10.7 Power Series

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

where:

$a)$ x is a variable

$b)$ The c_n's are constants called the coefficients of the series.

For each fixed x, the series above is a series of constants that we can test for convergence or divergence.

A power series may converge for some values of x and diverge for other values of x.

The sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots$$

whose domain is the set of all x for which the series converges.

$f(x)$ is reminiscent of a polynomial but it has infinitely many terms

If all c_n's $= 1$, we have

$$f(x) = 1 + x + x^2 + \ldots + x^n + \ldots = \sum_{n=0}^{\infty} x^n$$

This is the geometric series with $r = x$.

The power series will converge for $|x| < 1$ and diverge for all other x.

$$a = 1, r = x \Rightarrow s = \frac{a}{1-r} = \frac{1}{1-x} \qquad \boxed{\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \ldots = \sum_{n=0}^{\infty} x^n}$$

In general, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

is called a power series centered at a or a power series about a

We use the Ratio Test (or the Root Test) to find for what values of x the series converges.

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$ for convergence

solve for $|x-a|$ to get $|x-a| < R$

$\Rightarrow -R < x-a < R$

$\Rightarrow a-R < x < a+R$

R is called the **radius of convergence** (R.O.C.).

This is called the **interval of convergence** (I.O.C.).

Plug in the endpoints to check for convergence or divergence at the endpoints.

use square brackets [or]

use parentheses (or)

Find the radius of convergence and the interval of convergence.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2 x^n}{2^n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(-1)^n}\cdot\frac{(n+1)^2}{n^2}\cdot\frac{2^n}{2^{n+1}}\cdot\frac{x^{n+1}}{x^n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n\cdot(-1)}{(-1)^n}\cdot\frac{(n+1)^2}{n^2}\cdot\frac{2^n}{2^n\cdot 2}\cdot\frac{x^n\cdot x}{x^n}\right| = \left|\frac{-x}{2}\right|$$

($\frac{(n+1)^2}{n^2} \nearrow 1$)

For convergence, this limit needs to be less than 1

$$\left|\frac{-x}{2}\right| < 1 \;\Rightarrow \frac{1}{2}|x| < 1 \;\Rightarrow |x| < 2 \qquad \text{so, } -2 < x < 2$$

Now we need to solve this inequality for $|x|$.

This is the radius of convergence.

Plug in $x=2$ and $x=-2$ to see if there is conv. or div. at the endpoints.

$\underline{x=2}$

$$\sum_{n=1}^{\infty}\frac{(-1)^n n^2 2^n}{2^n} = \sum_{n=1}^{\infty}(-1)^n n^2$$

Diverges by the Test for Divergence since $\lim_{n\to\infty}(-1)^n n^2$ does not exist.

$\underline{x=-2}$

$$\sum_{n=1}^{\infty}\frac{(-1)^n n^2 (-2)^n}{2^n} = \sum_{n=1}^{\infty}\frac{\overbrace{((-1)\cdot(-2))^n}^{2^n} n^2}{2^n} = \sum_{n=1}^{\infty} n^2$$

Diverges by the Test for Divergence since $\lim_{n\to\infty} n^2 = \infty$.

Radius of convergence: $R = 2$

Interval of convergence: $(-2, 2)$

Find the radius of convergence and the interval of convergence.

$$\sum_{n=1}^{\infty} \frac{3^n (x+4)^n}{\sqrt{n}}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}}{3^n}\cdot\frac{\sqrt{n}}{\sqrt{n+1}}\cdot\frac{(x+4)^{n+1}}{(x+4)^n}\right| = \lim_{n\to\infty}\left|\frac{3^n\cdot 3}{3^n}\cdot\frac{\sqrt{n}}{\sqrt{n+1}}\cdot\frac{(x+4)^n(x+4)}{(x+4)^n}\right| = |3(x+4)|$$

($\frac{\sqrt{n}}{\sqrt{n+1}} \nearrow 1$)

For convergence, this limit needs to be less than 1

$$|3(x+4)| < 1 \Rightarrow 3|x+4| < 1 \Rightarrow |x+4| < \frac{1}{3}$$

Now we need to solve this inequality for $|x+4|$.

This is the radius of convergence.

$$\text{so, } -\frac{1}{3} < x+4 < \frac{1}{3}$$
$$-\frac{1}{3} - 4 < x < \frac{1}{3} - 4$$
$$\frac{-13}{3} < x < \frac{-11}{3}$$

Plug in $x = \frac{-13}{3}$ and $x = \frac{-11}{3}$ to see if there is conv. or div. at the endpoints.

$x = \frac{-13}{3}$

$$\sum_{n=1}^{\infty}\frac{3^n\left(\frac{-13}{3}+4\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{3^n\left(\frac{-1}{3}\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$$

Converges by the Alt. Series Test

$b_n = \frac{1}{\sqrt{n}}$ is decreasing and $\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0.$

$x = \frac{-11}{3}$

$$\sum_{n=1}^{\infty}\frac{3^n\left(\frac{-11}{3}+4\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{3^n\left(\frac{1}{3}\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$$

Divergent p – series with $p = \frac{1}{2}$.

R.O.C.: $R = \frac{1}{3}$

I.O.C. : $\left[\frac{-13}{3}, \frac{-11}{3}\right)$

Find the radius of convergence and the interval of convergence.

$$\sum_{n=1}^{\infty}\frac{(4x+1)^n}{n^2}$$

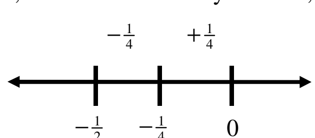
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2}\cdot\frac{(4x+1)^{n+1}}{(4x+1)^n}\right| = \lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2}\cdot\frac{(4x+1)^n\cdot(4x+1)}{(4x+1)^n}\right| = |4x+1|$$

($\frac{n^2}{(n+1)^2} \nearrow 1$)

For convergence, this limit needs to be less than 1

$|4x+1| < 1$ For this one, the value a isn't very obvious, so we will proceed as follows:

$$\left|4\left(x+\tfrac{1}{4}\right)\right| < 1$$
$$4\left|x+\tfrac{1}{4}\right| < 1$$
$$\left|x+\tfrac{1}{4}\right| < \tfrac{1}{4}$$



so in this case with $a = \frac{-1}{4}$ and the interval going from $\frac{-1}{2}$ to 0,

the radius of convergence is $R = \frac{1}{4}$

Check endpoints:

$x = \frac{-1}{2}$

$$\sum_{n=1}^{\infty}\frac{\left(4\left(\frac{-1}{2}\right)+1\right)^n}{n^2} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

convergent Alt. series

$x = 0$

$$\sum_{n=1}^{\infty}\frac{(1)^n}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

convergent p – series

R.O.C.: $R = \frac{1}{4}$

I.O.C. : $\left[\frac{-1}{2}, 0\right]$

Sometimes the Root Test can be used just as the Ratio Test.

When a_n can be written as $(b_n)^n$, then the Root Test should be used.

$$\sum_{n=1}^{\infty}\frac{3^n(x-5)^n}{n^n}=\sum_{n=1}^{\infty}\left(\frac{3(x-5)}{n}\right)^n$$

No value of x will make this limit >1 to give divergence

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{3(x-5)}{n}\right)^n\right|}=\lim_{n\to\infty}\left|\frac{3(x-5)}{n}\right|=0<1$$

We get convergence no matter what x is

R.O.C. $=\infty$
I.O.C. $=(-\infty,\infty)$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=0\Rightarrow R.O.C.=\infty\Rightarrow I.O.C.\ (-\infty,\infty)$$

$\left(\text{or}\lim_{n\to\infty}\sqrt[n]{|a_n|}=0\right)$

the power series converges for all x

$$\sum_{n=1}^{\infty}\frac{n!(x-7)^n}{2^n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(n+1)!}{n!}\cdot\frac{2^n}{2^{n+1}}\cdot\frac{(x-7)^{n+1}}{(x-7)^n}\right|=\lim_{n\to\infty}\left|\frac{(n+1)n!}{n!}\cdot\frac{2^n}{2^n\cdot 2}\cdot\frac{(x-7)^n\cdot(x-7)}{(x-7)^n}\right|$$

$$=\lim_{n\to\infty}\left|\tfrac{1}{2}(n+1)(x-7)\right|=\infty>1$$

R.O.C. $=0$
I.O.C. $=\{7\}$

No value of x will make this limit <1 to give convergence

We get divergence for all values of x except at $x=a$

at $x=a$, each term of the series is 0

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\infty\Rightarrow R.O.C.=0\Rightarrow I.O.C.\ \{a\}$$

$\left(\text{or}\lim_{n\to\infty}\sqrt[n]{|a_n|}=\infty\right)$

the power series only converges at the point $x=a$

Find the radius of convergence.

$$\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2 x^{2n}}{(2n)!}$$

$$[(n+1)n!]^2$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(-1)^n}\cdot\frac{[(n+1)!]^2}{(n!)^2}\cdot\frac{(2n)!}{[2(n+1)]!}\cdot\frac{x^{2(n+1)}}{x^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)^n(-1)}{(-1)^n}\cdot\frac{(n+1)^2(n!)^2}{(n!)^2}\cdot\frac{(2n)!}{(2n+2)(2n+1)(2n)!}\cdot\frac{x^{2n}x^2}{x^{2n}}\right| = \lim_{n\to\infty}\left|(-1)x^2\cdot\frac{n^2+2n+1}{4n^2+6n+2}\right| \quad \nearrow \frac{1}{4}$$

$$= \left|\frac{(-1)x^2}{4}\right|$$

For convergence, this limit needs to be less than 1

$$\left|\frac{(-1)x^2}{4}\right| < 1 \Rightarrow \frac{1}{4}\left|x^2\right| < 1 \Rightarrow |x|^2 < 4 \Rightarrow |x| < 2$$

This is the radius of convergence.

Radius of convergence: $R = 2$

Functions as Power Series

The very first function we have seen represented as a power series is the geometric series with $a = 1$ and $r = x$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad ,|x| < 1$$

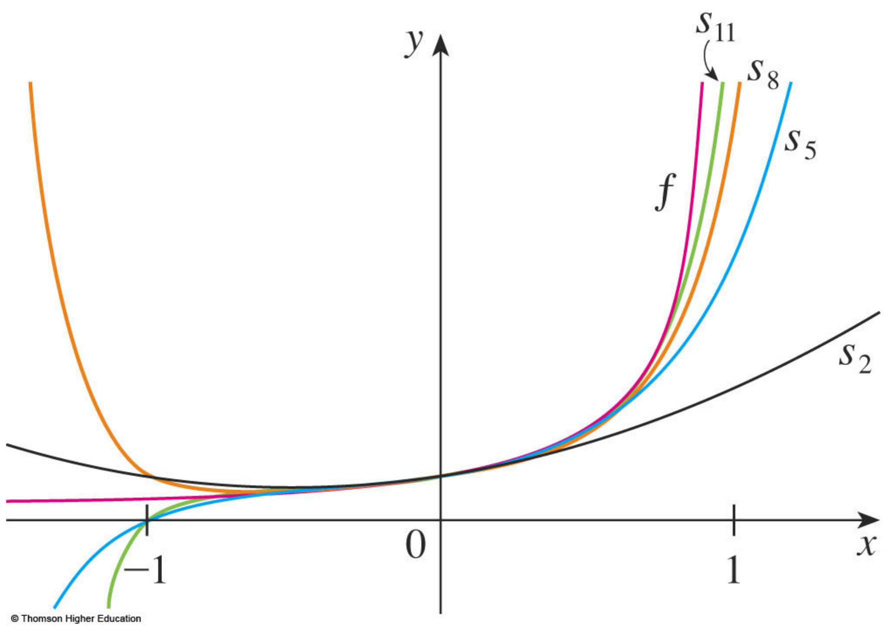
We can find the power series representation of other functions by algebraically manipulating them to to be some multiple of this series.

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n \quad ,|x| < 1$$

The interval of convergence remains unchanged since this is still a type of geometric series.

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n \quad ,|x| < 1$$

$$= 1 - x + x^2 - x^3 + \cdots$$



$$f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

If the power series representation of $f(x)$ has a radius of convergence $R>0$,

we can obtain a power series representation for $f'(x)$ by **term-by-term differentiation**:

$$f(x)=c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

$$f'(x)=c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

$$f'(x)=\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right)=\sum_{n=0}^{\infty}\frac{d}{dx}\left[c_n (x-a)^n\right]=\sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

starts at $n=1$

with the same radius of convergence R

we can obtain a power series representation for $\int f(x)\,dx$ by **term-by-term integration**:

$$\int f(x)\,dx = C + c_0 x + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + c_3\frac{(x-a)^4}{4} + \cdots$$

$$\int\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right)dx = \sum_{n=0}^{\infty}\int\left[c_n (x-a)^n\right]dx \; = C + \sum_{n=0}^{\infty}\frac{c_n (x-a)^{n+1}}{n+1}$$

with the same radius of convergence R

C is a constant of integration

$f(x)=\frac{1}{1-x}$ Find $f'(x)$.

$f(x)=(1-x)^{-1}$

$f'(x)=(-1)\cdot(1-x)^{-2}\cdot\underbrace{(-1)}_{\text{chain rule}}$

$f'(x)=\frac{1}{(1-x)^2}$

$$\frac{1}{1-x}=1+x+x^2+x^3+\cdots=\sum_{n=0}^{\infty}x^n \text{ ,convergent for } |x|<1$$

take the derivative

$$\frac{1}{(1-x)^2}=1+2x+3x^2+4x^3+\cdots=\sum_{n=1}^{\infty}nx^{n-1} \text{ ,convergent for } |x|<1$$

$f(x)=\frac{1}{1-x}$ Find $\int f(x)\,dx$.

$$\int\frac{1}{1-x}dx=-\ln(1-x)+C$$

$u=1-x$

$du=-dx$

$-1\cdot du=dx$

$=-1\cdot\int\frac{1}{u}du$

$=-\ln u$

$$\frac{1}{1-x}=1+x+x^2+x^3+\cdots=\sum_{n=0}^{\infty}x^n \text{ ,convergent for } |x|<1$$

$$-\ln(1-x)+C=x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots$$

plug in $x=0$ to find C

$$-\underbrace{\ln(1)}_{0}+C=0+0+0+\cdots\quad\Rightarrow C=0$$

$$-\ln(1-x)=x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots=\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1} \text{ ,convergent for } |x|<1$$

$$\frac{1}{1-x}=1+x+x^2+x^3+\cdots=\sum_{n=0}^{\infty}x^n \quad ,|x|<1$$

$$\frac{1}{1+x^2}=\frac{1}{1-\left(-x^2\right)}=1+\left(-x^2\right)+\left(-x^2\right)^2+\left(-x^2\right)^3+\cdots=\sum_{n=0}^{\infty}\left(-x^2\right)^n \quad ,|x|<1$$

$$\frac{1}{1+x^2}=1-x^2+x^4-x^6+\cdots=\sum_{n=0}^{\infty}(-1)^n x^{2n} \quad ,|x|<1$$

$$\int\frac{1}{1+x^2}\,dx=\arctan(x)+C$$

integrate

$$\arctan(x)+C= x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$$

plug in $x=0$ to find C $\quad \underbrace{\arctan(0)}_{0}+C=0+0+0+\cdots \Rightarrow C=0$

$$\arctan(x)=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1} \quad ,|x|<1$$

$$\frac{1}{1-x}=1+x+x^2+x^3+\cdots=\sum_{n=0}^{\infty}x^n \quad ,|x|<1$$

$$\frac{1}{(1-x)^2}=1+2x+3x^2+4x^3+\cdots=\sum_{n=1}^{\infty}nx^{n-1} \quad \text{,convergent for } |x|<1$$

$$-\ln(1-x)=x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots=\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1} \quad \text{,convergent for } |x|<1$$

$$\arctan(x)=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1} \quad ,|x|<1$$

Represent the function as a power series

$$f(x)=\frac{x^3}{(1-x)^2}=x^3\left[\frac{1}{(1-x)^2}\right]$$

$$\boxed{\frac{1}{(1-x)^2}=1+2x+3x^2+4x^3+\cdots=\sum_{n=1}^{\infty}nx^{n-1} \text{ ,convergent for } |x|<1}$$

$$x^3\left[\frac{1}{(1-x)^2}\right]=x^3\left(1+2x+3x^2+4x^3+\cdots\right)=x^3\cdot\sum_{n=1}^{\infty}nx^{n-1}$$

$$\frac{x^3}{(1-x)^2}=x^3+2x^4+3x^5+4x^6+\cdots=\sum_{n=1}^{\infty}nx^{n-1}\cdot x^3$$

$$x^{n-1}\cdot x^3=x^{n-1+3}=x^{n+2}$$

$$\frac{x^3}{(1-x)^2}=x^3+2x^4+3x^5+4x^6+\cdots=\sum_{n=1}^{\infty}nx^{n+2} \quad \text{with } R=1$$

Find the coeff. of x^6 in the power series representation of

$$\frac{3}{4+x^2}=3\left(\frac{1}{4+x^2}\right)=3\left(\frac{1}{4\left(1+\frac{x^2}{4}\right)}\right)=\frac{3}{4}\left(\frac{1}{1+\frac{x^2}{4}}\right)=\frac{3}{4}\left(\frac{1}{1-\left(-\frac{x^2}{4}\right)}\right)$$

$$\boxed{\frac{1}{1-x}=1+x+x^2+x^3+\cdots=\sum_{n=0}^{\infty}x^n \quad ,|x|<1}$$

$$\frac{3}{4}\left(\frac{1}{1-\left(-\frac{x^2}{4}\right)}\right)=\frac{3}{4}\left(1+\left(-\frac{x^2}{4}\right)+\left(-\frac{x^2}{4}\right)^2+\left(-\frac{x^2}{4}\right)^3+\cdots\right)$$

$$=\frac{3}{4}\left(1-\frac{x^2}{4}+\frac{x^4}{16}-\frac{x^6}{64}+\cdots\right)$$

$$=\frac{3}{4}-\frac{3}{16}x^2+\frac{3}{64}x^4-\frac{3}{256}x^6+\ldots$$

coeff on x^6

$$\boxed{-\frac{3}{256}}$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{,with } R = 1$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

$$\arctan\left(\frac{\sqrt{3}}{3}\right) = \frac{\sqrt{3}}{3} - \frac{\left(\frac{\sqrt{3}}{3}\right)^3}{3} + \frac{\left(\frac{\sqrt{3}}{3}\right)^5}{5} - \frac{\left(\frac{\sqrt{3}}{3}\right)^7}{7} + \ldots$$

$$\frac{\pi}{6} = \frac{\sqrt{3}}{3} - \frac{\left(\frac{\sqrt{3}}{3}\right)^3}{3} + \frac{\left(\frac{\sqrt{3}}{3}\right)^5}{5} - \frac{\left(\frac{\sqrt{3}}{3}\right)^7}{7} + \ldots$$

$$\pi = 6\left(\frac{\sqrt{3}}{3} - \frac{\left(\sqrt{3}\right)^3}{3 \cdot 3^3} + \frac{\left(\sqrt{3}\right)^5}{5 \cdot 3^5} - \frac{\left(\sqrt{3}\right)^7}{7 \cdot 3^7} + \ldots\right)$$

$$\boxed{\pi = 2\sqrt{3}\left(1 - \frac{\left(\sqrt{3}\right)^2}{3 \cdot 3^2} + \frac{\left(\sqrt{3}\right)^4}{5 \cdot 3^4} - \frac{\left(\sqrt{3}\right)^6}{7 \cdot 3^6} + \ldots\right)}$$

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \quad \text{,with } R = 1$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

$$\ln\left(1-\tfrac{1}{2}\right) = -\tfrac{1}{2} - \frac{\left(\frac{1}{2}\right)^2}{2} - \frac{\left(\frac{1}{2}\right)^3}{3} - \frac{\left(\frac{1}{2}\right)^4}{4} + \ldots$$

$$\ln\left(\tfrac{1}{2}\right) = -\tfrac{1}{2} - \frac{1}{2 \cdot 2^2} - \frac{1}{3 \cdot 2^3} - \frac{1}{4 \cdot 2^4} + \ldots$$

$$\ln 1 - \ln 2 = -\tfrac{1}{2} - \frac{1}{2 \cdot 2^2} - \frac{1}{3 \cdot 2^3} - \frac{1}{4 \cdot 2^4} + \ldots$$

$$\boxed{\ln 2 = \tfrac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \frac{1}{4 \cdot 2^4} + \ldots}$$