10.8 Taylor and Maclaurin Series

Suppose f is a function which has a power series representation

$$f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

for $|x-a| < R$

We can find the coefficients c_n in the following manner:

$$f(a) = c_0 + c_1 \underbrace{(a-a)}_{0} + c_2 \underbrace{(a-a)^2}_{0} + c_3 \underbrace{(a-a)^3}_{0} + \cdots \Rightarrow f(a) = c_0$$

Now let's take the derivative:

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$$

$$f'(a) = c_1 + 2c_2 \underbrace{(a-a)}_{0} + 3c_3 \underbrace{(a-a)^2}_{0} + 4c_4 \underbrace{(a-a)^3}_{0} + \cdots \Rightarrow f'(a) = c_1$$

Now let's take the second derivative:

$$f''(x) = 2c_2 + 2\cdot 3c_3(x-a) + 3\cdot 4c_4(x-a)^2 + \cdots$$

$$f''(a) = 2c_2 + 2\cdot 3c_3 \underbrace{(a-a)}_{0} + 3\cdot 4c_4 \underbrace{(a-a)^2}_{0} + 4c_5 \underbrace{(a-a)^3}_{0} + \cdots$$

$$\Rightarrow f''(a) = 2c_2$$

$$c_2 = \frac{f''(a)}{2}$$

Finally let's take the third derivative:

$$f'''(x) = 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4(x-a) + 3\cdot 4\cdot 5c_5(x-a)^2 + \cdots$$

$$f'''(a) = 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4(a-a) + 3\cdot 4\cdot 5c_5(a-a)^2 + \cdots \Rightarrow f'''(a) = 2\cdot 3c_3$$

$$c_3 = \frac{f'''(a)}{2\cdot 3}$$

Continuing in this manner, we'll obtain:

$$c_4 = \frac{f^{(4)}(a)}{2\cdot 3\cdot 4} \qquad c_5 = \frac{f^{(5)}(a)}{2\cdot 3\cdot 4\cdot 5} \qquad \cdots \qquad c_n = \frac{f^{(n)}(a)}{n!}$$

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots$$

Taylor series of the function f centered at a.

If $a=0$, then we call the series the **Maclaurin series** of the function f.

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots$$

Find the Maclaurin series for $f(x)=e^x$

$$\begin{array}{lll}
f(x)=e^x & f(0)=1 \ \div 0! & =1 \\
f'(x)=e^x & f'(0)=1 \ \div 1! & =1\,x \\
f''(x)=e^x & f''(0)=1 \ \div 2! & =\tfrac{1}{2!}x^2 \\
f'''(x)=e^x & f'''(0)=1 \ \div 3! & =\tfrac{1}{3!}x^3 \\
f^{(4)}(x)=e^x & f^{(4)}(0)=1 \ \div 4! & =\tfrac{1}{4!}x^4 \\
\vdots & \vdots & \vdots
\end{array}$$

$$\frac{f^{(n)}(0)}{n!}=\frac{1}{n!} \qquad \frac{f^{(n)}(0)}{n!}x^n=\frac{x^n}{n!}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\cdot\frac{x^{n+1}}{x^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{\cancel{n!}}{(n+1)\cdot\cancel{n!}}\cdot\frac{\cancel{x^n}\cdot x}{\cancel{x^n}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{x}{(n+1)}\right| = 0<1$$

for all x

$\Rightarrow R=\infty$

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots \quad \text{, with } R=\infty$$

Find the Maclaurin series for $f(x) = \sin x$

Math 104 – Rimmer
10.8-10.10
Taylor & Maclaurin Series

$$f(x) = \sin x \qquad f(0) = 0 \ \div 0! \ = 0$$
$$f'(x) = \cos x \qquad f'(0) = 1 \ \div 1! \ = 1\,x$$
$$f''(x) = -\sin x \qquad f''(0) = 0 \ \div 2! \ = 0\,x^2$$
$$f'''(x) = -\cos x \qquad f'''(0) = -1 \div 3! \ = -\tfrac{1}{3!}x^3$$
$$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = 0 \ \div 4! \ = 0\,x^4$$
$$f^{(5)}(x) = \cos x \qquad f^{(5)}(0) = 1 \ \div 5! \ = \tfrac{1}{5!}\,x^5$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2n+1)!}{(2(n+1)+1)!}\cdot\frac{x^{\overset{2n+3}{2(n+1)+1}}}{x^{2n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\cancel{(2n+1)!}}{(2n+3)(2n+2)\cdot\cancel{(2n+1)!}}\cdot\frac{\cancel{x^{2n+1}}\cdot x^2}{\cancel{x^{2n+1}}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^2}{(2n+3)(2n+2)}\right| = 0 < 1$$

for all x

$$\Rightarrow R = \infty$$

only odd powers so we should use $2n-1$ or $2n+1$

the first term (when $n=0$) is x^1 so we choose $2n+1$

$$\frac{f^{(n)}(0)}{n!}x^n = \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{, with } R = \infty$$

List of important Maclaurin series :

Math 104 – Rimmer
10.8-10.10
Taylor & Maclaurin Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad R = 1$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots \quad \text{with } R = 1$$

$$\ln(1-x) = -\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \quad \text{with } R = 1$$

$$\arctan x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad \text{,with } R = 1$$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad \text{,with } R = \infty$$

$$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{,with } R = \infty$$

$$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{,with } R = \infty$$

Find the third degree Taylor polynomial for $f(x)=\sqrt{x}$ centered at $x=1$.

$f(x)=x^{1/2}$ $\quad f(1)=1 \div 0! \quad =1$

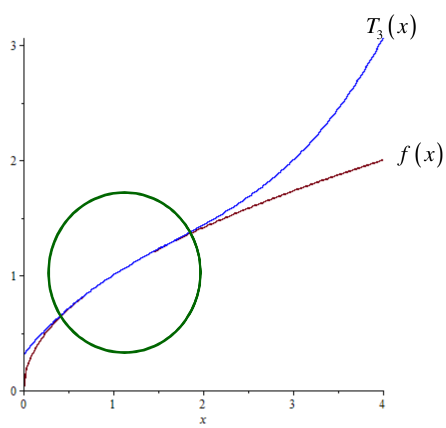
$f'(x)=\frac{1}{2}x^{-1/2}$ $\quad f'(1)=\frac{1}{2} \div 1! \quad =\frac{1}{2}(x-1)$

$f''(x)=\frac{-1}{4}x^{-3/2}$ $\quad f''(1)=\frac{-1}{4} \div 2! \quad =\frac{-1}{4\cdot 2!}=\frac{-1}{8}(x-1)^2$

$f'''(x)=\frac{3}{8}x^{-5/2}$ $\quad f'''(1)=\frac{3}{8} \div 3! \quad =\frac{3}{8\cdot 3!}=\frac{1}{16}(x-1)^3$

$$T_3(x)=1+\tfrac{1}{2}(x-1)-\tfrac{1}{8}(x-1)^2+\tfrac{1}{16}(x-1)^3 \approx \sqrt{x}$$

Math 104 – Rimmer
10.8-10.10
Taylor & Maclaurin Series



$T_3(1.5)=1.226562500$

$\sqrt{1.5}\approx 1.224744871$

$T_3(1.5)-\sqrt{1.5}=0.001817629$

10.9/10.10 Applications of Taylor and Maclaurin Series

Use a power series to integrate a function when there is no integration technique you could use.

Integrate and find the coefficient on x^{10}

$$\int x^3 e^{-x^3}\,dx$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

$$e^{-x^3} = 1 + \left(-x^3\right) + \frac{\left(-x^3\right)^2}{2} + \frac{\left(-x^3\right)^3}{6} + \dots$$

$$e^{-x^3} = 1 - x^3 + \frac{x^6}{2} - \frac{x^9}{6} + \dots$$

$$x^3 e^{-x^3} = x^3\left(1 - x^3 + \frac{x^6}{2} - \frac{x^9}{6} + \dots\right) = x^3 - x^6 + \frac{x^9}{2} - \frac{x^{12}}{6} + \dots$$

$$\int x^3 e^{-x^3}\,dx = \int\left(x^3 - x^6 + \frac{x^9}{2} - \frac{x^{12}}{6} + \dots\right)dx = \frac{x^4}{4} - \frac{x^7}{7} + \frac{x^{10}}{20} - \frac{x^{13}}{78} + \dots$$

coefficient on $x^{10} = \dfrac{1}{20}$

Math 104 – Rimmer
10.8-10.10
Taylor & Maclaurin Series

Use a power series to evaluate a limit.

$$\lim_{x\to 0}\frac{\sin x - x + \frac{x^3}{6}}{x^5}$$

$$\lim_{x\to 0}\frac{\left(\cancel{x} - \cancel{\frac{x^3}{3!}} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \cancel{-x} + \cancel{\frac{x^3}{6}}}{x^5} = \lim_{x\to 0}\frac{\frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots}{x^5}$$

$$= \lim_{x\to 0}\left(\frac{x^5}{5!\,x^5} - \frac{x^7}{7!\,x^5} + \frac{x^9}{9!\,x^5} - \frac{x^{11}}{11!\,x^5} + \cdots\right)$$

$$= \lim_{x\to 0}\left(\frac{1}{5!} - \underbrace{\frac{x^2}{7!}}_{0} + \underbrace{\frac{x^4}{9!}}_{0} - \underbrace{\frac{x^6}{11!}}_{0} + \cdots\right) = \boxed{\frac{1}{120}}$$

Use a power series to find the sum of a series.

$$\sum_{n=0}^{\infty}\frac{\pi^n}{n!}=1+\pi+\frac{\pi^2}{2!}+\frac{\pi^3}{3!}+\frac{\pi^4}{4!}+\cdots=\boxed{e^{\pi}}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{2^{2n+1}(2n+1)!}=\frac{\pi}{2}-\frac{\pi^3}{3!\cdot 8}+\frac{\pi^5}{5!\cdot 32}-\frac{\pi^7}{7!\cdot 128}+\cdots$$

$$=\left(\tfrac{\pi}{2}\right)-\frac{\left(\frac{\pi}{2}\right)^3}{3!}+\frac{\left(\frac{\pi}{2}\right)^5}{5!}-\frac{\left(\frac{\pi}{2}\right)^7}{7!}+\cdots=\sin\frac{\pi}{2}=\boxed{1}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots=\arctan 1=\boxed{\frac{\pi}{4}}$$

Math 104 – Rimmer
10.8-10.10
Taylor & Maclaurin Series

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\cdots=\frac{\pi}{4}$$

$$4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\cdots\right)=\pi$$

$$\pi=4-\frac{4}{3}+\frac{4}{5}-\frac{4}{7}+\frac{4}{9}-\frac{4}{11}+\frac{4}{13}-\frac{4}{15}+\cdots$$

This is a series that sums to π but it does it very slowly.

$S_{50}=3.121594653$

Find the first three non-zero terms of the Maclaurin series for $f\left(x\right)=e^{x}\ln\left(1-x\right)$ by **multiplying** two series.

$$e^{x}=\sum_{n=0}^{\infty}\frac{x^{n}}{n!}=1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\frac{x^{4}}{4!}+\cdots \text{ , with } R=\infty \text{ and } I=\left(-\infty,\infty\right)$$

$$\ln\left(1-x\right)=-\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}=-x-\frac{x^{2}}{2}-\frac{x^{3}}{3}-\frac{x^{4}}{4}-\cdots \text{ , with } R=1 \text{ and } I=|x|<1$$

$$e^{x}\ln\left(1-x\right)=\left(1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\frac{x^{4}}{4!}+\cdots\right)\left(-x-\frac{x^{2}}{2}-\frac{x^{3}}{3}-\frac{x^{4}}{4}-\cdots\right)$$

$$-x-\frac{x^{2}}{2}-\frac{x^{3}}{3}\cdots$$

$$-x^{2}-\frac{x^{3}}{2}\cdots$$

$$-\frac{x^{3}}{2}\cdots$$

$$e^{x}\ln\left(1-x\right)=-x-\frac{3}{2}x^{2}-\frac{4}{3}x^{3}-\cdots \text{ , with } R=1 \text{ and } I=|x|<1$$

the overlap b/w
the two intervals

Find the first three non-zero terms of the Maclaurin series for $f\left(x\right)=\tan x$ by **dividing** two series.

$$\tan x=\frac{\sin x}{\cos x}=\frac{x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-\frac{x^{7}}{7!}+\cdots}{1-\frac{x^{2}}{2!}+\frac{x^{4}}{4!}-\frac{x^{6}}{6!}+\cdots}$$

$$\begin{array}{r}
x+\frac{x^{3}}{3}+\frac{2x^{5}}{15}+\cdots \\
1-\frac{x^{2}}{2}+\frac{x^{4}}{24}-\frac{x^{6}}{720}+\cdots\ \Big)\ x-\frac{x^{3}}{6}+\frac{x^{5}}{120}-\frac{x^{7}}{5040}+\cdots \\
-\left(x-\frac{x^{3}}{2}+\frac{x^{5}}{24}-\frac{x^{7}}{720}+\cdots\right) \\
\hline
\frac{x^{3}}{3}-\frac{x^{5}}{30}+\frac{x^{7}}{840}+\cdots \\
-\left(\frac{x^{3}}{3}-\frac{x^{5}}{6}+\frac{x^{7}}{72}-\cdots\right) \\
\hline
\frac{2x^{5}}{15}-\frac{4x^{7}}{315}+\cdots
\end{array}$$

$$-\frac{x^{3}}{6}+\frac{x^{3}}{2}=\frac{-1+3}{6}x^{3}=\frac{x^{3}}{3}$$

$$\frac{x^{5}}{120}-\frac{x^{5}}{24}=\frac{1-5}{120}x^{5}=-\frac{x^{5}}{30}$$

$$-\frac{x^{7}}{5040}+\frac{x^{7}}{720}=\frac{-1+7}{5040}x^{7}=\frac{x^{5}}{840}$$

$$-\frac{x^{5}}{30}+\frac{x^{5}}{6}=\frac{-1+5}{30}x^{5}=\frac{2x^{5}}{15}$$

$$\frac{x^{7}}{840}-\frac{x^{7}}{72}=\frac{3-35}{2520}x^{7}=\frac{-4x^{7}}{315}$$

$$\tan x=x+\frac{x^{3}}{3}+\frac{2x^{5}}{15}+\cdots$$