8.3 Trig. Substitution

Math 104 – Rimmer
8.3 Trig. Substitution

1. $\sqrt{a^2 - x^2}$

Let $x = a\sin\theta$

Assume $a > 0$.
$-a \le x \le a$

$x = a\sin\theta \Rightarrow \theta = \arcsin\left(\frac{x}{a}\right)$

$\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$

Quadrant 1

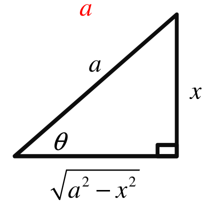
Quadrant 4 but with θ from $\frac{-\pi}{2}$ to 0.

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - (a\sin\theta)^2}$$
$$= \sqrt{a^2 - a^2\sin^2\theta}$$
$$= \sqrt{a^2\left(1 - \sin^2\theta\right)}$$
$$= \sqrt{a^2\cos^2\theta}$$
$$= \left|a\cos\theta\right|$$

$a > 0$ and $\cos\theta \ge 0$ for $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$ so we can drop the absolute value sign

$$\sqrt{a^2 - x^2} = a\cos\theta$$

$x = a\sin\theta \Rightarrow \frac{x}{a} = \sin\theta$



Reference Triangle

8.3 Trig. Substitution

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2. $\sqrt{a^2 + x^2}$ or $\sqrt{x^2 + a^2}$

Let $x = a\tan\theta$

Assume $a > 0$.
$-\infty < x < \infty$

$x = a\tan\theta \Rightarrow \theta = \arctan\left(\frac{x}{a}\right)$

$\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$

Quadrant 1

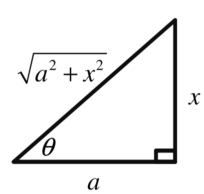
Quadrant 4 but with θ from $\frac{-\pi}{2}$ to 0.

$$\sqrt{a^2 + x^2} = \sqrt{a^2 + (a\tan\theta)^2}$$
$$= \sqrt{a^2 + a^2\tan^2\theta}$$
$$= \sqrt{a^2\left(1 + \tan^2\theta\right)}$$
$$= \sqrt{a^2\sec^2\theta}$$
$$= \left|a\sec\theta\right|$$

$a > 0$ and $\sec\theta \ge 0$ for $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$ so we can drop the absolute value sign

$$\sqrt{a^2 + x^2} = a\sec\theta$$

$x = a\tan\theta \Rightarrow \frac{x}{a} = \tan\theta$



Reference Triangle

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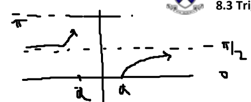
3. $\sqrt{x^2 - a^2}$

Let $x = a\sec\theta$

Assume $a > 0$.
$x < -a$ or $x > a$



$x = a\sec\theta \Rightarrow \theta = \operatorname{arcsec}\left(\frac{x}{a}\right)$

$0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$

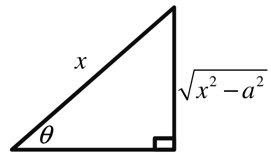
Quadrant 1 Quadrant 2.

$$\sqrt{x^2 - a^2} = \sqrt{(a\sec\theta)^2 - a^2}$$
$$= \sqrt{a^2\sec^2\theta - a^2}$$
$$= \sqrt{a^2(\sec^2\theta - 1)}$$
$$= \sqrt{a^2\tan^2\theta}$$
$$= |a\tan\theta|$$

$a > 0$ but t $\tan\theta \ge 0$ for $0 \le \theta < \frac{\pi}{2}$ and $\tan\theta \le 0$ for $\frac{\pi}{2} < \theta \le \pi$
so we can't drop the absolute value sign

$\sqrt{x^2 - a^2} = a\tan\theta$ for $x > a$
$\sqrt{x^2 - a^2} = -a\tan\theta$ for $x < -a$

$x = a\sec\theta \Rightarrow \frac{x}{a} = \sec\theta$



Reference Triangle

8.3 Trig. Substitution

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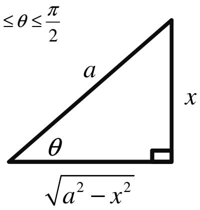
1. $\sqrt{a^2 - x^2}$ $\boxed{\text{Let } x = a\sin\theta}$

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - (a\sin\theta)^2}$$
$$= \sqrt{a^2 - a^2\sin^2\theta}$$
$$= \sqrt{a^2(1 - \sin^2\theta)}$$
$$= \sqrt{a^2\cos^2\theta}$$
$$\boxed{\sqrt{a^2 - x^2} = a\cos\theta}$$

$x = a\sin\theta \Rightarrow \frac{x}{a} = \sin\theta$

$\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$

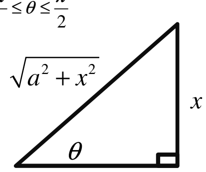


2. $\sqrt{a^2 + x^2}$ $\boxed{\text{Let } x = a\tan\theta}$

$$\sqrt{a^2 + x^2} = \sqrt{a^2 + (a\tan\theta)^2}$$
$$= \sqrt{a^2 + a^2\tan^2\theta}$$
$$= \sqrt{a^2(1 + \tan^2\theta)}$$
$$= \sqrt{a^2\sec^2\theta}$$
$$\boxed{\sqrt{a^2 + x^2} = a\sec\theta}$$

$x = a\tan\theta \Rightarrow \frac{x}{a} = \tan\theta$

$\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$

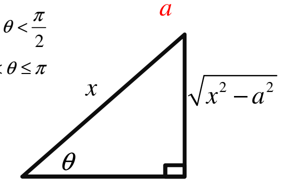


3. $\sqrt{x^2 - a^2}$ $\boxed{\text{Let } x = a\sec\theta}$

$$\sqrt{x^2 - a^2} = \sqrt{(a\sec\theta)^2 - a^2}$$
$$= \sqrt{a^2\sec^2\theta - a^2}$$
$$= \sqrt{a^2(\sec^2\theta - 1)}$$
$$= \sqrt{a^2\tan^2\theta}$$
$$\boxed{\begin{array}{ll} \sqrt{x^2 - a^2} = a\tan\theta & \text{for } x > a \\ \sqrt{x^2 - a^2} = -a\tan\theta & \text{for } x < -a \end{array}}$$

$x = a\sec\theta \Rightarrow \frac{x}{a} = \sec\theta$

$0 \le \theta < \frac{\pi}{2}$
$\frac{\pi}{2} < \theta \le \pi$



$$\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}} \qquad \int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}}$$

$x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta \qquad \sqrt{4-x^2} = 2\cos\theta$

$x^2 = 4\sin^2\theta \qquad \sqrt{4-4\sin^2\theta} = \sqrt{4(1-\sin^2\theta)} = \sqrt{4\cos^2\theta} = 2\cos\theta$

Limit Switching

$x = \sqrt{2} \Rightarrow \sqrt{2} = 2\sin\theta \quad \sin\theta = \frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{\pi}{4}$

$x = 1 \Rightarrow 1 = 2\sin\theta \quad \sin\theta = \frac{1}{2} \quad \Rightarrow \theta = \frac{\pi}{6}$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cancel{2\cos\theta}\, d\theta}{4\sin^2\theta \cdot \cancel{2\cos\theta}}$$

$$= \frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \csc^2\theta\, d\theta = -\frac{1}{4}\left[\cot\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= -\frac{1}{4}\left[\frac{\cos\theta}{\sin\theta}\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}} = -\frac{1}{4}\left(1 - \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right)$$

$$= \boxed{\frac{1}{4}\left(\sqrt{3}-1\right)}$$

Using Triangle

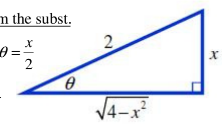
$$= \int \frac{\cancel{2\cos\theta}\, d\theta}{4\sin^2\theta \cdot \cancel{2\cos\theta}} \qquad = \frac{1}{4}\int \csc^2\theta\, d\theta$$

$$= -\frac{1}{4}\left[\cot\theta\right]$$

from the subst.

$\sin\theta = \frac{x}{2}$

$\cot\theta = \frac{adj.}{opp.}$



$$= -\frac{1}{4}\left[\frac{\sqrt{4-x^2}}{x}\right]_1^{\sqrt{2}} = -\frac{1}{4}\left[\frac{\sqrt{2}}{\sqrt{2}} - \frac{\sqrt{3}}{1}\right]$$

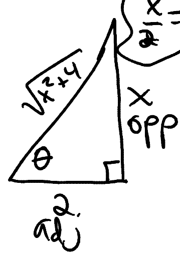
$$= -\frac{1}{4}\left[1-\sqrt{3}\right] = \boxed{\frac{1}{4}\left(\sqrt{3}-1\right)}$$

Spring 12 #10

$$\int_0^1 \frac{dx}{\left(x^2+4\right)^{3/2}}$$

$\sqrt{x^2+a^2} \to x = a\tan\theta$

$\frac{x}{2} = \tan\theta \Leftarrow \boxed{x = 2\tan\theta}$



$\boxed{\bigstar\ dx = 2\sec^2\theta\, d\theta}$

numerator

$(x^2+4)^{3/2} = \left(\sqrt{x^2+4}\right)^3 = (2\sec\theta)^3$

$\boxed{(x^2+4)^{3/2} = 8\sec^3\theta}$

denominator

$$\int \frac{\cancel{2}\cancel{\sec^2\theta}\, d\theta}{\cancel{8}\sec^{\cancel{3}}\theta}$$

$$= \frac{1}{4}\int \frac{1}{\sec\theta}\, d\theta$$

$$= \frac{1}{4}\int \cos\theta\, d\theta$$

$$= \frac{1}{4}\sin\theta \Big|_0^1$$

$$= \frac{1}{4}\left(\frac{x}{\sqrt{x^2+4}}\right)\Bigg|_0^1$$

$$= \frac{1}{4}\left(\frac{1}{\sqrt{5}} - 0\right) = \boxed{\frac{1}{4\sqrt{5}}}$$

$x = 1 \quad 1 = 2\tan\theta$

abandon limit switch $\frac{1}{2} = \tan\theta$?

$$\int \frac{5dx}{\sqrt{25x^2-9}} = \int \frac{5dx}{\sqrt{25\left(x^2-\frac{9}{25}\right)}} = \int \frac{\cancel{5}dx}{\cancel{5}\sqrt{x^2-\frac{9}{25}}} = \int \frac{dx}{\sqrt{x^2-\frac{9}{25}}}$$

$$= \int \frac{dx}{\sqrt{x^2-\frac{9}{25}}}$$

$$x = \frac{3}{5}\sec\theta$$

$$dx = \frac{3}{5}\sec\theta\tan\theta\, d\theta$$

$$\sqrt{x^2-\frac{9}{25}} = \frac{3}{5}\tan\theta$$

$$\sqrt{\tfrac{9}{25}\sec^2\theta-\tfrac{9}{25}} = \sqrt{\tfrac{9}{25}\left(\sec^2\theta-1\right)} = \sqrt{\tfrac{9}{25}\tan^2\theta} = \frac{3}{5}\tan\theta$$

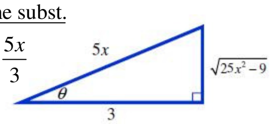
$$= \int \frac{\cancel{\tfrac{3}{5}}\sec\theta\,\cancel{\tan\theta}\,d\theta}{\cancel{\tfrac{3}{5}}\,\cancel{\tan\theta}} = \int \sec\theta\, d\theta$$

$$\boxed{\int \sec x\,dx = \ln\left|\sec x+\tan x\right| + C}$$

$$= \ln\left|\sec\theta+\tan\theta\right| + C$$

from the subst.

$$\sec\theta = \frac{5x}{3}$$



$$= \boxed{\ln\left|\frac{5x}{3}+\frac{\sqrt{25x^2-9}}{3}\right| + C}$$

$$\sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\int_1^2 \frac{dx}{\sqrt{4x-x^2}}$$

Complete the square

$$4x-x^2 = -x^2+4x$$

$$= -\left(x^2-4x+\underline{4}\right)+\underline{4}$$

$$= -\left(x-2\right)^2+\underline{4}$$

$$4x-x^2 = 4-\left(x-2\right)^2$$

$$= \int_1^2 \frac{dx}{\sqrt{4-\left(x-2\right)^2}}$$

$$u = x-2 \qquad x=2 \Rightarrow u=0$$

$$du = dx \qquad x=1 \Rightarrow u=-1$$

LIMIT SWITCH

$$= \int_{-1}^{0} \frac{du}{\sqrt{4-u^2}} \qquad \int_{-1}^{0} \frac{du}{\sqrt{4-u^2}}$$

$$u = 2\sin\theta$$

$$du = 2\cos\theta\, d\theta$$

$$\sqrt{4-u^2} = 2\cos\theta$$

$$\sqrt{4-4\sin^2\theta} = \sqrt{4\left(1-\sin^2\theta\right)} = \sqrt{4\cos^2\theta} = 2\cos\theta$$

$$= \int_{-\frac{\pi}{6}}^{0} \frac{\cancel{2\cos\theta}\,d\theta}{\cancel{2\cos\theta}}$$

$$u=0 \Rightarrow 0 = 2\sin\theta \quad \sin\theta = 0 \quad \Rightarrow \theta = 0$$

$$u=-1 \Rightarrow -1 = 2\sin\theta \quad \sin\theta = -\frac{1}{2} \quad \Rightarrow \theta = -\frac{\pi}{6}$$

$$= \int_{-\frac{\pi}{6}}^{0} d\theta = \left[\theta\right]_{-\frac{\pi}{6}}^{0} = \left(0-\left(-\frac{\pi}{6}\right)\right) = \boxed{\frac{\pi}{6}}$$

NO LIMIT SWITCH

$$\int_1^2 \frac{dx}{\sqrt{4-(x-2)^2}}$$

$u = x-2$
$du = dx$

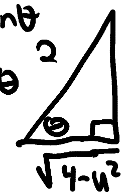
$$\int \frac{dx}{\sqrt{4-u^2}}$$

$u = 2\sin\theta$
$\boxed{du = 2\cos\theta\, d\theta}$
$\boxed{\sqrt{4-u^2} = 2\cos\theta}$

$$\int \frac{\cancel{2\cos\theta}\, d\theta}{\cancel{2\cos\theta}} = \int d\theta$$

$$= \theta = \arcsin\left(\frac{u}{2}\right)$$

$u = 2\sin\theta$
$\frac{\text{opp}}{\text{hyp}}\quad \frac{u}{2} = \sin\theta$



← actually didn't need this

$$= \arcsin\left(\frac{x-2}{2}\right)\Bigg|_1^2$$

$$= \arcsin(0) - \arcsin\left(\frac{-1}{2}\right)$$

$$= 0 - \left(-\frac{\pi}{6}\right)$$

$$= \boxed{\frac{\pi}{6}}$$

W/O Trig. Substitution

$$\int_{-1}^{0} \frac{1}{\sqrt{4-u^2}}\, du$$

$$\frac{1}{2}\int_{-1}^{0} \frac{1}{\sqrt{1-\left(\frac{u}{2}\right)^2}}\, du$$

$$\cancel{\frac{1}{2}}\cdot\cancel{2}\int_{-1/2}^{0} \frac{dw}{\sqrt{1-w^2}}$$

Algebra: Factor out 4

$$\sqrt{4\left(1-\frac{u^2}{4}\right)}$$

$$\sqrt{4\left(1-\left(\frac{u}{2}\right)^2\right)}$$

$$\frac{\sqrt{4}\,\sqrt{1-\left(\frac{u}{2}\right)^2}}{}$$

$2dw = du$
$w = \frac{u}{2}$
$u = 0 \quad w = 0$
$u = -1 \quad w = \frac{-1}{2}$
$2dw = \frac{1}{2}du \cdot 2$

$$= \left[\arcsin(w)\right]\Big|_{-1/2}^{0}$$

$$= \left(\arcsin(0) - \arcsin\left(-\tfrac{1}{2}\right)\right)$$

$$= 0 - \frac{-\pi}{6}$$

$$= \pi/6$$

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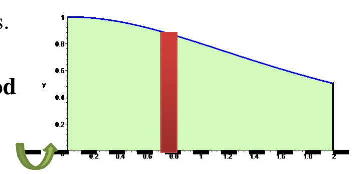
Find the volume of the solid generated by revolving the region bounded by the curves $y=\dfrac{4}{x^2+4}$, $y=0, x=0,$ and $x=2$ about the $x-$axis.



no gap b/w axis of rotation and the region $\Rightarrow$ **Disk Method**

$$\text{Radius: } r(x)=\frac{4}{x^2+4}$$

$$Volume=\pi\int_a^b\left[r(x)\right]^2dx$$

$$Volume=\pi\int_0^2\left[\frac{4}{x^2+4}\right]^2dx = 16\pi\int_0^2\frac{dx}{\left(x^2+4\right)^2}$$

$$16\pi\int_0^2\frac{dx}{\left(x^2+4\right)^2}$$

$$x=2\tan\theta$$

$$dx=2\sec^2\theta\,d\theta$$

$$x^2+4=4\tan^2\theta+4=4\left(1+\tan^2\theta\right)=4\sec^2\theta$$

$$\left(x^2+4\right)^2=16\sec^4\theta$$

LIMIT SWITCH

$x=2\Rightarrow 2=2\tan\theta$	$x=0\Rightarrow 0=2\tan\theta$
$\tan\theta=1$	$\tan\theta=0$
$\Rightarrow\theta=\dfrac{\pi}{4}$	$\Rightarrow\theta=0$

$$=\cancel{16}\pi\int_0^{\frac{\pi}{4}}\frac{\cancel{2\sec^2\theta\,d\theta}}{\cancel{16\sec^4\theta}\ \sec^2\theta}=2\pi\int_0^{\frac{\pi}{4}}\cos^2\theta\,d\theta$$

$$=2\pi\int_0^{\frac{\pi}{4}}\frac{1}{2}\left(1+\cos2\theta\right)d\theta=\pi\int_0^{\frac{\pi}{4}}\left(1+\cos2\theta\right)d\theta$$

$$=\pi\left[\theta+\tfrac{1}{2}\sin(2\theta)\right]_0^{\frac{\pi}{4}}=\pi\left[\left(\frac{\pi}{4}+\tfrac{1}{2}\sin\left(\frac{\pi}{2}\right)\right)-0\right]=\boxed{\pi\left(\frac{\pi}{4}+\frac{1}{2}\right)}$$