8.2 Integrating Powers of Trig. Functions

1. $\int \cos^m x \sin^n x \, dx \quad (m, n \text{ positive integers})$

2. $\int \tan^m x \sec^n x \, dx \quad (m, n \text{ positive integers})$

3. $\int \sin(mx)\sin(nx)\,dx \qquad \int \cos(mx)\cos(nx)\,dx \qquad \int \sin(mx)\cos(nx)\,dx$

$(m, n \text{ rational with } m \neq n)$

1. $\int \cos^m x \sin^n x \, dx \quad (m, n \text{ positive integers})$

A) m, n: **one (or both) odd** (greater than 1)

1. Factor out one power from the trig. function that has the odd power,
if both have the odd power, just pick one of them and factor out one power.

2. Use $\cos^2 x + \sin^2 x = 1$ to transform the remaining even power
of the above trig function into the other trig. function

3. Use u-substitution to finish the problem (let $u =$ "other" trig function)

B) m, n: **both even**

Replace all even powers using the half-angle identities:
$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

C) m, n: **one or both $= 1$**

Use u-substitution. Let $u =$ the trig function with power $\neq 1$
If both $= 1$, choose either one to be u

2. $\int \tan^m x \sec^n x \, dx \quad (m, n \text{ positive integers})$

A) m (the power of $\tan x$): **odd**

1. Factor out one power of $\sec x$ and one power of $\tan x$
2. Use $\tan^2 x = \sec^2 x - 1$ to transform the remaining even power of $\tan x$ to be in terms of $\sec x$.
3. Use u-substitution to finish the problem (let $u = \sec x$)

B) n (the power of $\sec x$): **even**

1. Factor out $\sec^2 x$
2. If $n > 2$, use $\sec^2 x = 1 + \tan^2 x$ to transform the remaining even power of $\sec x$ to be in terms of $\tan x$
3. Use u-substitution to finish the problem (let $u = \tan x$)

C) Both m **odd** and n **even** : Pick either of the above methods.

D) If any other combination, then there is no set method.

3. $\int \sin(mx)\sin(nx)\,dx$

$$\int \cos(mx)\cos(nx)\,dx \qquad (m, n \text{ rational with } m \neq n)$$

$$\int \sin(mx)\cos(nx)\,dx$$

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We change the product into a sum using the following identities:

$$\sin(mx)\sin(nx) = \frac{1}{2}\left[\cos([m-n]x) - \cos([m+n]x)\right]$$

$$\cos(mx)\cos(nx) = \frac{1}{2}\left[\cos([m-n]x) + \cos([m+n]x)\right]$$

$$\sin(mx)\cos(nx) = \frac{1}{2}\left[\sin([m-n]x) + \sin([m+n]x)\right]$$

8.2 Integrating Powers of Trig. Functions

1. $\int \cos^m x \sin^n x\, dx \quad (m, n \text{ positive integers})$

A) m, n : **one (or both) odd**

$ex.1$ one odd : $\int \cos^5 x \sin^2 x dx$

1. Factor out one power from the trig. function that has the odd power, if both have the odd power, just pick one of them and factor out one power.

$$ex : \cos^5 x = \cos^4 x \cdot \cos x$$

2. use $\cos^2 x + \sin^2 x = 1$ to transform the remaining even power of the above trig function into the other trig. function

$$ex : \cos^5 x = \left(\cos^2 x\right)^2 \cos x = \left(1 - \sin^2 x\right)^2 \cos x$$

3. use u – substitution to finish the problem (let u = "other" trig function)

$$\int \cos^5 x \sin^2 x dx = \int \sin^2 x \underbrace{\left(1 - \sin^2 x\right)^2 \cos x}_{\cos^5 x} dx \qquad u = \sin x, \quad du = \cos x dx$$

$$= \int u^2 \left(1 - u^2\right)^2 du = \int u^2 \left(1 - 2u^2 + u^4\right) du = \int \left(u^2 - 2u^4 + u^6\right) du$$

$$= \tfrac{u^3}{3} - \tfrac{2u^5}{5} + \tfrac{u^7}{7} + C = \boxed{\tfrac{1}{3}\sin^3 x - \tfrac{2}{5}\sin^5 x + \tfrac{1}{7}\sin^7 x + C}$$

$\int \cos^m x \sin^n x\, dx \quad (m, n \text{ positive integers})$

$ex.2$ both odd : $\int \cos^3 x \sin^3 x dx$

1. Factor out one power from the trig. function that has the odd power, if both have the odd power, just pick one of them and factor out one power.

$$ex : \sin^3 x = \sin^2 x \cdot \sin x$$

2. Use $\cos^2 x + \sin^2 x = 1$ to transform the remaining even power of the above trig function into the other trig. function

$$ex : \sin^3 x = \sin^2 x \cdot \sin x = \left(1 - \cos^2 x\right) \sin x$$

3. Use u – substitution to finish the problem (let u = "other" trig function)

$$\int \cos^3 x \sin^3 x dx = \int \cos^3 x \underbrace{\left(1 - \cos^2 x\right) \sin x}_{\sin^3 x} dx \qquad u = \cos x, \quad du = -\sin x dx$$

$$= -\int u^3 \left(1 - u^2\right) du = \int u^3 \left(u^2 - 1\right) du = \int \left(u^5 - u^3\right) du$$

$$= \tfrac{u^6}{6} - \tfrac{u^4}{4} + C = \boxed{\tfrac{1}{6}\cos^6 x - \tfrac{1}{4}\cos^4 x + C}$$

$$\int \cos^m x \sin^n x \, dx \quad (m,n \text{ positive integers})$$

B) m,n: **both even** $\qquad ex: \int_0^{\pi} \cos^4 x \sin^2 x dx$

1. replace all even powers using the half-angle identities:
$\sin^2 x = \frac{1}{2}(1-\cos 2x)$ and $\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$$ex: \int_0^{\pi} \cos^4 x \sin^2 x dx = \int_0^{\pi} \cos^2 x \cos^2 x \sin^2 x dx$$

$$= \int_0^{\pi} \tfrac{1}{2}(1+\cos 2x)\tfrac{1}{2}(1+\cos 2x)\tfrac{1}{2}(1-\cos 2x) dx$$

$$= \tfrac{1}{8}\int_0^{\pi} (1+\cos 2x)(1+\cos 2x)(1-\cos 2x) dx$$

$$= \tfrac{1}{8}\int_0^{\pi} \left(1+\cos 2x-\cos^2 2x-\cos^3 2x\right) dx$$

$$= \tfrac{1}{8}\Bigg[\underbrace{\int_0^{\pi} dx}_{A} + \underbrace{\int_0^{\pi} \cos 2x dx}_{B} - \underbrace{\int_0^{\pi} \cos^2 2x dx}_{C} - \underbrace{\int_0^{\pi} \cos^3 2x dx}_{D}\Bigg]$$

think of $\cos 2x$ as w:

$$(1+w)(1+w)(1-w)$$

$$= (1+w)(1-w^2)$$

$$= 1 - w^2 + w - w^3$$

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$$\int_0^{\pi} \cos^4 x \sin^2 x dx = \tfrac{1}{8}\Bigg[\underbrace{\int_0^{\pi} dx}_{A} + \underbrace{\int_0^{\pi} \cos 2x dx}_{B} - \underbrace{\int_0^{\pi} \cos^2 2x dx}_{C} - \underbrace{\int_0^{\pi} \cos^3 2x dx}_{D}\Bigg]$$

$$A = \int_0^{\pi} dx = \pi$$

$$B = \int_0^{\pi} \cos(2x) dx = \left[\tfrac{1}{2}\sin(2x)\right]_0^{\pi} = \tfrac{1}{2}[\sin 2\pi - \sin 0] = 0$$

$$C = \int_0^{\pi} \cos^2 2x dx = \int_0^{\pi} \tfrac{1}{2}(1+\cos 4x) dx = \tfrac{1}{2}\left[x + \tfrac{1}{4}\sin 4x\right]_0^{\pi}$$

$$= \tfrac{1}{2}\left[\left(\pi + \tfrac{1}{4}\sin 4\pi\right) - \left(0 + \tfrac{1}{4}\sin 0\right)\right] = \frac{\pi}{2}$$

$$D = \int_0^{\pi} \cos^3 2x dx = \int_0^{\pi} \cos^2 2x \cdot \cos 2x dx = \int_0^{\pi} \left(1-\sin^2 2x\right)\cdot \cos 2x dx$$

$$= \tfrac{1}{2}\left[\sin(2x) - \tfrac{[\sin(2x)]^3}{3}\right]_0^{\pi}$$

$$= \tfrac{1}{2}\left[\left(\sin(2\pi) - \tfrac{[\sin(2\pi)]^3}{3}\right) - \left(\sin(0) - \tfrac{[\sin(0)]^3}{3}\right)\right] = 0$$

$u = \sin 2x$

$du = 2\cos 2x dx \quad \tfrac{1}{2} du = \cos 2x dx$

$\tfrac{1}{2}\int (1-u^2) du \quad = \tfrac{1}{2}\left[u - \tfrac{u^3}{3}\right]$

$= \tfrac{1}{2}\left[\sin(2x) - \tfrac{[\sin(2x)]^3}{3}\right]$

$$\int_0^{\pi} \cos^4 x \sin^2 x dx = \tfrac{1}{8}\left[\pi - \tfrac{\pi}{2}\right] = \boxed{\frac{\pi}{16}}$$

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$\int \cos^m x \sin^n x \, dx \quad (m, n \text{ positive integers})$

C) m,n: **one or both = 1** $\quad ex: \int_0^{\pi} \cos^{10} x \sin x dx$

1. Just use u – substitution (let u = the trig function with power $\neq 1$)
(if both $=1$, choose either)

$ex: \int_0^{\pi} \cos^{10} x \sin x dx$

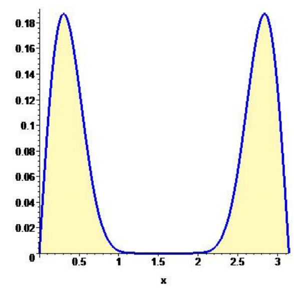
$u = \cos x$

$du = -\sin x dx$

$= -\int u^{10} du = -\frac{u^{11}}{11}$

$= \frac{-1}{11}\left[\cos^{11} x\right]_0^{\pi} = \frac{-1}{11}\left[(-1)^{11} - 1^{11}\right]$

$= \frac{-1}{11}[-1-1] = \boxed{\frac{2}{11}}$



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2. $\int \tan^m x \sec^n x \, dx \quad (m, n \text{ positive integers})$

A) m (the power of $\tan x$): **odd** $\quad ex: \int \tan^3 x \sec^3 x dx$

1. Factor out one power of $\sec x$ and one power of $\tan x$

$ex: \tan^3 x \sec^3 x = \tan^2 x \sec^2 x \sec x \tan x$

2. Use $\tan^2 x = \sec^2 x - 1$ to transform the remaining even power of $\tan x$ to be in terms of $\sec x$

$ex: \tan^2 x \sec^2 x \sec x \tan x = \left(\sec^2 x - 1\right)\sec^2 x \sec x \tan x$

3. Use u – substitution to finish the problem (let $u = \sec x$)

$ex: \int \tan^3 x \sec^3 x dx = \int \left(\sec^2 x - 1\right)\sec^2 x \sec x \tan x \, dx$

$u = \sec x$

$du = \sec x \tan x dx$

$= \int u^2\left(u^2 - 1\right)du = \int \left(u^4 - u^2\right)du$

$= \frac{u^5}{5} - \frac{u^3}{3} + C = \boxed{\frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C}$

$\int \tan^m x \sec^n x\,dx \quad (m, n \text{ positive integers})$

B) n (the power of $\sec x$): **even** $\qquad ex: \int \tan^2 x \sec^4 x\,dx$

1. Factor out $\sec^2 x$

$$ex: \tan^2 x \sec^4 x = \tan^2 x \sec^2 x \sec^2 x$$

2. If $n > 2$, use $\sec^2 x = 1 + \tan^2 x$ to transform the remaining even power of $\sec x$ to be in terms of $\tan x$

$$ex: \tan^2 x \sec^2 x \sec^2 x = \tan^2 x (1 + \tan^2 x) \sec^2 x$$

3. use u-substitution to finish the problem (let $u = \tan x$)

$$ex: \int \tan^2 x \sec^4 x\,dx = \int \tan^2 x (1 + \tan^2 x) \sec^2 x\,dx$$

$u = \tan x$

$du = \sec^2 x\,dx$

$$= \int u^2 (1 + u^2)\,du \quad = \int (u^2 + u^4)\,du$$

$$= \frac{u^3}{3} + \frac{u^5}{5} + C \quad = \boxed{\tfrac{1}{3}\tan^3 x + \tfrac{1}{5}\tan^5 x + C}$$

$\int \tan^m x \sec^n x\,dx$

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C) For all other cases, there is no set method ☹

Here are some examples:

$$ex: \int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

$$= -\ln|\cos x| + C$$

$$= \ln|\cos x|^{-1} + C$$

$$\boxed{\int \tan x\,dx = \ln|\sec x| + C}$$

$u = \cos x$

$du = -\sin x\,dx$

$\int \frac{-1}{u}\,du$

$= -\ln|u| + C$

$$ex: \int \sec x\,dx \quad = \int \sec x \cdot \underbrace{\frac{\sec x + \tan x}{\sec x + \tan x}}_{\text{convenient version of 1}}\,dx \quad = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$$

$u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x)\,dx$

$\int \frac{1}{u}\,du \quad = \ln|u| + C$

$$\boxed{\int \sec x\,dx = \ln|\sec x + \tan x| + C}$$

$$ex: \int \tan^3 x dx = \int \tan^2 x \tan x dx = \int (\sec^2 x - 1) \tan x dx$$

$$= \int \tan x \sec^2 x dx - \int \tan x dx$$

$$= \frac{1}{2}\tan^2 x - \int \tan x dx$$

$$\boxed{\int \tan^3 x dx = \frac{1}{2}\tan^2 x - \ln|\sec x| + C}$$

$u = \tan x$

$du = \sec^2 x dx$

$\int u \, du = \frac{u^2}{2}$

$\boxed{\int \tan x dx = \ln|\sec x| + C}$

$$ex: \int \sec^3 x dx = \int \sec x \sec^2 x dx$$

$u = \sec x \qquad dv = \sec^2 x$

$du = \sec x \tan x \qquad v = \tan x$

$$= \sec x \tan x - \int \sec x \tan^2 x dx$$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1) dx$$

$\boxed{\int \sec x dx = \ln|\sec x + \tan x| + C}$

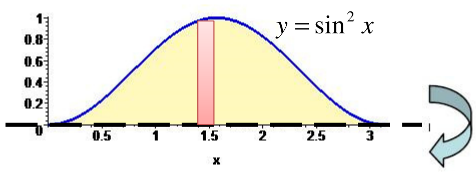
$$= \sec x \tan x - \int (\sec^3 x - \sec x) dx$$

$$\int \sec^3 x dx = \sec x \tan x - \int \sec^3 x dx + \int \sec x dx$$

$$2\int \sec^3 x dx = \sec x \tan x + \ln|\sec x \tan x|$$

$$\boxed{\int \sec^3 x dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x \tan x|) + C}$$

Find the volume of the solid obtained by rotating the region bounded by $y = \sin^2 x$ and $y = 0$ for $0 \le x \le \pi$ about the x-axis.



no gap b/w axis of rotation and the region

$\Rightarrow$ **Disk Method**

Radius: $r(x) = \sin^2 x$

$$Volume = \pi \int_a^b [r(x)]^2 dx$$

$$Volume = \pi \int_0^\pi [\sin^2 x]^2 dx$$

$$= \pi \int_0^\pi \sin^2 x \cdot \sin^2 x \, dx = \pi \int_0^\pi \tfrac{1}{2}(1 - \cos 2x) \cdot \tfrac{1}{2}(1 - \cos 2x) dx$$

$$= \tfrac{\pi}{4} \int_0^\pi (1 - \cos 2x)(1 - \cos 2x) dx = \tfrac{\pi}{4} \int_0^\pi (1 - 2\cos 2x + \cos^2 2x) dx$$

$$= \tfrac{\pi}{4} \int_0^\pi (1 - 2\cos 2x + \tfrac{1}{2}(1 + \cos 4x)) dx = \tfrac{\pi}{4} \int_0^\pi (\tfrac{3}{2} - \underbrace{2\cos 2x}_{0} + \underbrace{\tfrac{1}{2}\cos 4x}_{0}) dx$$

$$= \tfrac{\pi}{4} \int_0^\pi \tfrac{3}{2} dx = \boxed{\frac{3\pi^2}{8}}$$

m, n rational with $m \neq n$

3. $\int \sin(mx)\sin(nx)\,dx \qquad \int \cos(mx)\cos(nx)\,dx \qquad \int \sin(mx)\cos(nx)\,dx$

We change the product into a sum using the following identities:

$$\sin(mx)\sin(nx) = \frac{1}{2}\Big[\cos([m-n]x) - \cos([m+n]x)\Big]$$

$$\cos(mx)\cos(nx) = \frac{1}{2}\Big[\cos([m-n]x) + \cos([m+n]x)\Big]$$

$$\sin(mx)\cos(nx) = \frac{1}{2}\Big[\sin([m-n]x) + \sin([m+n]x)\Big]$$

$$\int \sin(3x)\cos(5x)\,dx = \frac{1}{2}\int\Big[\sin([3-5]x) + \sin([3+5]x)\Big]dx$$

$$= \frac{1}{2}\int\Big[\sin(-2x) + \sin(8x)\Big]dx = \frac{1}{2}\int\Big[\sin(8x) - \sin(2x)\Big]dx$$

$$= \frac{1}{2}\Big[-\tfrac{1}{8}\cos(8x) + \tfrac{1}{2}\cos(2x)\Big] + C = \boxed{-\tfrac{1}{16}\cos(8x) + \tfrac{1}{4}\cos(2x) + C}$$