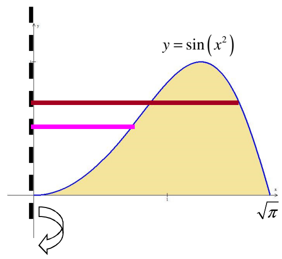
6.2 Volumes by Cylindrical Shells

Sometimes finding the volume of a solid of revolution is **impossible** by the disk or washer method



Since there is a gap b/w the region and the axis of rotation, we would try washer method

We would have to solve for x as a function of y since the axis of rotation is vertical.
Sometimes this is the problem, but we can do it here.

$$x = \sqrt{\sin^{-1} y}$$

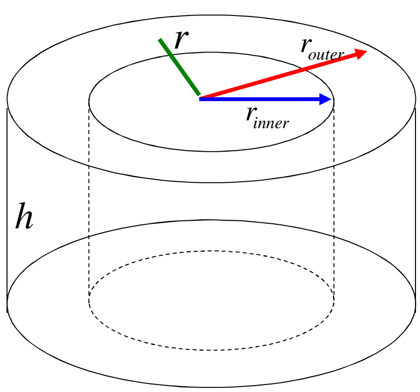
Our problem is that the outer radius and the inner radius use the **same curve**.

In order to find the volume of this solid of revolution we need a different technique.

Math 104 – Rimmer
6.2 Volumes by Cylindrical Shells

The **Method of Cylindrical Shells** uses the volume of **nested cylinders** to find the volume of a solid of revolution.

To understand the formula, lets first look at one of the cylindrical shells:



$\dfrac{(r_{outer} + r_{inner})}{2} = r_{average}$

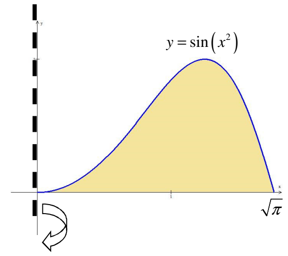
Let $r = r_{average}$ Let $\Delta r = r_{outer} - r_{inner}$

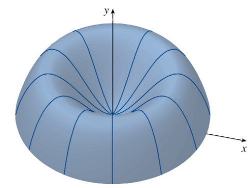
There are two cylinders, an outer cylinder and an inner cylinder.

The volume of the "shell" we use is found by taking the volume of the inner cylinder and subtracting it from the volume of the outer cylinder.

$$V_{shell} = V_{outer} - V_{inner}$$
$$V_{shell} = \pi (r_{outer})^2 h - \pi (r_{inner})^2 h$$
$$V_{shell} = \pi h\left[(r_{outer})^2 - (r_{inner})^2\right]$$
$$V_{shell} = \pi h\left[(r_{outer} + r_{inner})(r_{outer} - r_{inner})\right]$$
$$V_{shell} = 2\pi h \frac{(r_{outer} + r_{inner})}{2}(r_{outer} - r_{inner})$$

$$V_{shell} = \underbrace{2\pi r}_{\text{circumference}} \cdot \underbrace{h}_{\text{height}} \cdot \underbrace{\Delta r}_{\text{thickness}}$$





$$V_{shell} = \underset{\text{circumference}}{2\pi r} \cdot \underset{\text{height}}{h} \cdot \underset{\text{thickness}}{\Delta r}$$

$$V_i = 2\pi\left(\overline{x_i}\right) \cdot f\left(\overline{x_i}\right) \cdot \Delta x$$

now add up the volume of all the shells

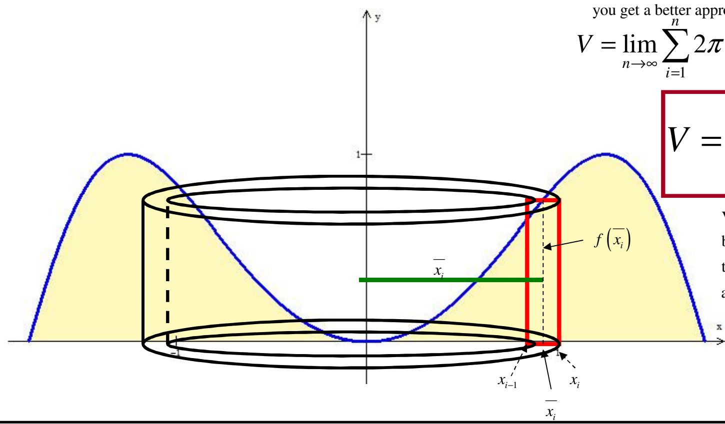
$$V \approx \sum_{i=1}^{n} V_i \approx \sum_{i=1}^{n} 2\pi\left(\overline{x_i}\right) \cdot f\left(\overline{x_i}\right) \cdot \Delta x$$

you get a better approx. as the number of shells $\to \infty$

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi\left(\overline{x_i}\right) \cdot f\left(\overline{x_i}\right) \cdot \Delta x$$

$$V = \int_a^b 2\pi x f(x)\, dx$$

Volume of the solid obtained by rotating the region under the curve $f(x)$ from $x = a$ to $x = b$ about the $y - axis$



Math 104 – Rimmer
6.2 Volumes by Cylindrical Shells

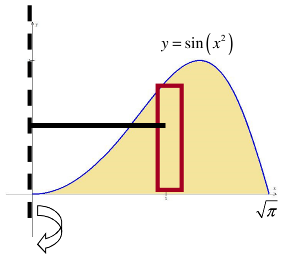
In general,

$$V = \int_a^b 2\pi(\text{radius})(\text{height})\, dx$$

- radius: distance from axis of rotation to a typical rectangle
- height: height of a typical rectangle
- dx: for a vertical axis of rotation; dy when the axis is horizontal

	Typical rectangle	Vertical axis of rotation	Horizontal axis of rotation
Disk or Washer	perpendicular to axis of rotation	Integral is dy	Integral is dx
Cylindrical Shells	parallel to axis of rotation	Integral is dx	Integral is dy

Math 104 – Rimmer
6.2 Volumes by Cylindrical Shells



radius $= x$

height $= \sin\left(x^2\right)$

$$V = \int_a^b 2\pi(\text{radius})(\text{height})\,dx$$

$$V = 2\pi \int_0^{\sqrt{\pi}} x\sin\left(x^2\right)dx$$

$$= 2\pi\left[-\tfrac{1}{2}\cos x^2\right]_0^{\sqrt{\pi}}$$

$$= -\pi\left[\cos\pi - \cos 0\right]$$

$$= -\pi\left[-1-1\right]$$

$$= \boxed{2\pi}$$

$u = x^2$

$du = 2xdx \quad \tfrac{1}{2}du = xdx$

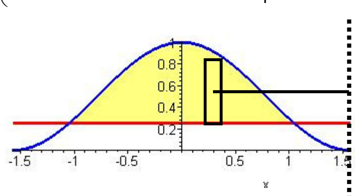
$\int \tfrac{1}{2}\sin u\,du = \; -\tfrac{1}{2}\cos u$

Math 104 – Rimmer
6.2 Volumes by Cylindrical Shells

Set up, but do not evaluate, an integral for the volume obtained by rotating the region bounded by

$y=\cos^2 x,\;\; y=\dfrac{1}{4},$ about the line $x=\dfrac{\pi}{2}$

$\left(\text{below } y=\cos^2 x \text{ and above } y=\frac{1}{4}, \text{from } -a \text{ to } a \text{ where these are the intersection pts. closest to the } y\text{-axis}\right)$



radius $= \dfrac{\pi}{2} - x$

height $= \cos^2 x - \dfrac{1}{4}$

limits of integration $\Rightarrow \quad x=\dfrac{\pi}{3} \quad x=-\dfrac{\pi}{3}$

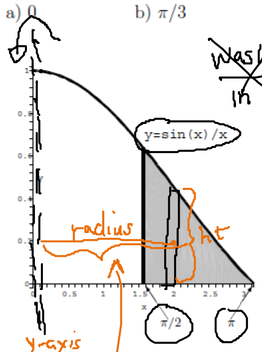
$\cos^2 x = \dfrac{1}{4} \Rightarrow \; \cos x = \dfrac{1}{2}$

$$V = \int_a^b 2\pi(\text{radius})(\text{height})\,dx$$

$$\boxed{V = \int_{-\pi/3}^{\pi/3} 2\pi\left(\frac{\pi}{2}-x\right)\left(\cos^2 x - \frac{1}{4}\right)dx}$$

2. Find the volume of the solid obtained by revolving the region below $y = \frac{\sin(x)}{x}$ and above the x-axis from $x = \pi/2$ to $x = \pi$ about the y-axis.

a) 0 b) $\pi/3$ c) $\pi/2$ d) π e) $3\pi/2$ f) 2π



~~Washer in y~~

Shell in x

radius = x

height = $\frac{\sin x}{x}$

$V = \int_a^b 2\pi \cdot (\text{rad.})(\text{ht.})\,dx$

$V = 2\pi \int_{\pi/2}^{\pi} x \cdot \frac{\sin x}{x}\,dx$

$V = 2\pi \int_{\pi/2}^{\pi} \sin x\,dx$

$V = 2\pi \left[-1 \cdot \cos x\right]_{\pi/2}^{\pi} = -2\pi\left[\cos\pi - \cos\frac{\pi}{2}\right]$ ($\cos\pi = -1$, $\cos\frac{\pi}{2} = 0$)

$= \boxed{2\pi}$

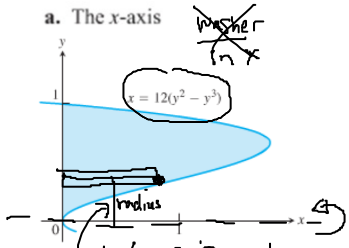
6.2 Set up

In Exercise 27, use the shell method to find the volumes of the solids generated by revolving the shaded regions about the indicated axes.

27. **a.** The x-axis **b.** The line $y = 1$

c. The line $y = 8/5$ **d.** The line $y = -2/5$

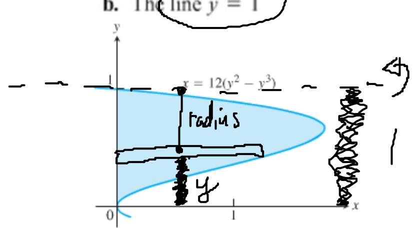
a. The x-axis ~~Washer in x~~



radius = y

height = $12(y^2 - y^3)$

b. The line $y = 1$



radius = $1 - y$

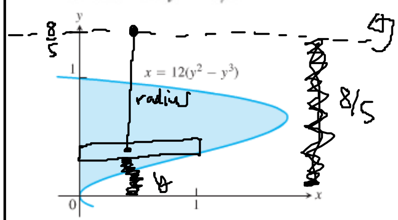
height = $12(y^2 - y^3)$

6.2 set up

In Exercise 27, use the shell method to find the volumes of the solids generated by revolving the shaded regions about the indicated axes.

27. a. The x-axis **b.** The line $y = 1$
c. The line $y = 8/5$ **d.** The line $y = -2/5$

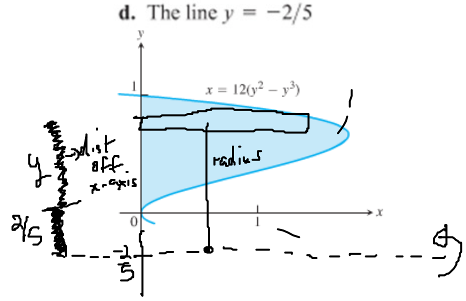
c. The line $y = 8/5$



radius = $\frac{8}{5} - y$

height = $12(y^2 - y^3)$

d. The line $y = -2/5$



radius = $\frac{2}{5} + y$

height = $12(y^2 - y^3)$

Website with volumes by shells animation:

http://mathdemos.gcsu.edu/mathdemos/shellmethod/gallery/gallery.html

Rihanna rides into the 2008 VMA Awards Show on a shell volume float

Type: “rihanna disturbia 2008 VMA performance” into the Google search, I think it’s the one on Vimeo