7.2 Separable Differential Eq.

$$\frac{dy}{dx} = f(x, y)$$

1) Take the right hand side and use algebra to represent it as a product of functions one of x only and the other of y only.

$$\frac{dy}{dx} = g(x) \cdot h(y)$$

2) Multiply by dx and divide by $h(y)$

$$\frac{dy}{h(y)} = g(x)\,dx$$

3) Integrate both sides.

4) If possible solve for y in terms of x.

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$$\frac{dy}{dx} = \overset{\text{constant}}{k}y$$

g(x) h(y)

① sep.

$$\int \frac{dy}{y} = \int k \cdot dx$$

② Int. both sides

$$\ln y + c_1 = kx + c_2$$

$$\ln y = kx + C \quad (C = c_2 - c_1)$$

$$e^{a+b} = e^a \cdot e^b$$

$$\cancel{e}^{\ln y} = e^{kx + C}$$

$$y = e^{kx} \cdot e^{C}$$

another random const. call it A

$$\boxed{y = Ae^{kx}}$$

$\underbrace{\frac{dy}{dx}}$ $= \underset{h(y)}{y^2} \cdot \underset{g(x)}{x}$ $y(1)=6$ ← initial cond. → to solve for C   Find $y(5)$

$x=1$ $y=6$

① $\int \frac{dy}{y^2} = \int x\,dx$ → $\int y^{-2}\,dy = \frac{y^{-1}}{-1}$

② $-\frac{1}{y} = \frac{x^2}{2} + C$ solve for C using $x=1$, $y=6$

$-\frac{1}{6} = \frac{1}{2} + C$

$C = -\frac{1}{6} - \frac{1}{2}$

$-\frac{1}{6} - \frac{3}{6} = -\frac{4}{6} = -\frac{2}{3}$

$-\frac{1}{y} = \frac{3\cdot x^2}{2\cdot 3} - \frac{2\cdot 2}{3\cdot 2}$

$-\frac{1}{y} = \frac{3x^2-4}{6} \Rightarrow -6 = y(3x^2-4)$ $\boxed{y = \frac{-6}{3x^2-4}}$

$$\frac{dy}{dx} = \frac{3x^2y^3 - 6x^2}{y^2}$$

$\frac{3x^2(y^3-2)}{y^2}$

$\frac{dy}{dx} = \underset{g(x)}{3x^2} \underbrace{\left(\frac{y^3-2}{y^2}\right)}_{h(y)}$

① $\int \frac{y^2}{y^3-2}\,dy = \int 3x^2\,dx$

② $\frac{1}{3}\ln(y^3-2) = x^3 + C$

$\frac{dy}{dx} = f(x,y)$ → $g(x) \cdot h(y)$

$\frac{a\cdot b}{c} = a \cdot \frac{b}{c}$

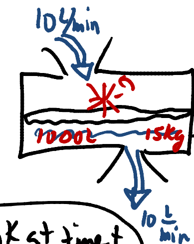
$u = y^3 - 2$
$du = 3y^2\,dy$
$\frac{1}{3}du = y^2\,dy$
$\frac{1}{3}\int \frac{1}{u}\,du \Rightarrow \frac{1}{3}\ln u$

A tank contains 1000 L of brine with 15 kg of dissolved salt. Pure water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank (a) after t minutes and (b) after 20 minutes?

Salt water

0 kg Salt

$$\boxed{(a)\ y = 15e^{-\frac{t}{100}}}$$



$$\boxed{\begin{array}{l} t=0 \\ y=15 \end{array}}$$

$y(t)$ = Amt. of salt (in kg) in the tank at time t

Setup

$$\frac{kg}{min}\quad \frac{dy}{dt} = \underbrace{(0)\cdot 10}_{\text{rate in}} - \underbrace{\left(\frac{y}{1000}\frac{kg}{L}\right)\cdot 10\frac{L}{min}}_{\text{rate out}}$$

conc. × rate of pour — conc. × rate of pour

$$e^{a+b} = e^a \cdot e^b$$

Solving

$$\frac{dy}{dt} = 0 - \frac{10y}{1000} \Rightarrow \boxed{\frac{dy}{dt} = \frac{-y}{100}} \Rightarrow$$

$$e^{\ln y} = e^{\frac{-1}{100}t + \ln 15}$$

$$y = e^{-\frac{t}{100}} \cdot 15$$

① Sep.

$$\frac{dy}{y} = \frac{-1}{100}\,dt$$

② Int

$$\int \frac{dy}{y} = \int \frac{-1}{100}\,dt \Rightarrow \ln y = \frac{-1}{100}t + C$$

$$\ln 15 = 0 + C \Rightarrow C = \ln 15$$

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(b)
$$y(t) = 15e^{-\frac{t}{100}}$$

$$y(20) = 15e^{-\frac{1}{5}} = \frac{15}{e^{\frac{1}{5}}} \text{ kg of salt}$$

$$\approx 12.28 \text{ kg}$$

Plutonium-239 The half-life of the plutonium isotope is 24,360 years. If 10 g of plutonium is released into the atmosphere by a nuclear accident, how many years will it take for 80% of the isotope to decay?

Radioactive Decay

$$\frac{dy}{dt} = -k \cdot y$$

$$\int \frac{dy}{y} = -k\int dt$$

$$\ln y = -kt + C$$

$$y = e^{-kt} \cdot e^{C}$$

$$\boxed{y = Ae^{-kt}}$$

$$\boxed{k = \frac{\ln 2}{\text{half-life}}}$$

half-life = 24,360 yrs.

$\boxed{t = 0 \quad y = M}$

$\boxed{t = \tfrac{1}{2}\text{life} \quad y = \tfrac{1}{2}M}$

$t = 24{,}360 \quad y = \frac{1}{2}M$

$M = Ae^{0} \Rightarrow A = M$

$$y = Me^{-kt}$$

$$\frac{\frac{1}{2}\cancel{M}}{\cancel{M}} = \frac{\cancel{M}e^{-k(24{,}360)}}{\cancel{M}}$$

$$\frac{1}{2} = e^{-24360k}$$

$$\ln\tfrac{1}{2} = \cancel{\ln e}^{-24360k}$$

$$\underbrace{\ln 1}_{0} - \ln 2 = -24{,}360k \Rightarrow k = \frac{\ln 2}{24{,}360}$$

$$y = Me^{-\frac{\ln 2}{24360}t}$$

M initial amount

$\boxed{t = 0 \quad M = 10\text{ g}}$

$t = ?$

$y = 2g$

M_1 = 80% of original decays

20% still there

$10g \cdot \frac{1}{5} = 2g$

$$\frac{2}{10} = \frac{10e^{-\frac{\ln 2}{24360}t}}{10}$$

$$\frac{1}{5} = e^{-\frac{\ln 2}{24360}t}$$

$$\ln\tfrac{1}{5} = \cancel{\ln}\left(\cancel{e}^{-\frac{\ln 2}{24360}t}\right)$$

$$\ln 1 - \ln 5$$

$$-\ln 5 = \frac{-\ln 2}{24{,}360} \cdot t$$

$$t = \frac{\ln 5}{\ln 2}(24{,}360)$$

$$t \approx 56{,}562 \text{ years}$$

Cooling soup Suppose that a cup of soup cooled from 90°C to 60°C after 10 min in a room whose temperature was 20°C. Use Newton's law of cooling to answer the following questions.

a. How much longer would it take the soup to cool to 35°C?

b. Instead of being left to stand in the room, the cup of 90°C soup is put in a freezer whose temperature is −15°C. How long will it take the soup to cool from 90°C to 35°C?

$y(t) =$ temp. of the object at any time t

°C   min

Newton's Law of Cooling — the rate at which an object's temperature is changing at any given time is roughly proportional to the difference between its temperature and the temperature of the surrounding medium.

$$\boxed{\frac{dy}{dt} = -k(y - M)}$$

$$\frac{dy}{y-M} = -k\,dt$$

$$\ln(y-M) = -kt + C$$

$$e^{-kt} \cdot \boxed{e^{C}}$$

$$y - M = Ae^{-kt}$$

$$\boxed{y = Ae^{-kt} + M}$$

$t = 0$, $y = y_0$ (init. temp)

$$y_0 = Ae^0 + M$$

$$y_0 = A + M \rightarrow \boxed{A = y_0 - M}$$

a is prop. to b

$$a = k \cdot b$$

some constant

$$\boxed{y = (y_0 - M)e^{-kt} + M}$$

y_0 = init temp

M = surrounding medium temp. (usually room temp.)

(a) $t = 0$ $y = 90$ $\boxed{y_0 = 90 \quad M = 20}$

$\boxed{t = 10 \quad y = 60}$

$M = 20$

Solve for k

$$y = 70e^{-kt} + 20$$

$$60 = 70e^{-10k} + 20$$

$$40 = 70e^{-10k}$$

$$\frac{4}{7} = e^{-10k}$$

$$\ln\left(\frac{4}{7}\right) = \cancel{\ln e}^{-10k}$$

$$-10k$$

$$\boxed{\frac{-\ln\left(\frac{4}{7}\right)}{10} = k}$$

$$y = 70e^{-\frac{-\ln(4/7)}{10}\cdot t} + 20$$

$$\boxed{y = 70e^{\frac{t}{10}\cdot\ln\left(\frac{4}{7}\right)} + 20}$$

$$y = 70e^{\frac{t}{10}\ln\left(\frac{4}{7}\right)} + 20$$

$$y = 70\cancel{e^{\ln}}\left(\tfrac{4}{7}\right)^{\frac{t}{10}} + 20$$

$$\boxed{y = 70\left(\frac{4}{7}\right)^{\frac{t}{10}} + 20}$$

(a) $y = 35$
$t = ?$

how much longer?

17.5 min

$$35 = 70\left(\tfrac{4}{7}\right)^{t/10} + 20$$
$$-20 \qquad\qquad -20$$

$$\frac{15}{70} = \frac{70\left(\frac{4}{7}\right)^{t/10}}{70}$$

$$\frac{3}{14} = \left(\tfrac{4}{7}\right)^{t/10}$$

$$\ln\left(\tfrac{3}{14}\right) = \tfrac{t}{10}\ln\left(\tfrac{4}{7}\right)$$

$$10 \cdot \ln\left(\tfrac{3}{14}\right) = t \cdot \ln\left(\tfrac{4}{7}\right)$$

$$t = \frac{10 \cdot \ln\left(\frac{3}{14}\right)}{\ln\left(\frac{4}{7}\right)} \approx 27.5 \text{ min}$$

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b) $y = (y_0 - M)e^{-kt} + M$

$M = -15°C$

$y_0 = 90°C$

$y_0 - M = 90 - {-15}$
$= 105$

K is the same

$$k = \frac{-\ln\left(\frac{4}{7}\right)}{10}$$

$$y = 105e^{-kt} + -15$$

$$y = 105e^{-\frac{-\ln(4/7)}{10}t} - 15$$

$$y = 105e^{\frac{t}{10}\ln\left(\frac{4}{7}\right)} - 15$$

$$y = 105\cancel{e^{\ln}}\left(4/7\right)^{t/10} - 15$$

$$y = 105\left(\tfrac{4}{7}\right)^{\frac{t}{10}} - 15$$

(b) $y = 105\left(\frac{4}{7}\right)^{t/10} - 15$

$y = 35$
$t = ?$

$$35 = 105\left(\frac{4}{7}\right)^{t/10} - 15$$
$$+15 \qquad\qquad +15$$

$$\frac{50}{105} = \frac{105\left(\frac{4}{7}\right)^{t/10}}{105}$$

$$\frac{10}{21} = \left(\frac{4}{7}\right)^{t/10}$$

$$\ln\left(\frac{10}{21}\right) = \frac{t}{10}\ln\left(\frac{4}{7}\right)$$

$$\frac{10\cdot\ln\left(\frac{10}{21}\right)}{\ln\left(\frac{4}{7}\right)} = \frac{t\cdot\ln\left(\frac{4}{7}\right)}{\ln\left(\frac{4}{7}\right)}$$

$$t = \frac{10\cdot\ln\left(\frac{10}{21}\right)}{\ln\left(\frac{4}{7}\right)}$$

$t \approx 13.26$ min