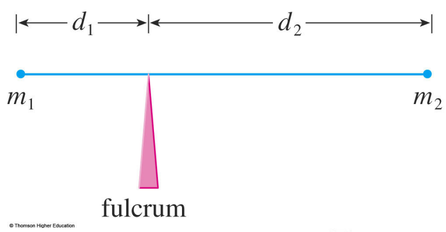
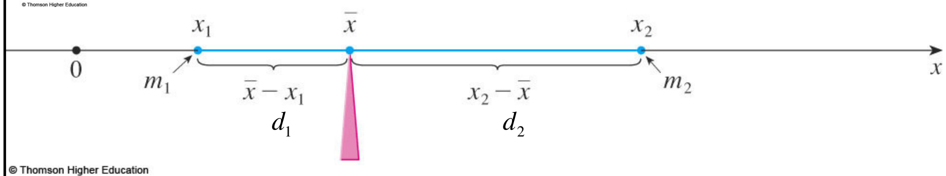
6.6 Center of Mass 1-d



Archimedes' Law of the Lever

the rod will balance if

$$\boxed{m_1 d_1 = m_2 d_2}$$



$$m_1\left(\bar{x} - x_1\right) = m_2\left(x_2 - \bar{x}\right)$$

$$m_1\bar{x} - m_1 x_1 = m_2 x_2 - m_2\bar{x}$$

$$\bar{x}\left(m_1 + m_2\right) = m_1 x_1 + m_2 x_2$$

$$\boxed{\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}}$$

moment of m_1 (with respect to the origin)

moment of m_2 (with respect to the origin)

moment of the system about the origin

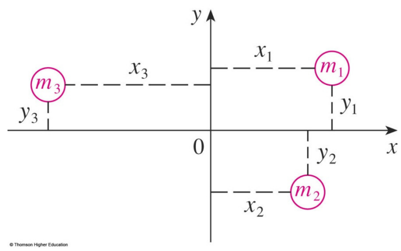
total mass

$$\boxed{\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\text{total mass}}}$$

$$\underbrace{\bar{x}\cdot\left(\text{total mass}\right)}_{\text{moment for the total mass}} = \underbrace{\sum_{i=1}^{n} m_i x_i}_{\text{moment for the system}}$$

If the total mass was concentrated at $\bar{x}$, then its moment would be the same as the moment for the system.

6.6 Center of Mass 2-d



$M_y =$ moment of the system about the $y-$axis
measures the tendency of the system
to rotate about the $y-$axis

$$M_y = m_1x_1 + m_2x_2 + m_3x_3$$

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

$$\boxed{\bar{x} = \frac{M_y}{\text{total mass}}}$$

$M_x =$ moment of the system about the $x-$axis
measures the tendency of the system
to rotate about the $x-$axis

$$M_x = m_1y_1 + m_2y_2 + m_3y_3$$

$$\bar{y} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3}$$

$$\boxed{\bar{y} = \frac{M_x}{\text{total mass}}}$$

$$M_y = \bar{x}(\text{total mass})$$

$$M_x = \bar{y}(\text{total mass})$$

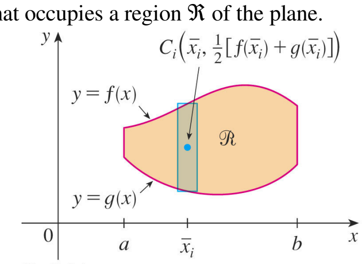
The center of mass is the point $(\bar{x}, \bar{y})$ where a single particle with the same mass as the total mass would have the same moments as the system

Math 104 – Rimmer
6.6 Center of Mass

6.6 Center of Mass 3-d

Consider a flat plate (called a lamina) with uniform density ρ that occupies a region $\Re$ of the plane.

The center of mass of the plate is called the **centroid** of $\Re$.



length $= f(x) - g(x)$
width $= dx$

area $=$ (length)(width)
area $= [f(x) - g(x)]dx$
area $= dA$

$\bar{x}_i =$ average x on the interval

$$\bar{x}_i = \frac{x_i + x_{i+1}}{2}$$

$\bar{y}_i =$ average y on the interval

$$\bar{y}_i = \frac{f(\bar{x}_i) + g(\bar{x}_i)}{2}$$

density $= \rho$ (a constant)

mass $=$ (density)(area)
mass $= \rho[f(x) - g(x)]dx$
mass $= dm$

6.6 Center of Mass 3-d

$\text{length} = f(x) - g(x)$ $\quad \text{area} = [f(x) - g(x)]dx$ $\quad \text{mass} = \rho[f(x) - g(x)]dx$

$\text{width} = dx$ $\quad \text{area} = dA$ $\quad \text{mass} = dm$

Moment about the y – axis

$$M_y = m_1\overline{x_1} + m_2\overline{x_2} + \cdots + m_n\overline{x_n} = \sum_{i=1}^{n} m_i\overline{x_i}$$

let $n \to \infty$ $\quad M_y = \lim_{n\to\infty}\sum_{i=1}^{n}\overline{x_i}dm = \int x\,dm \Rightarrow \boxed{M_y = \int x\rho[f(x) - g(x)]dx}$

Moment about the x – axis

$$M_x = m_1\overline{y_1} + m_2\overline{y_2} + \cdots + m_n\overline{y_n} = \sum_{i=1}^{n} m_i\overline{y_i}$$

let $n \to \infty$ $\quad M_x = \lim_{n\to\infty}\sum_{i=1}^{n}\overline{y_i}dm = \int y\,dm = \int \tfrac{1}{2}[f(x) + g(x)]\rho[f(x) - g(x)]dx$

$$\Rightarrow \boxed{M_x = \int \tfrac{1}{2}\rho\left[[f(x)]^2 - [g(x)]^2\right]dx}$$

Total mass

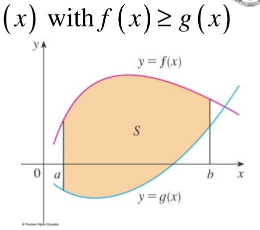
$M = m_1 + m_2 + m_3 + \cdots = \sum_{i=1}^{n} m_i = \int dm \quad \Rightarrow \boxed{M = \int \rho[f(x) - g(x)]dx}$

let $n \to \infty$

General Formulas

Thin plate : region between $y = f(x)$ and $y = g(x)$ with $f(x) \geq g(x)$

Constant density function $\rho(x) = \rho$



Moment about the y – *axis*

$$M_y = \rho\int_a^b x\cdot(f(x) - g(x))dx$$

Moment about the x – *axis*

$$M_x = \frac{\rho}{2}\int_a^b\left([f(x)]^2 - [g(x)]^2\right)dx$$

Mass

$$M = \rho\cdot\int_a^b (f(x) - g(x))dx$$

$$\boxed{\overline{x} = \frac{M_y}{M}}$$

Center of Mass

$(\overline{x}, \overline{y})$

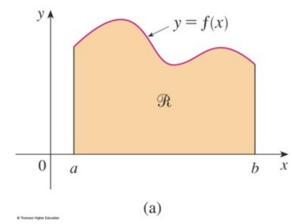
$$\boxed{\overline{y} = \frac{M_x}{M}}$$

General Formulas

Thin plate : region under the graph of $y = f(x)$ and above the $x-$axis

Constant density function $\rho(x) = \rho$

Set $g(x) = 0$ in the previous formulas.



(a)

Moment about the $x-axis$

$$M_x = \frac{\rho}{2}\int_a^b [f(x)]^2 \, dx$$

Moment about the $y-axis$

$$M_y = \rho \int_a^b x f(x) \, dx$$

Mass

$$M = \rho \cdot \int_a^b f(x) \, dx$$

$$\overline{x} = \frac{M_y}{M}$$

Center of Mass

$(\overline{x}, \overline{y})$

$$\overline{y} = \frac{M_x}{M}$$

Thin plate : region between $y = x - x^2$ and $y = -x$

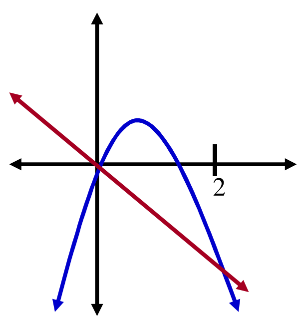
Constant density function $\rho(x) = \rho$

Limits of integration: $x - x^2 = -x$

$$2x - x^2 = 0$$

$$x(2 - x) = 0$$

$$x = 0, \quad x = 2$$



Calculate the three integrals:

$$M_y = \rho \int_a^b x \cdot \left(f(x) - g(x) \right) dx$$

$$M_x = \frac{\rho}{2}\int_a^b \left([f(x)]^2 - [g(x)]^2 \right) dx$$

$$M = \rho \int_a^b \left(f(x) - g(x) \right) dx$$

and plug into the formulas.

$$\overline{x} = \frac{M_y}{M} \qquad \overline{y} = \frac{M_x}{M}$$

$$M_y = \rho\int_a^b x\cdot\left(f(x)-g(x)\right)dx = \rho\int_0^2 x\left[\left(x-x^2\right)-\left(-x\right)\right]dx$$

$$= \rho\int_0^2 x\cdot\left(2x-x^2\right)dx \ = \rho\int_0^2\left(2x^2-x^3\right)dx = \rho\left(\frac{2x^3}{3}-\frac{x^4}{4}\right)\Bigg|_0^2$$

$$= \rho\left(\frac{16}{3}-\frac{16}{4}\right) = \rho\left(\frac{16}{3}-4\right) = \frac{16-12}{3}\rho = \boxed{\frac{4\rho}{3}}$$

$$M_x = \frac{\rho}{2}\int_a^b\left(\left[f(x)\right]^2-\left[g(x)\right]^2\right)dx = \frac{\rho}{2}\int_0^2\left[\left(x-x^2\right)^2-\left(-x\right)^2\right]dx$$

$$= \frac{\rho}{2}\int_0^2\left[\left(\cancel{x^2}-2x^3+x^4\right)-\cancel{x^2}\right]dx = \frac{\rho}{2}\int_0^2\left(-2x^3+x^4\right)dx = \frac{\rho}{2}\left(\frac{-x^4}{2}+\frac{x^5}{5}\right)\Bigg|_0^2 = \frac{\rho}{2}\left(-8+\frac{32}{5}\right)$$

$$= \frac{\rho}{2}\left(\frac{-40+32}{5}\right) = \frac{\rho}{2}\left(\frac{-8}{5}\right) = \frac{\rho}{2}\left(\frac{-8}{5}\right) = \boxed{\frac{-4\rho}{5}}$$

$$M = \rho\int_a^b\left(f(x)-g(x)\right)dx = \rho\int_0^2\left(2x-x^2\right)dx = \rho\left(x^2-\frac{x^3}{3}\right)\Bigg|_0^2 = \rho\left(4-\frac{8}{3}\right) = \frac{12-8}{3}\rho = \boxed{\frac{4\rho}{3}}$$

$$\bar{x} = \frac{M_y}{M} \qquad \bar{y} = \frac{M_x}{M}$$

$$\bar{x} = \frac{\frac{4\rho}{3}}{\frac{4\rho}{3}} \qquad \bar{y} = \frac{\frac{-4\rho}{5}}{\frac{4\rho}{3}} = \frac{-4\rho}{5}\,\frac{3}{4\rho}$$

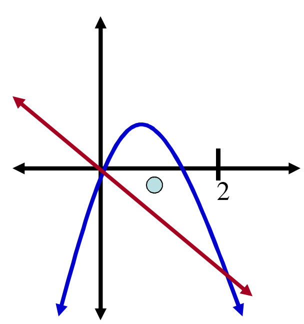
$$\Rightarrow \bar{x} = 1 \qquad \Rightarrow \bar{y} = \frac{-3}{5}$$

$$\left(\bar{x},\bar{y}\right) = \left(1,\frac{-3}{5}\right)$$

$$\boxed{M_y = \frac{4\rho}{3}}$$

$$\boxed{M_x = \frac{-4\rho}{5}}$$

$$\boxed{M = \frac{4\rho}{3}}$$



13. The region between the curve $y = 1/\sqrt{x}$ and the x-axis from $x = 1$ to $x = 16$

$$M_y = \rho\int_a^b x\cdot f(x)\,dx = \rho\int_1^{16} x\cdot\frac{1}{\sqrt{x}}\,dx = \rho\int_1^{16}\sqrt{x}\,dx = \rho\int_1^{16} x^{1/2}\,dx$$

$$= \frac{2\rho}{3}\Big[x^{3/2}\Big]_1^{16} = \frac{2\rho}{3}\Big[16^{3/2} - 1^{3/2}\Big] = \frac{2\rho}{3}\Big[\left(4^2\right)^{3/2} - 1\Big] = \frac{2\rho}{3}[63] = 42\rho \qquad (4^3 = 64,\ 21)$$

$$M_x = \frac{\rho}{2}\int_a^b\big[f(x)\big]^2\,dx = \frac{\rho}{2}\int_1^{16}\left(\frac{1}{\sqrt{x}}\right)^2 dx = \frac{\rho}{2}\int_1^{16}\frac{1}{x}\,dx = \frac{\rho}{2}[\ln x]_1^{16} = \frac{\rho}{2}(\ln 16 - \ln 1) = \frac{\rho}{2}\ln 16$$

$$M = \rho\int_a^b f(x)\,dx = \rho\int_1^{16}\frac{1}{\sqrt{x}}\,dx = \rho\int_1^{16} x^{-1/2}\,dx = 2\rho\Big[x^{1/2}\Big]_1^{16} = 2\rho\Big[\sqrt{16} - \sqrt{1}\Big] = 2\rho[4-1] = 6\rho$$

$$\bar{x} = \frac{42\rho}{6\rho} \qquad \bar{y} = \frac{\frac{\rho}{2}\ln 16}{6\rho}$$

$$\boxed{\bar{x} = 7 \qquad \bar{y} = \frac{\ln 16}{12}}$$