\infty$ and $g(x)\to\pm\infty$ as $x\to a$

$\lim_{x\to a}\frac{f(x)}{g(x)} = \pm\text{"}\frac{\infty}{\infty}\text{"}$ ← this is called an indeterminate form

These two types of indeterminate forms can be simplified using **L'Hopital's Rule**

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} \leftarrow \text{assuming that this limit exists}$$

8.7 Improper Integrals

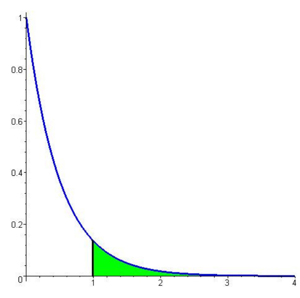
Infinite Upper Limit

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

$$\int_1^\infty e^{-2x}dx = \lim_{t\to\infty}\int_1^t e^{-2x}dx = \lim_{t\to\infty}\left[\frac{-1}{2}e^{-2x}\right]_1^t = \lim_{t\to\infty}\left[\frac{-1}{2e^{2x}}\right]_1^t$$

$$= \lim_{t\to\infty}\frac{-1}{2e^{2t}} + \frac{1}{2e^2}$$

$$= \boxed{\frac{1}{2e^2}} \quad \text{since } \lim_{t\to\infty}\frac{-1}{2e^{2t}} = 0$$



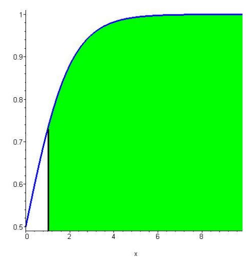
Infinite Upper Limit

$$\int_1^{\infty} \frac{e^x}{1+e^x}\,dx = \lim_{t\to\infty}\int_1^{t} \frac{e^x}{1+e^x}\,dx$$

$\boxed{\begin{array}{l} u = 1+e^x \\ du = e^x dx \end{array}}$ $\boxed{\int \frac{1}{u}\,du = \ln|u| + C}$

$$= \lim_{b\to\infty}\Big[\ln\left(1+e^x\right)\Big]_1^b$$

$$= \lim_{b\to\infty} \ln\left(1+e^b\right) - \ln\left(1+e\right) = \boxed{\infty}$$ **DIVERGENT**



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Infinite Lower Limit

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$

$\underline{D}$	$\underline{I}$
x (+)	e^x
1 (−)	e^x
0	e^x

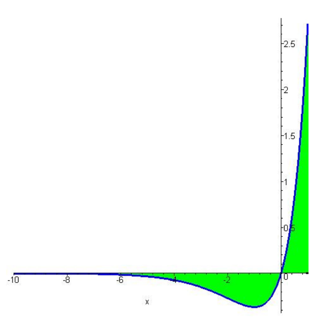
$$\int_{-\infty}^{1} xe^x\,dx = \lim_{a\to-\infty}\int_a^1 xe^x\,dx = \lim_{a\to-\infty}\Big[xe^x - e^x\Big]_a^1$$

$$= \lim_{a\to-\infty}\left(e-e\right) - \left(ae^a - e^a\right)$$

$$= \lim_{a\to-\infty} e^a\left(1-a\right) = 0\cdot\infty \text{ (indeterminate)}$$

$$= \lim_{a\to-\infty}\frac{1-a}{e^{-a}} = \frac{\infty}{\infty} \text{ (L'Hospital)}$$

$$\overset{L'H}{=} \lim_{a\to-\infty}\frac{-1}{-e^{-a}} = \boxed{0}$$



Infinite Upper and Lower Limit

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx \text{ ;} c \text{ any real number}$$

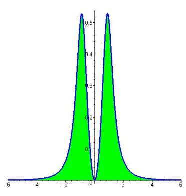
$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to-\infty}\int_{a}^{c} f(x)\,dx + \lim_{b\to\infty}\int_{c}^{b} f(x)\,dx$$

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^6}\,dx = \lim_{a\to-\infty}\int_{a}^{0} \frac{x^2}{1+x^6}\,dx + \lim_{b\to\infty}\int_{0}^{b} \frac{x^2}{1+x^6}\,dx$$

$u = x^3$

$du = 3x^2 dx \quad \frac{1}{3}du = x^2 dx$

$\frac{1}{3}\int \frac{1}{1+u^2}\,du = \frac{1}{3}\arctan u + C$



$$= \lim_{a\to-\infty}\left[\tfrac{1}{3}\arctan\left(x^3\right)\right]_a^0 + \lim_{b\to\infty}\left[\tfrac{1}{3}\arctan\left(x^3\right)\right]_0^b$$

$$= \lim_{a\to-\infty} -\tfrac{1}{3}\arctan\left(a^3\right) + \lim_{b\to\infty}\tfrac{1}{3}\arctan\left(b^3\right)$$

$$= -\tfrac{1}{3}\left(-\tfrac{\pi}{2}\right) + \tfrac{1}{3}\left(\tfrac{\pi}{2}\right) = \boxed{\frac{\pi}{3}}$$

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Infinite Discontinuity at Lower Limit

$$\int_{a}^{b} f(x)\,dx = \lim_{t\to a^+}\int_{t}^{b} f(x)\,dx$$

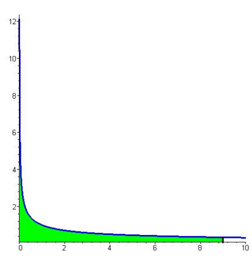
$f(a)\to$infinite discontinuity

$$\int_{0}^{9} \frac{dx}{\sqrt{x}} = \lim_{t\to0^+}\int_{t}^{9} \frac{dx}{\sqrt{x}} = \lim_{t\to0^+}\int_{t}^{9} x^{-1/2}\,dx = \lim_{t\to0^+}\left[2x^{1/2}\right]_t^9$$

$f(0)\to$infinite discontinuity

$$= \lim_{t\to0^+}\left[2\sqrt{x}\right]_t^9$$

$$= \lim_{t\to0^+} 6-2\sqrt{t} = \boxed{6}$$



Infinite Discontinuity at Upper Limit

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

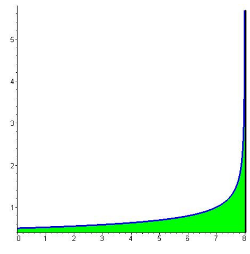
$f(b)\to$infinite discontinuity

$$\int_0^8 \frac{dx}{\sqrt[3]{8-x}} = \lim_{t\to 8^-}\int_0^t \frac{dx}{\sqrt[3]{8-x}} = \lim_{t\to 8^-}\int_0^t (8-x)^{-1/3}\,dx$$

$f(8)\to$infinite discontinuity

$u = 8-x$
$du = -dx$

$-\int u^{-1/3}du = \frac{-3}{2}u^{2/3} + C$



$$= \lim_{t\to 8^-}\left[\tfrac{-3}{2}(8-x)^{2/3}\right]_0^t$$

$$= \lim_{t\to 8^-}\tfrac{-3}{2}(8-t)^{2/3} + \tfrac{3}{2}(8)^{2/3}$$

$$= \tfrac{3}{2}\left[(8)^{1/3}\right]^2 \quad \text{since } \lim_{t\to 8^-}\tfrac{-3}{2}(8-t)^{2/3} = 0$$

$$= \boxed{6}$$

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Infinite Discontinuity inside the interval

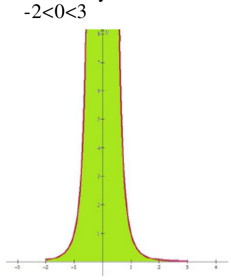
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \lim_{t\to c^-}\int_a^t f(x)\,dx + \lim_{t\to c^+}\int_t^b f(x)\,dx$$

$f(c)\to$infinite discontinuity
$a<c<b$

$$\int_{-2}^3 \frac{dx}{x^4} = \int_{-2}^0 \frac{dx}{x^4} + \int_0^3 \frac{dx}{x^4} = \lim_{t\to 0^-}\int_{-2}^t \frac{dx}{x^4} + \lim_{t\to 0^+}\int_t^3 \frac{dx}{x^4}$$

$f(0)\to$infinite discontinuity
-2<0<3

Both are **DIVERGENT**
(actually only need one of them to be divergent for the entire integral to be divergent)



$$= \lim_{t\to 0^-}\left[\frac{-1}{3x^3}\right]_{-2}^t + \lim_{t\to 0^+}\left[\frac{-1}{3x^3}\right]_t^3$$

$$= \underbrace{\lim_{t\to 0^-}\left[\frac{-1}{3t^3} - \frac{1}{24}\right]}_{\to\infty} + \underbrace{\lim_{t\to 0^+}\left[\frac{-1}{81} + \frac{1}{3t^3}\right]}_{\to\infty} = \boxed{\infty}$$

Doubly Improper

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$$\int_0^{\infty} f(x)\,dx = \int_0^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx = \lim_{a\to 0^+}\int_a^{c} f(x)\,dx + \lim_{b\to\infty}\int_c^{b} f(x)\,dx$$

$f(0)\to$infinite discontinuity

$$\int_0^{\infty} \frac{e^{-1/x}}{x^2}\,dx = \int_0^{1} \frac{e^{-1/x}}{x^2}\,dx + \int_1^{\infty} \frac{e^{-1/x}}{x^2}\,dx = \lim_{a\to 0^+}\int_a^{1} \frac{e^{-1/x}}{x^2}\,dx + \lim_{b\to\infty}\int_1^{b} \frac{e^{-1/x}}{x^2}\,dx$$

$f(0)\to$infinite discontinuity

$u = \frac{-1}{x}$

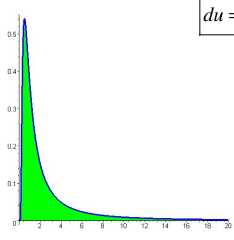
$du = \frac{1}{x^2}dx \quad \int e^u du = e^u + C$

$$= \lim_{a\to 0^+}\left[e^{-1/x}\right]_a^1 + \lim_{b\to\infty}\left[e^{-1/x}\right]_1^b$$

$$= \lim_{a\to 0^+}\left[\frac{1}{e^{1/x}}\right]_a^1 + \lim_{b\to\infty}\left[\frac{1}{e^{1/x}}\right]_1^b$$

$$= \left[\frac{1}{e} - \lim_{a\to 0^+}\left[\frac{1}{e^{1/a}}\right]\right] + \left[\lim_{b\to\infty}\left[\frac{1}{e^{1/b}}\right] - \frac{1}{e}\right]$$

$$= \frac{1}{e^{\frac{1}{\lim_{b\to\infty} b}}} - \frac{1}{e^{\frac{1}{\lim_{a\to 0^+} a}}} = 1 - 0 = \boxed{1}$$



Direct Comparison Theorem

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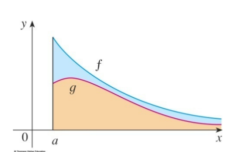
Suppose that $f(x)$ and $g(x)$ are continuous functions with $f(x) \ge g(x) \ge 0$ for $x \ge a$.

$a)$ If $\int_a^{\infty} f(x)\,dx$ is **convergent**, then $\int_a^{\infty} g(x)\,dx$ is **convergent**.

If your function is **smaller** than a function with a **finite** area, then your function will also have finite area



$b)$ If $\int_a^{\infty} g(x)\,dx$ is **divergent**, then $\int_a^{\infty} f(x)\,dx$ is **divergent**.

If your function is **larger** than a function with **infinite area**, then your function will also have infinite area.

Two examples worked on the next slides: $\int_1^{\infty} \frac{x}{x^3+1}\,dx \qquad \int_1^{\infty} \frac{2+e^{-x}}{x}\,dx$

Two examples worked out on my YouTube videos:

$\int_2^{\infty} \frac{x+1}{\sqrt{x^4 - x}}\,dx$ http://youtu.be/ZYMlAlFeDvc

$\int_0^{\infty} \frac{\arctan x}{2+e^x}\,dx$ http://youtu.be/FzXKyf0_b0c

$\int_1^\infty \frac{x}{x^3+1}\,dx$ The 1 won't matter as x gets big, so we can drop this term

$\frac{x}{x^3+1} \le \frac{x}{x^3} = \frac{1}{x^2}$ for $x \ge 1$ since $\frac{x}{x^3+1}$ has a larger denominator

$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b x^{-2}\,dx = \lim_{b\to\infty}\left[\frac{-1}{x}\right]_1^b = \lim_{b\to\infty}\frac{-1}{b}+1 = \boxed{1}$ **Convergent**

Your function is **smaller** than a function with a **finite** area.

So your function must have **finite** area.

Thus $\boxed{\int_1^\infty \frac{x}{x^3+1}\,dx \text{ is } \textbf{convergent}}$ by the Direct Comparison Theorem.

$\int_1^\infty \frac{2+e^{-x}}{x}\,dx$ e^{-x} gets small as x gets big, so we can drop this term

$\frac{2}{x} \le \frac{2+e^{-x}}{x}$ since $\frac{2+e^{-x}}{x}$ has a larger numerator

$\int_1^\infty \frac{2}{x}\,dx = \lim_{b\to\infty} 2\left[\ln x\right]_1^b = \lim_{b\to\infty} 2\left[\ln b\right] - 0 = \infty$ **Divergent**

Your function is **larger** than a function with a **infinite** area.

So your function must have **infinite** area.

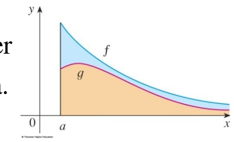
Thus $\boxed{\int_1^\infty \frac{2+e^{-x}}{x}\,dx \text{ is } \textbf{divergent}}$ by the Direct Comparison Theorem.

What do you do if the inequality goes the wrong direction?

$$\int_2^\infty \frac{x}{\sqrt{x^4+3x}}\,dx \qquad \frac{x}{\sqrt{x^4+3x}} \le \frac{x}{\sqrt{x^4}} = \frac{1}{x} \quad \text{and} \quad \int_2^\infty \frac{1}{x}\,dx \text{ diverges}$$

So your function is smaller than a function that has infinite area

For the direct comparison to work your function needs to be larger than the one with infinite area or smaller than one with finite area.



The good news is that you can still recover for some cases using:

Limit Comparison Theorem

Suppose that $f(x)$ and $g(x)$ are continuous positive functions for $x \ge a$,

and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$ with $0 < L < \infty$,

then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ behave alike.

(They either both converge or they both diverge.)

$\int_2^\infty \frac{x}{\sqrt{x^4+3x}}\,dx$ $\quad x^4$ will be so much larger than $3x$ as x gets big, so we can drop the $3x$ term

$$\frac{x}{\sqrt{x^4+3x}} \le \frac{x}{\sqrt{x^4}} = \frac{1}{x} \quad \text{and} \quad \int_2^\infty \frac{1}{x}\,dx \textbf{ diverges}$$

This is going in the wrong direction since your function is smaller than one that has infinite area.

Try the Limit Comparison Test.

$$\lim_{x\to\infty} \frac{\frac{x}{\sqrt{x^4+3x}}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x^2}{\sqrt{x^4+3x}} = \lim_{x\to\infty} \frac{x^2}{\sqrt{x^4+3x}} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{1}{\frac{\sqrt{x^4+3x}}{x^2}} = \lim_{x\to\infty} \frac{1}{\frac{\sqrt{x^4+3x}}{\sqrt{x^4}}} = \lim_{x\to\infty} \frac{1}{\sqrt{\frac{x^4+3x}{x^4}}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{3}{x^3}}} = 1$$

($\frac{3}{x^3} \to 0$)

So by the Limit Comparison Theorem, they behave alike.

Thus $\boxed{\int_2^\infty \frac{x}{\sqrt{x^4+3x}}\,dx \textbf{ diverges}}$ by the Limit Comparison Theorem.

$\int_1^\infty \frac{1}{\sqrt{e^x - x}}\,dx$ $\quad e^x$ will be so much larger than x as x gets big, so we can drop the x term

$$\frac{1}{\sqrt{e^x}} \le \frac{1}{\sqrt{e^x - x}} \qquad \frac{1}{\sqrt{e^x}} = e^{-x/2} \qquad \int_1^\infty e^{-x/2}dx = \lim_{b\to\infty}\left[\frac{-2}{e^{x/2}}\right]_1^b = \lim_{b\to\infty}\left[\frac{-2}{e^{b/2}} - \frac{-2}{e^{1/2}}\right] = \frac{2}{\sqrt{e}}$$

$\frac{-2}{e^{b/2}} \to 0$ as $b \to \infty$

So $\int_1^\infty e^{-x/2}dx$ **converges**.

This is going in the wrong direction since your function is bigger than one that has finite area.

Try the Limit Comparison Test.

$$\lim_{x\to\infty} \frac{\frac{1}{\sqrt{e^x - x}}}{\frac{1}{\sqrt{e^x}}} = \lim_{x\to\infty} \frac{\sqrt{e^x}}{\sqrt{e^x - x}} = \lim_{x\to\infty} \sqrt{\frac{e^x}{e^x - x}} = \sqrt{\lim_{x\to\infty} \frac{e^x}{e^x - x}}$$

$$= \sqrt{\lim_{x\to\infty} \frac{e^x}{e^x - x} \cdot \frac{\frac{1}{e^x}}{\frac{1}{e^x}}} = \sqrt{\lim_{x\to\infty} \frac{1}{1 - \frac{x}{e^x}}} = 1$$

$\frac{x}{e^x} \to 0$

Since $\lim_{x\to\infty} \frac{x}{e^x} = \text{"}\frac{\infty}{\infty}\text{"} \overset{L'H}{=} \lim_{x\to\infty} \frac{1}{e^x} = 0$

So by the Limit Comparison Theorem, they behave alike.

Thus $\boxed{\int_1^\infty \frac{1}{\sqrt{e^x - x}}\,dx \textbf{ converges}}$ by the Limit Comparison Theorem.