10.6 Alternating Series Test

An **alternating series** is of the form $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ or $\sum_{n=1}^{\infty} (-1)^n b_n$, (where $b_n > 0$)

(it has successive terms of opposite signs) $\qquad b_n = |a_n|$

Example: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \cdots$

Example: $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n+5} = -\frac{1}{6} + \frac{4}{7} - \frac{9}{8} + \frac{16}{9} - \cdots$

Forms for the term that makes the series alternate in sign:

$(-1)^{n-1} \qquad (-1)^n \qquad (-1)^{n+1}$

$\cos(n\pi) \qquad \sin\left((2n-1)\frac{\pi}{2}\right)$

The Alternating Series Test

If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ (where $b_n > 0$) satisfies:

$i)$ $\lim_{n\to\infty} b_n = 0$

$ii)$ $\{b_n\}$ is a decreasing sequence, and

,then the series is **convergent**.

<u>Note:</u>

$a)$ This test is for convergence only. It says nothing about divergence.

$b)$ Like the function in the Integral Test, the sequence $\{b_n\}$ needs to be decreasing "eventually" i.e., for all $n > N$ for some N

Example 1:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} \qquad b_n=\frac{1}{n} \qquad \lim_{n\to\infty}\frac{1}{n}=0$$

consider $f(x)=\frac{1}{x}$

$$f'(x)=\frac{-1}{x^2} \qquad f'(x)<0 \text{ for all positive } x \quad \Rightarrow \{b_n\} \text{ is decreasing}$$

$\underbrace{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}}$ is **convergent** by the Alternating Series Test

The Alternating Harmonic Series converges.

Example 2:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n^2}{n^2+5} \qquad b_n=\frac{n^2}{n^2+5} \qquad \lim_{n\to\infty}\frac{n^2}{n^2+5}=1$$

the Alternating Series Test Does not apply

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{(-1)^{n+1}n^2}{n^2+5}=\lim_{n\to\infty}(-1)^{n+1}\cdot\lim_{n\to\infty}\frac{n^2}{n^2+5}=\lim_{n\to\infty}(-1)^{n+1}\cdot 1 \Rightarrow \text{The limit does not exist.}$$

The series **diverges** by the Test For Divergence, since does not exist.

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and Absolute Convergence

Example 3:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\ln n}{n} \qquad b_n=\frac{\ln n}{n}$$

$$\lim_{n\to\infty}b_n=\lim_{n\to\infty}\frac{\ln n}{n}=\frac{"\infty"}{\infty} \quad \text{Indeterminate form} \Rightarrow \text{Use L'Hopistals Rule}$$

$$\overset{L'H}{=}\lim_{n\to\infty}\frac{\frac{1}{n}}{1}=0$$

consider $f(x)=\frac{\ln x}{x}$

$$f'(x)=\frac{x\cdot\frac{1}{x}-\ln x}{x^2}=\frac{1-\ln x}{x^2}$$

$f'(x)$ will be negative when $1-\ln x<0 \quad \Rightarrow \ln x>1$

$$e^{\ln x}>e^1$$

$$x>e$$

$\{b_n\}$ is decreasing for $n>2$

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\ln n}{n}$ is **convergent** by the Alternating Series Test

10.6 Absolute Convergence

An infinite series

$\sum_{n=1}^{\infty} a_n$ is called **absolutely convergent** if the positive series $\sum_{n=1}^{\infty} |a_n|$ converges.

Absolute convergence implies converges.
(If the series of absolute value converges, then the original series also converges)

If the series of absolute value **diverges**, it is still possible for the original series to converge.

Use the Alternating Series Test on the original series.
If the Alternating Series Test gives convergence, then this is a special type of convergence.

An infinite series

$\sum_{n=1}^{\infty} a_n$ is called **conditionally convergent** if it converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

$\sum_{n=1}^{\infty} |a_n|$ → **convergent**: $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent**

$\sum_{n=1}^{\infty} a_n$ is **convergent**

$\sum_{n=1}^{\infty} |a_n|$ → **divergent**

On $\sum_{n=1}^{\infty} a_n$ try :

a) the Alternating Series Test, or
b) the Test for Divergence

If the Alternating Series Test gives convergence,
$\sum_{n=1}^{\infty} a_n$ is **conditionally convergent**

If the Test for Divergence gives divergence,
$\sum_{n=1}^{\infty} a_n$ is **divergent**

A major difference between absolutely convergent and conditionally convergent comes in the rearrangement of the terms.

If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent with sum s,
then no matter how the terms are rearranged, the sum will always be s.

If $\sum_{n=1}^{\infty} a_n$ is conditionally convergent and r is any real number,
then there is a rearrangement of the sum $\sum_{n=1}^{\infty} a_n$ that has the sum r.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \ln 2 \quad \text{(We will show this later)}$$

$$\frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\right) = \frac{1}{2}\ln 2$$

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} + \cdots = \frac{1}{2}\ln 2$$

$$0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + \cdots = \frac{1}{2}\ln 2$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots = \ln 2$$

$$+ \quad 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 + -\frac{1}{8} + \cdots = \frac{1}{2}\ln 2$$

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots = \frac{3}{2}\ln 2$$

different sums

same terms

Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$i)\ \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n^3}}$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

convergent $p-$series

$\Rightarrow \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n^3}}$ is absolutely convergent

$ii)\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

divergent $p-$series

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ use $A.S.T.$:

$b_n = \frac{1}{\sqrt{n}}$ is decreasing,

and $\lim_{n\to\infty} b_n = 0$

convergent by $A.S.T.$

$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ is conditionally convergent

$iii)\ \sum_{n=1}^{\infty} \frac{(-4)^{n+1}}{3^n}$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n} = 4\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$$

divergent geom. series

$\sum_{n=1}^{\infty} \frac{(-4)^{n+1}}{3^n}$ use $T.F.D.$:

$\lim_{n\to\infty} \frac{(-4)^{n+1}}{3^n}$ does not exist

$\lim_{n\to\infty} \frac{(-4)^{n+1}}{3^n} = -\infty$, for n even

$\lim_{n\to\infty} \frac{(-4)^{n+1}}{3^n} = \infty$, for n odd

$\Rightarrow \sum_{n=1}^{\infty} \frac{(-4)^{n+1}}{3^n}$ is divergent