Section 8.1 Integration By Parts

Goal: To be able to integrate more functions.

Chapter 8: Techniques of Integration

- **8.1**: Integration By Parts
- **8.2**: Integrating Powers of Trig. Functions
- **8.3**: Trig. Substitution
- **8.4**: Partial Fraction Decomposition

Integration using **substitution** can be thought of as the **chain rule** in reverse.

Integration by parts can be thought of as the **product rule** in reverse.

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right]=f'(x)\cdot g(x)+f(x)\cdot g'(x)$$

$$\int\frac{d}{dx}\left[f(x)\cdot g(x)\right]dx=\int\left[f'(x)\cdot g(x)\right]dx+\int\left[f(x)\cdot g'(x)\right]dx$$

$$f(x)\cdot g(x)=\int\left[f'(x)\cdot g(x)\right]dx+\int\left[f(x)\cdot g'(x)\right]dx$$

$$f(x)\cdot g(x)-\int\left[f'(x)\cdot g(x)\right]dx=\int\left[f(x)\cdot g'(x)\right]dx$$

$$\int\left[f(x)\cdot g'(x)\right]dx=f(x)\cdot g(x)-\int\left[f'(x)\cdot g(x)\right]dx$$

$$\int \overset{u}{f(x)}\cdot \overset{dv}{g'(x)dx} = \overset{u}{f(x)}\cdot \overset{v}{g(x)} - \int \overset{v}{g(x)}\cdot \overset{du}{f'(x)dx}$$

$$u = f(x) \qquad v = g(x)$$

$$du = f'(x)dx \qquad dv = g'(x)dx$$

$$\boxed{\int udv = u\cdot v - \int vdu}$$

Choose u and choose dv

Big Picture: We are trading in one integral for another

$\int udv$ $\int vdu$

Goal: To get a **simpler** integral than the original one

1. Choose u to be a function that becomes simpler when **differentiated**

2. Make sure dv can be readily **integrated**

Hierarchy Mneumonic to aid in choosing u

L : logarithmic functions

I : inverse trigonometric functions

A : algebraic functions

T : trigonometric functions

E : exponential functions

(T and E are interchangeable)

$$\int x^2e^{5x}dx$$

Wrong Way

$u = e^{5x}$ $\quad dv = x^2dx$

$du = 5e^{5x}dx$ $\quad v = \frac{x^3}{3}$

$$\overbrace{\int x^2e^{5x}dx}^{\text{traded in this}} = \tfrac{1}{3}x^3e^{5x} - \overbrace{\int \tfrac{5}{3}x^3e^{5x}dx}^{\text{for this}}$$

Correct Way

$u = x^2$ $\quad dv = e^{5x}dx$

$du = 2xdx$ $\quad v = \frac{1}{5}e^{5x}$

$$\overbrace{\int x^2e^{5x}dx} = \tfrac{1}{5}x^2e^{5x} - \overbrace{\int \tfrac{2}{5}xe^{5x}dx}$$

$$\int x^2e^{5x}dx = \tfrac{1}{5}x^2e^{5x} - \tfrac{2}{5}\boxed{\int xe^{5x}dx}$$

I.B.P. again

$u = x$ $\quad dv = e^{5x}dx$

$du = dx$ $\quad v = \frac{1}{5}e^{5x}$

$$\int x^2e^{5x}dx = \tfrac{1}{5}x^2e^{5x} - \tfrac{2}{5}\overbrace{\left[\tfrac{1}{5}xe^{5x} - \tfrac{1}{5}\int e^{5x}dx\right]}$$

$$\int x^2e^{5x}dx = \tfrac{1}{5}x^2e^{5x} - \tfrac{2}{25}xe^{5x} + \tfrac{2}{25}\int e^{5x}dx$$

$$\boxed{\int x^2e^{5x}dx = \tfrac{1}{5}x^2e^{5x} - \tfrac{2}{25}xe^{5x} + \tfrac{2}{125}e^{5x} + C}$$

Shortcut: Works when you have one of the following two situations :

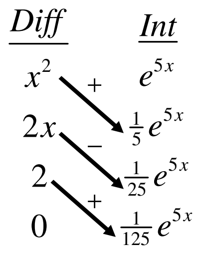
1. $\int(\text{polynomial})(\text{exponential})\,dx$
2. $\int(\text{polynomial})(\text{trig.})\,dx$

$$\int x^2e^{5x}dx$$

Step 1: Differentiate the polynomial down to 0.

Step 2: Integrate the trig. or exponential the same amount of times

Step 3: Multiply along diagonals going down to the right applying an alternating sign starting with +



Diff	*Int*
x^2	e^{5x}
$2x$	$\frac{1}{5}e^{5x}$
2	$\frac{1}{25}e^{5x}$
0	$\frac{1}{125}e^{5x}$

$$\boxed{\int x^2e^{5x}dx = \tfrac{1}{5}x^2e^{5x} - \tfrac{2}{25}xe^{5x} + \tfrac{2}{125}e^{5x} + C}$$

$$\int_0^1 x\cos(\pi x)\,dx$$

Diff		*Int*
x	$+$	$\cos(\pi x)$
1	$-$	$\frac{1}{\pi}\sin(\pi x)$
0		$\frac{-1}{\pi^2}\cos(\pi x)$

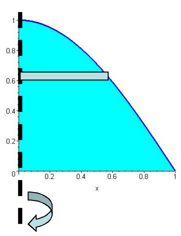
$$\int_0^1 x\cos(\pi x)\,dx=\left[\frac{x}{\pi}\sin(\pi x)+\frac{1}{\pi^2}\cos(\pi x)\right]_0^1$$

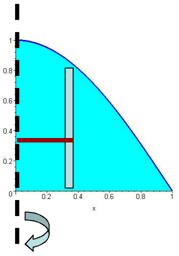
$$=\left[\frac{1}{\pi}\sin(\pi)+\frac{1}{\pi^2}\cos(\pi)\right]-\left[0\sin 0+\frac{1}{\pi^2}\cos(0)\right]$$

$$=\left[0-\frac{1}{\pi^2}\right]-\left[0+\frac{1}{\pi^2}\right]=\boxed{-\frac{2}{\pi^2}}$$

Find the volume of the solid of revolution formed by rotating the region bounded by the curves

$y=\cos\left(\frac{\pi x}{2}\right)$, $y=0,\ 0\le x\le 1$ about the $y-$axis.





Try disk method

Problem : need to solve for x in terms of y

$$x=\frac{2}{\pi}\cos^{-1} y$$

Use Shells

radius : x

height : $\cos\left(\frac{\pi x}{2}\right)$

$$V=\int_a^b 2\pi(\text{radius})(\text{height})\,dx$$

$$V=2\pi\int_0^1 x\cos\left(\frac{\pi x}{2}\right)dx$$

$$V = 2\pi\int_0^1 x\cos\left(\frac{\pi x}{2}\right)dx$$

$$\begin{array}{lcl} \underline{Diff} & & \underline{Int} \\ x & + \searrow & \cos\left(\frac{\pi x}{2}\right) \\ 1 & - \searrow & \frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right) \\ 0 & & \frac{-4}{\pi^2}\cos\left(\frac{\pi x}{2}\right) \end{array}$$

$$V = 2\pi\left[\frac{2x}{\pi}\sin\left(\frac{\pi x}{2}\right)+\frac{4}{\pi^2}\cos\left(\frac{\pi x}{2}\right)\right]_0^1$$

$$V = 2\pi\left(\left[\frac{2}{\pi}\sin\left(\frac{\pi}{2}\right)+\frac{4}{\pi^2}\cos\left(\frac{\pi}{2}\right)\right]-\left[0+\frac{4}{\pi^2}\cos(0)\right]\right)$$

$$V = 2\pi\left(\frac{2}{\pi}-\frac{4}{\pi^2}\right) \qquad V = \boxed{4-\frac{8}{\pi}}$$

$$\int_4^9 \frac{\ln y}{\sqrt{y}}dy$$

× **Short-cut doesn't work here**

Wrong Way

$$u = \frac{1}{\sqrt{y}} \qquad dv = \ln y \; dy$$

$$du = \frac{-1}{2y^{3/2}}dy \qquad v = ????$$

$$\boxed{\begin{array}{l} f = y^{-1/2} \\ f' = -\frac{1}{2}y^{-3/2} \quad \Rightarrow f' = \frac{-1}{2y^{3/2}} \end{array}}$$

Correct Way

$$u = \ln y \qquad dv = \frac{1}{\sqrt{y}}\,dy$$

$$du = \frac{1}{y}dy \qquad v = 2y^{1/2}$$

$$\boxed{\int y^{-1/2}dy = \frac{y^{1/2}}{1/2} = 2y^{1/2}}$$

$$\int\frac{\ln y}{\sqrt{y}}dy = 2y^{1/2}\ln y - \int\frac{2y^{1/2}}{y}dy = 2y^{1/2}\ln y - 2\int y^{-1/2}dy = 2y^{1/2}\ln y - 2\cdot 2y^{1/2}$$

$$\int_4^9\frac{\ln y}{\sqrt{y}}dy = \left[2\sqrt{y}\ln y - 4\sqrt{y}\right]_4^9 = \left[2\sqrt{9}\ln 9 - 4\sqrt{9}\right]-\left[2\sqrt{4}\ln 4 - 4\sqrt{4}\right]$$

$$= \left[6\ln 9 - 12\right]-\left[4\ln 4 - 8\right] = \boxed{6\ln 9 - 4\ln 4 - 4}$$

Evaluate

$\int \ln x \, dx$

we need to use Integration by Parts

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x} dx \qquad v = x$

$uv - \int v \, du$

$\int \ln x \, dx = x \ln x - \int \frac{1}{x} \cdot x \, dx$

$\int \ln x \, dx = x \ln x - \int dx$

$$\boxed{\int \ln x \, dx = x \ln x - x + C}$$

$\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right) dx$ ✖ **Short-cut doesn't work here**

$f = \arctan\left(\frac{1}{x}\right)$

$f' = \frac{1}{1+\left(\frac{1}{x}\right)^2}\left(\frac{-1}{x^2}\right) \Rightarrow f' = \frac{1}{1+\frac{1}{x^2}}\left(\frac{-1}{x^2}\right) \Rightarrow f' = \frac{-1}{x^2+1}$

Only Way

$u = \arctan\left(\frac{1}{x}\right) \qquad dv = dx$

$du = \frac{-1}{x^2+1} dx \qquad v = x$

U – substitution

$u = x^2 + 1$

$du = 2x dx \qquad \frac{1}{2} du = x dx$

$\int \frac{x}{x^2+1} dx \Rightarrow \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln u \Rightarrow \frac{1}{2} \ln\left(x^2+1\right)$

$$\int \arctan\left(\frac{1}{x}\right) dx = x \arctan\left(\frac{1}{x}\right) + \int \frac{x}{x^2+1} dx$$

$$\int \arctan\left(\frac{1}{x}\right) dx = x \arctan\left(\frac{1}{x}\right) + \frac{1}{2} \ln\left(x^2+1\right)$$

$$\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right) dx = \left[x \arctan\left(\frac{1}{x}\right) + \frac{1}{2} \ln\left(x^2+1\right)\right]_1^{\sqrt{3}}$$

$$= \left[\sqrt{3} \arctan\left(\frac{1}{\sqrt{3}}\right) + \frac{1}{2} \ln(4)\right] - \left[\arctan(1) + \frac{1}{2} \ln(2)\right]$$

$$= \sqrt{3} \frac{\pi}{6} + \frac{1}{2}(\ln 4 - \ln 2) - \frac{\pi}{4} = \boxed{\frac{\pi\sqrt{3}}{6} + \frac{1}{2} \ln 2 - \frac{\pi}{4}}$$

$\int e^{-t}\sin(t)\,dt$

× **Short-cut doesn't work here**

$u=\sin(t) \quad dv=e^{-t}dt$

$du=\cos(t)dt \quad v=-e^{-t}$

$$\Rightarrow \int e^{-t}\sin(t)\,dt = -e^{-t}\sin(t)+\underbrace{\int e^{-t}\cos(t)\,dt}$$

$u=\cos(t) \quad dv=e^{-t}dt$

$du=-\sin(t)dt \quad v=-e^{-t}$

$$-e^{-t}\cos(t)-\int e^{-t}\sin(t)\,dt$$

$$\Rightarrow \int e^{-t}\sin(t)\,dt = -e^{-t}\sin(t)-e^{-t}\cos(t)-\int e^{-t}\sin(t)\,dt$$

$$+\int e^{-t}\sin(t)\,dt \qquad\qquad +\int e^{-t}\sin(t)\,dt$$

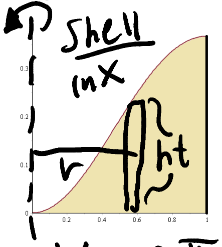
$$\Rightarrow 2\int e^{-t}\sin(t)\,dt = -e^{-t}\sin(t)-e^{-t}\cos(t)$$

$$\boxed{\Rightarrow \int e^{-t}\sin(t)\,dt = \frac{1}{2}\left[-e^{-t}\sin(t)-e^{-t}\cos(t)\right]+C}$$

Fall 2012

8. The volume of the solid of revolution obtained by rotating the region bounded by $y=x^2e^{-x^2}$ and the x-axis for $0\le x\le 1$ about the y-axis is:

a) $\frac{2}{3}\pi$ (b) $\frac{1}{2}\pi$ (c) $\frac{3}{2}$ (d) $2\pi(e-1)$ (e) $\frac{5}{3}\pi e$ (f) $\pi-\frac{2\pi}{e}$



$r = x$

$ht = x^2e^{-x^2}$

$$V = 2\pi\int_0^1 \underbrace{x}_{r}\cdot\underbrace{x^2e^{-x^2}}_{ht}\,dx$$

$$= 2\pi\int_0^1 x^3e^{-x^2}\,dx$$

$$= 2\pi\int_0^1 x^2\cdot e^{-x^2}\cdot x\,dx$$

$u=-x^2 \Rightarrow x^2=-u$

$du=-2x\,dx$

$-\frac{1}{2}du = x\,dx$

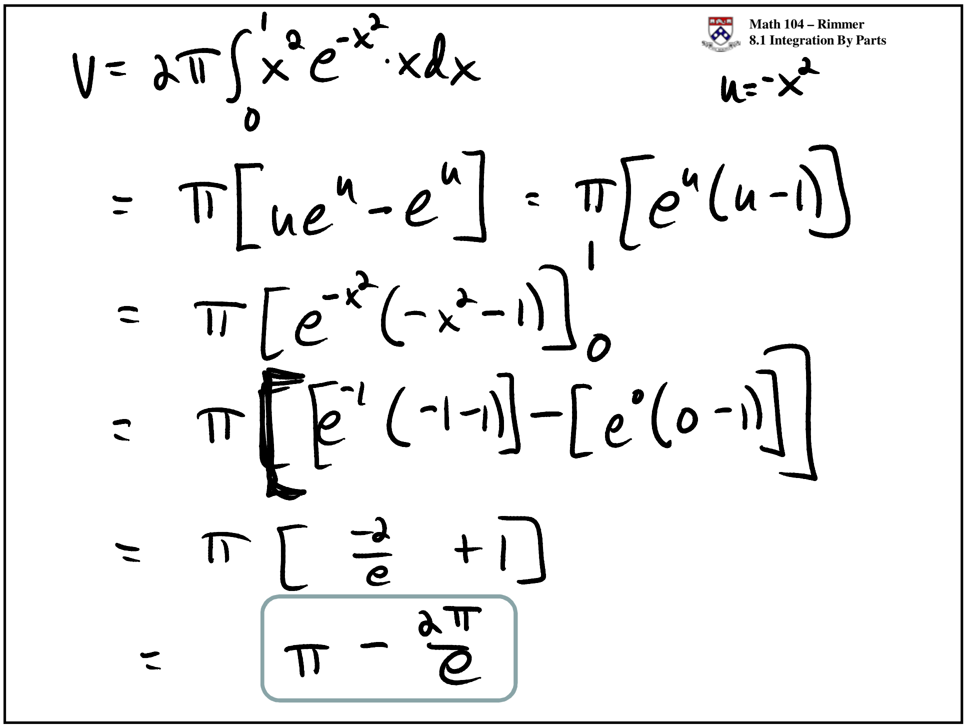
$2\pi\int -u\,\frac{-1}{2}e^u\,du$

$\cancel{2}\pi\frac{1}{\cancel{2}}\int ue^u\,du$

$\pi\int ue^u\,du$

D	I
u	$+\ e^u$
1	$-\ e^u$
0	e^u

$\pi\left[ue^u-e^u\right]$



$$V = 2\pi \int_0^1 x^2 e^{-x^2} \cdot x\,dx \qquad u = -x^2$$

$$= \pi\left[ue^u - e^u\right] = \pi\left[e^u(u-1)\right]$$

$$= \pi\left[e^{-x^2}(-x^2-1)\right]_0^1$$

$$= \pi\left[\left[e^{-1}(-1-1)\right] - \left[e^0(0-1)\right]\right]$$

$$= \pi\left[\frac{-2}{e} + 1\right]$$

$$= \boxed{\pi - \frac{2\pi}{e}}$$