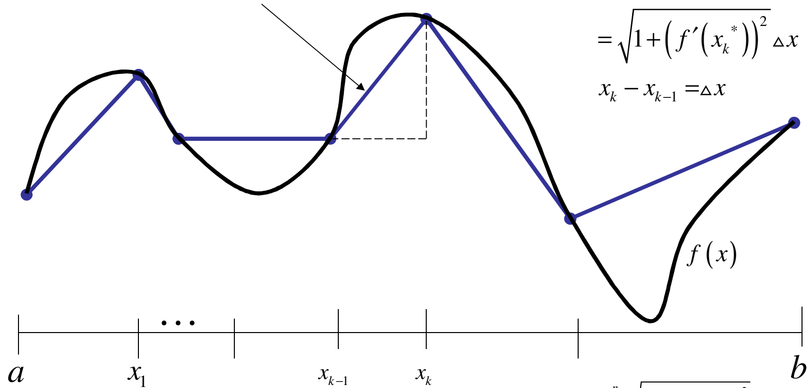
6.3 Arc Length

$$\sqrt{(x_k - x_{k-1})^2 + \left(f(x_k) - f(x_{k-1})\right)^2} = \sqrt{(x_k - x_{k-1})^2 + \left(f'(x_k^*)(x_k - x_{k-1})\right)^2}$$

$$= \sqrt{1 + \left(f'(x_k^*)\right)^2}\,\Delta x$$

$$x_k - x_{k-1} = \Delta x$$



by the Mean Value Theorem: $\exists\, x_k^* \in (x_{k-1}, x_k)$ such that

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(x_k^*)$$

$$\Rightarrow f(x_k) - f(x_{k-1}) = f'(x_k^*)(x_k - x_{k-1})$$

$$\text{Arc Length} = \sum_{k=1}^{n} \sqrt{1 + \left(f'(x_k^*)\right)^2}\,\Delta x$$

$$\text{Arc Length} = \lim_{\Delta x \to 0} \sum_{k=1}^{n} \sqrt{1 + \left(f'(x_k^*)\right)^2}\,\Delta x$$

$$\boxed{\text{Arc Length} = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\,dx}$$

Find the length of the curve.

$$y = \frac{2}{3}\left(1 + x^2\right)^{3/2} \quad 0 \le x \le 3$$

$$\boxed{\text{Arc Length} = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\,dx}$$

$$y' = 2x\left(1 + x^2\right)^{1/2}$$

$$(y')^2 = 4x^2\left(1 + x^2\right)$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + 4x^2\left(1 + x^2\right)} = \sqrt{4x^4 + 4x^2 + 1} = \sqrt{\left(2x^2 + 1\right)^2} = 2x^2 + 1$$

$$\int_0^3 \sqrt{1 + (y')^2}\,dx = \int_0^3 \left(2x^2 + 1\right)dx = \left[\frac{2}{3}x^3 + x\right]_0^3 = \boxed{21}$$

Find the length of the curve $y = 1 + \frac{2}{3}(x-1)^{3/2}$ for $1 \le x \le 4$.

$$y = 1 + \frac{2}{3}(x-1)^{3/2}$$

$$\text{Arc Length} = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\, dx$$

$$\frac{dy}{dx} = (x-1)^{1/2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = x - 1$$

$$\sqrt{1 + \left(\tfrac{dy}{dx}\right)^2} = \sqrt{1 + x - 1} = \sqrt{x}$$

$$\text{Arc Length} = \int_a^b \sqrt{1 + \left(\tfrac{dy}{dx}\right)^2}\, dx = \int_1^4 \sqrt{x}\, dx = \int_1^4 x^{1/2} dx$$

$$= \frac{2}{3}\left[x^{3/2}\right]_1^4 = \frac{2}{3}\left[\left(4^{1/2}\right)^3 - 1\right] = \boxed{\frac{14}{3}}$$

Find the length of the curve.

$y = x^2 - \frac{\ln x}{8}$ for $1 \le x \le 2$.

$$\text{Arc Length} = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\, dx$$

Find the derivative.

$$\Rightarrow y' = 2x - \frac{1}{8x}$$

Square the derivative.

$$(y')^2 = \left(2x - \frac{1}{8x}\right)\left(2x - \frac{1}{8x}\right) = 4x^2 - \frac{1}{4} - \frac{1}{4} + \frac{1}{64x^2}$$

$$\Rightarrow (y')^2 = \boxed{4x^2 - \frac{1}{2} + \frac{1}{64x^2}}$$

Add 1 and attempt to represent the expression as a perfect square.

$$1 + (y')^2 = 4x^2 - \frac{1}{2} + 1 + \frac{1}{64x^2} = \boxed{4x^2 + \frac{1}{2} + \frac{1}{64x^2}} = \left(2x + \frac{1}{8x}\right)\left(2x + \frac{1}{8x}\right)$$

Take the square root and integrate.

$$\sqrt{1 + (y')^2} = \sqrt{\left(2x + \frac{1}{8x}\right)^2} = 2x + \frac{1}{8x}$$

$$\int_1^2 \sqrt{1 + (y')^2}\, dx = \int_1^2 \left(2x + \frac{1}{8x}\right) dx = \left[x^2 + \frac{\ln x}{8}\right]_1^2 = \left(4 + \frac{\ln 2}{8}\right) - (1 + 0) = \boxed{3 + \frac{1}{8}\ln 2}$$