8.4 Partial Fraction Decomposition

Algebra Review:

The **degree** of a polynomial is the highest exponent on x

$$\left.\begin{matrix} x-4 \\ 2x+3 \end{matrix}\right\}\text{linear polynomials} \qquad \left.\begin{matrix} 2x^2-5x-12 \\ x^2-x+3 \end{matrix}\right\}\text{quadratic polynomials}$$

Polynomials that can be factored (over the reals) are called **reducible**.

Polynomials that **can't** be factored (over the reals) are called **irreducible**.

Fundamental Theorem of Algebra

Every polynomial of degree $n > 0$ with real coefficients can be written as a product of linear and/or irreducible quadratic factors.

How can you tell whether $ax^2 + bx + c$ is reducible?

$$b^2 - 4ac \geq 0 \Rightarrow \; ax^2 + bx + c \text{ is reducible}$$

$$b^2 - 4ac < 0 \Rightarrow \; ax^2 + bx + c \text{ is irreducible}$$

8.4 Partial Fraction Decomposition

Algebra Review:

Reducible

Example 1: $x^2 + 3x - 18 \qquad b^2 - 4ac = 9 + 72 = 81$

When $b^2 - 4ac > 0$ and is a perfect square, the polynomial should factor nicely because it will have rational roots.

$$x^2 + 3x - 18 = (x+6)(x-3)$$

Example 2: $x^2 - 4x + 4 \qquad b^2 - 4ac = 16 - 16 = 0$

When $b^2 - 4ac = 0$, the polynomial will have a double root.

$$x^2 - 4x + 4 = (x-2)(x-2) = (x-2)^2$$

Example 3: $x^2 + 2x - 10 \qquad b^2 - 4ac = 4 + 40 = 44$

When $b^2 - 4ac > 0$ but not a perfect square, the polynomial doesn't factor nicely because it will have irrational roots. These rarely show up in the context of partial fractions.

Irreducible

Example 4: $x^2 - 4x + 13 \qquad b^2 - 4ac = 16 - 52 = -36$

When $b^2 - 4ac < 0$, the polynomial does not factor because it will have imaginary roots.

8.4 Partial Fraction Decomposition

Rational Function : $\dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials

Goal of the Partial Fraction technique: To integrate rational functions.

⊙ Write $q(x)$ as a product of linear factors and irreducible quadratic factors.

⊙ Use algebra to express the rational function as a sum of simpler fractions.

⊙ The simpler fractions should be integrable without too much trouble.

Examples of simpler fractions that can be integrated quickly:

$$\frac{1}{x-4} \qquad \frac{1}{(x-4)^2} \qquad \frac{1}{x^2+4}$$

8.4 Partial Fraction Decomposition

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8.4 Partial Fraction Decomposition

$$\int \frac{1}{x-4}\,dx = \ln|x-4| + C$$

$u = x-4$, $du = dx$ $\qquad \int \frac{1}{u}\,du = \ln|u| + C$

$$\int \frac{1}{(x-4)^2}\,dx = \frac{-1}{x-4} + C$$

$u = x-4$, $du = dx$ $\qquad \int \frac{1}{u^2}\,du = \int u^{-2}\,du = \frac{-1}{u} + C$

$$\int \frac{1}{x^2+4}\,dx = \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$$

$$\frac{1}{x^2+4} = \frac{1}{4\left(\frac{x^2}{4}+1\right)} = \frac{1}{4}\cdot\frac{1}{\left(\frac{x}{2}\right)^2+1}$$

$$\int \frac{1}{x^2+4}\,dx = \frac{1}{4}\int \frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx$$

$u = \frac{x}{2}$, $du = \frac{1}{2}dx$ $\Rightarrow 2du = dx$

$$= \frac{1}{4}\int \frac{2}{u^2+1}\,du = \frac{1}{2}\arctan u + C$$

$$\boxed{\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C}$$

8.4 Partial Fraction Decomposition

1. The degree of the denominator **must** be greater than the degree of the numerator $\frac{p(x)}{q(x)}$
   If it is not, then **long divide** the denominator into the numerator.

2. Decompose the fraction in the following manner: $(A, B, C,$ and D are constants$)$

   i) $q(x)$ can be written as a product of only linear polynomials

   $$\frac{5x}{(x-4)(2x+3)} = \frac{A}{x-4} + \frac{B}{2x+3}$$

   ii) $q(x)$ can be written as a product involving powers of linear polynomials

   $$\frac{x^2+6x-4}{(x-3)^3(x+5)} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{(x-3)^3} + \frac{D}{x+5}$$

   iii) $q(x)$ can be written as a product involving irreducible quadratic polynomials

   $$\frac{16x-5}{(x^2+2x+10)(x-7)} = \frac{Ax+B}{x^2+2x+10} + \frac{C}{x-7}$$

3. Use algebra to find the constants and then integrate the simpler fractions.

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8.4 Partial Fraction Decomposition

$$\int_0^2 \frac{x-12}{x^2+3x-18}\,dx = \int_0^2 \frac{x-12}{(x+6)(x-3)}\,dx = \int_0^2 \frac{A}{x+6} + \frac{B}{x-3}\,dx$$

$$\frac{x-12}{(x+6)(x-3)} = \frac{A(x-3)}{(x+6)(x-3)} + \frac{B(x+6)}{(x-3)(x+6)}$$

$$\frac{x-12}{(x+6)(x-3)} = \frac{A(x-3)+B(x+6)}{(x+6)(x-3)}$$

denom. match

$$x-12 = \boxed{A(x-3)} + B(x+6) \quad \boxed{\text{true for all } x}$$

Choose $x=3$: $3-12 = B(9)$

$-9 = 9B \Rightarrow B = -1$

Choose $x=-6$: $-6-12 = -9A$

$-18 = -9A \Rightarrow A = +2$

$$= \int_0^2 \left(\frac{2}{x+6} - \frac{1}{x-3}\right) dx$$

$$= \left(2\ln|x+6| - \ln|x-3|\right)\Big|_0^2$$

$$= (2\ln 8 - \underbrace{\ln 1}_{0}) - (\underbrace{2\ln 6 - \ln 3}_{\ln 36 - \ln 3})$$

$\ln 36 - \ln 3$

$\ln\left(\frac{36}{3}\right)$

$= \ln 64 - \ln 12$

$$= \ln\left(\frac{64}{12}\right) = \boxed{\ln\frac{16}{3}}$$

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8.4 Partial Fraction Decomposition

$$\int \frac{2x+8}{x^3-4x^2+4x}\,dx = \int \frac{2x+8}{\underbrace{x(x^2-4x+4)}_{x(x-2)^2}}\,dx = \int \frac{A}{x}+\frac{B}{x-2}+\frac{C}{(x-2)^2}\,dx$$

$$= \int \frac{2}{x} + \frac{-2}{x-2} + \frac{6}{(x-2)^2}\,dx$$

$$= \boxed{2\ln|x| - 2\ln|x-2| + \frac{-6}{x-2} + C}$$

$$\frac{2x+8}{\boxed{x(x-2)^2}} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$2x+8 = A(x-2)^2 + Bx(x-2) + Cx$$

let $x=2$

$$12 = 2C \Rightarrow C=6$$

let $x=0$

$$8 = 4A \Rightarrow A=2$$

★ let $x=1$ (random)

$$10 = \underset{2}{A} - B + \underset{6}{C} \qquad 10 = 8 - B \qquad B = -2$$

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8.4 Partial Fraction Decomposition

$$\int \frac{6x^2-23x+58}{(x-2)\underbrace{(x^2-4x+13)}_{\text{irreducible}}}\,dx = \int\left(\frac{A}{x-2} + \frac{Bx+C}{x^2-4x+13}\right)dx$$

$$= \int\left(\frac{4}{x-2} + \frac{2x-3}{x^2-4x+13}\right)dx$$

$$6x^2-23x+58 = A(\underbrace{x^2-4x+13}_{2}) + (Bx+C)(x-2)$$

let $x=2$

$$\underbrace{24-46+58}_{36} = A(4-8+13)$$

$$36 = A\cdot 9 \qquad \boxed{A=4}$$

let $x=1$

$$6-23+58 = A(1-4+13)$$

$$41 = 10\underset{4}{A} + -1(B+C) \qquad 41 = 40 - (B+C)$$

$$1 = -(B+C) \Rightarrow B+C = -1$$

let $x=0$

$$58 = 13\underset{4}{A} - 2C \qquad 58 = 52 - 2C \qquad 6 = -2C \qquad C=-3 \qquad B-3=-1 \qquad B=2$$

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8.4 Partial Fraction Decomposition

$$\int\left(\frac{4}{x-2}+\frac{2x-3}{x^2-4x+13}\right)dx$$

$$\int\left(\frac{4}{x-2}+\frac{\overbrace{2x-3-1}^{2x-4}+1}{x^2-4x+13}\right)dx$$

$$\int\left(\frac{4}{x-2}+\frac{2x-4}{x^2-4x+13}+\frac{1}{x^2-4x+13}\right)$$

$$=4\ln|x-2|+\ln|x^2-4x+13|$$

$$=\boxed{4\ln|x-2|+\ln|x^2-4x+13|+\frac{1}{3}\arctan\left(\frac{x-2}{3}\right)+C}$$

$u=x^2-4x+13$

$du=(2x-4)\,dx$

$\int\frac{1}{u}du$

$=\ln|u|$

$x^2-4x+13$

$\underbrace{x^2-4x+4}_{(x-2)^2}\;\underbrace{+13-4}_{+9}$

$(x-2)^2+9$

$\int\frac{1}{(x-2)^2+9}dx$ $\quad u=x-2$, $du=dx$

$\int\frac{1}{u^2+9}du$

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8.4 Partial Fraction Decomposition

$$\int\frac{x^3-2x^2+18x-29}{x^2+16}dx$$

deg den < deg. num ⇒ Long Divide

$$\begin{array}{r} x-2 \\ x^2+16\;\overline{)\;x^3-2x^2+18x-29} \\ -(x^3+0x^2+16x) \\ \hline -2x^2+2x-29 \\ -(-2x^2+0x-32) \\ \hline 2x+3 \end{array}$$

$\int\frac{1}{x^2+a^2}dx$

$u=x^2+16$

$du=2x\,dx$

$\int\frac{1}{u}du$

$\ln|u|$

u subst. arctan formula

$$=\int\left(x-2+\frac{2x+3}{x^2+16}\right)dx \quad \int\frac{2x}{x^2+16}dx+\int\frac{3}{x^2+16}dx$$

$$=\frac{x^2}{2}-2x+\ln|x^2+16|+3\cdot\frac{1}{4}\arctan\left(\frac{x}{4}\right)+C$$