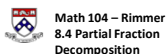


8.4 Partial Fraction Decomposition



Algebra Review:

The **degree** of a polynomial is the highest exponent on x

$$\left. \begin{array}{l} x-4 \\ 2x+3 \end{array} \right\} \text{linear polynomials} \qquad \left. \begin{array}{l} 2x^2-5x-12 \\ x^2-x+3 \end{array} \right\} \text{quadratic polynomials}$$

Polynomials that can be factored (over the reals) are called **reducible**.

Polynomials that **can't** be factored (over the reals) are called **irreducible**.

Fundamental Theorem of Algebra

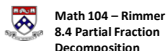
Every polynomial of degree $n > 0$ with real coefficients can be written as a product of linear and/or irreducible quadratic factors.

How can you tell whether $ax^2 + bx + c$ is reducible?

$$b^2 - 4ac \geq 0 \Rightarrow ax^2 + bx + c \text{ is reducible}$$

$$b^2 - 4ac < 0 \Rightarrow ax^2 + bx + c \text{ is irreducible}$$

8.4 Partial Fraction Decomposition



Algebra Review:

Reducible

Example 1: $x^2 + 3x - 18$ $b^2 - 4ac = 9 + 72 = 81$

When $b^2 - 4ac > 0$ and is a perfect square, the polynomial should factor nicely because it will have rational roots.

$$x^2 + 3x - 18 = (x + 6)(x - 3)$$

Example 2: $x^2 - 4x + 4$ $b^2 - 4ac = 16 - 16 = 0$

When $b^2 - 4ac = 0$, the polynomial will have a double root.

$$x^2 - 4x + 4 = (x - 2)(x - 2) = (x - 2)^2$$

Example 3: $x^2 + 2x - 10$ $b^2 - 4ac = 4 + 40 = 44$

When $b^2 - 4ac > 0$ but not a perfect square, the polynomial doesn't factor nicely because it will have irrational roots. These rarely show up in the context of partial fractions.

Irreducible

Example 4: $x^2 - 4x + 13$ $b^2 - 4ac = 16 - 52 = -36$

When $b^2 - 4ac < 0$, the polynomial does not factor because it will have imaginary roots.

8.4 Partial Fraction Decomposition

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Decomposition

Rational Function : $\frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials

Goal of the Partial Fraction technique: To integrate rational functions.

- ⊙ Write $q(x)$ as a product of linear factors and irreducible quadratic factors.
- ⊙ Use algebra to express the rational function as a sum of simpler fractions.
- ⊙ The simpler fractions should be integrable without too much trouble.

Examples of simpler fractions that can be integrated quickly:

$$\frac{1}{x-4} \quad \frac{1}{(x-4)^2} \quad \frac{1}{x^2+4}$$

8.4 Partial Fraction Decomposition

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$$\int \frac{1}{x-4} dx = \ln|x-4| + C$$

$$u = x-4 \quad \frac{du}{dx} = 1 \quad \int \frac{1}{u} du = \ln|u| + C$$

$$\int \frac{1}{(x-4)^2} dx = \frac{-1}{x-4} + C$$

$$u = x-4 \quad \frac{du}{dx} = 1 \quad \int \frac{1}{u^2} du = \int u^{-2} du = \frac{-1}{u} + C$$

$$\int \frac{1}{x^2+4} dx = \frac{1}{2} \arctan\left(\frac{x}{2}\right) + C$$

$$\frac{1}{x^2+4} = \frac{1}{4\left(\frac{x^2}{4}+1\right)} = \frac{1}{4} \cdot \frac{1}{\left(\frac{x}{2}\right)^2+1}$$

$$\int \frac{1}{x^2+4} dx = \frac{1}{4} \int \frac{1}{\left(\frac{x}{2}\right)^2+1} dx \quad \begin{matrix} u = \frac{x}{2} \\ du = \frac{1}{2} dx \end{matrix} \Rightarrow 2du = dx$$

$$\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$

$$= \frac{1}{4} \int \frac{2}{u^2+1} du = \frac{1}{2} \arctan u + C$$

8.4 Partial Fraction Decomposition

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Decomposition

- The degree of the denominator **must** be greater than the degree of the numerator $\frac{p(x)}{q(x)}$.
If it is not, then **long divide** the denominator into the numerator.
- Decompose the fraction in the following manner: (A, B, C , and D are constants)
 - $q(x)$ can be written as a product of **only linear** polynomials

$$\frac{5x}{(x-4)(2x+3)} = \frac{A}{x-4} + \frac{B}{2x+3}$$
 - $q(x)$ can be written as a product involving **powers of linear** polynomials

$$\frac{x^2+6x-4}{(x-3)^3(x+5)} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{(x-3)^3} + \frac{D}{x+5}$$
 - $q(x)$ can be written as a product involving **irreducible quadratic** polynomials

$$\frac{16x-5}{(x^2+2x+10)(x-7)} = \frac{Ax+B}{x^2+2x+10} + \frac{C}{x-7}$$
- Use algebra to find the constants and then integrate the simpler fractions.

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Decomposition

$$\int_0^2 \frac{x-12}{x^2+3x-18} dx = \int_0^2 \frac{x-12}{(x+6)(x-3)} dx = \int_0^2 \frac{A}{x+6} + \frac{B}{x-3} dx$$

$$= \int_0^2 \left(\frac{2}{x+6} - \frac{1}{x-3} \right) dx$$

$$= \left(2 \ln|x+6| - \ln|x-3| \right) \Big|_0^2$$

$$= (2 \ln 8 - \ln 1) - (2 \ln 6 - \ln 3)$$

$$= \ln 8^2 - \ln 1 - \ln 6^2 + \ln 3$$

$$= \ln \left(\frac{64}{36} \right) = \ln \frac{16}{9}$$

denom. match

$$\frac{x-12}{(x+6)(x-3)} = \frac{A(x-3)}{(x+6)(x-3)} + \frac{B(x+6)}{(x-3)(x+6)}$$

$$\frac{x-12}{(x+6)(x-3)} = \frac{A(x-3) + B(x+6)}{(x+6)(x-3)}$$

denom. match

$$x-12 = A(x-3) + B(x+6)$$

true for all x

choose $x=3$: $3-12 = B(9)$
 $-9 = 9B \Rightarrow B = -1$

choose $x=6$: $6-12 = -9A$
 $-6 = -9A \Rightarrow A = \frac{2}{3}$

$$\int \frac{2x+8}{x^3-4x^2+4x} dx = \int \frac{2x+8}{x(x^2-4x+4)} dx = \int \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2} dx$$

$$= \int \frac{2}{x} + \frac{-2}{x-2} + \frac{6}{(x-2)^2} dx$$

$$\frac{2x+8}{x(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$2x+8 = A(x-2)^2 + Bx(x-2) + Cx$$

$$= 2 \ln|x| - 2 \ln|x-2| + \frac{6}{x-2} + C$$

let x=2
 $12 = 2C \Rightarrow C=6$

let x=0
 $8 = 4A \Rightarrow A=2$

* let x=1 (random)
 $10 = \frac{A}{2} - \frac{B}{6} + C$ $10 = 8 - B$
 $B = -2$

$$\int \frac{6x^2-23x+58}{(x-2)(x^2-4x+13)} dx = \int \left(\frac{A}{x-2} + \frac{Bx+C}{x^2-4x+13} \right) dx$$

irreducible

$$= \int \left(\frac{4}{x-2} + \frac{2x-3}{x^2-4x+13} \right) dx$$

$$6x^2-23x+58 = A(x^2-4x+13) + (Bx+C)(x-2)$$

let x=2
 $24-46+58 = A(4-8+13)$
 $36 = A \cdot 9$ $A=4$

let x=1
 $6-23+58 = A(1-4+13)$
 $41 = 10A + -1(B+C)$
 $41 = 40 - (B+C)$
 $1 = -(B+C) \Rightarrow B+C = -1$

let x=0
 $58 = 13A - 2C$ $58 = 52 - 2C$ $6 = -2C$ $C = -3$
 $B = -1 - C = -1 - (-3) = 2$

$$\int \left(\frac{4}{x-2} + \frac{2x-3}{x^2-4x+13} \right) dx$$

$$\int \left(\frac{4}{x-2} + \frac{\overbrace{2x-3}^{2x-4} + 1}{x^2-4x+13} \right) dx$$

$$\int \left(\frac{4}{x-2} + \frac{2x-4}{x^2-4x+13} + \frac{1}{x^2-4x+13} \right)$$

$$\downarrow$$

$$= 4 \ln|x-2| + \ln|x^2-4x+13|$$

$$= 4 \ln|x-2| + \ln|x^2-4x+13| + \frac{1}{3} \arctan\left(\frac{x-2}{3}\right) + C$$

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Decomposition

$$u = x^2 - 4x + 13$$

$$du = (2x - 4) dx$$

$$\int \frac{1}{u} du = \ln|u|$$

$$x^2 - 4x + 13$$

$$\underbrace{x^2 - 4x + 4}_{(x-2)^2} + \underbrace{13 - 4}_{9}$$

$$(x-2)^2 + 9$$

$$\int \frac{1}{(x-2)^2 + 9} dx \quad (u = x-2)$$

$$\int \frac{1}{u^2 + 9} du \quad du = dx$$

$$\int \frac{x^3 - 2x^2 + 18x - 29}{x^2 + 16} dx$$

deg. den < deg. num \Rightarrow Long Divide

$$\begin{array}{r} x-2 \\ x^2+16 \overline{) x^3-2x^2+18x-29} \\ \underline{-(x^3+0x^2+16x)} \\ -2x^2+2x-29 \\ \underline{-(-2x^2+0x-32)} \\ 2x+3 \end{array}$$

$$= \int \left(x-2 + \frac{2x+3}{x^2+16} \right) dx$$

$$= \frac{x^2}{2} - 2x + \ln|x^2+16| + 3 \cdot \frac{1}{4} \arctan\left(\frac{x}{4}\right) + C$$

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8.4 Partial Fraction
Decomposition

$$\int \frac{1}{x^2+a^2} dx$$

$$u = x^2 + 16$$

$$du = 2x dx$$

$$\int \frac{1}{u} du = \ln|u|$$

$$\int \frac{2x}{x^2+16} dx + \int \frac{3}{x^2+16} dx$$

$\xrightarrow{u \text{ subst.}}$ $\xrightarrow{\text{arctan formula}}$