10.5 The Ratio Test

Let $\{a_n\}$ be a sequence and assume that the following limit exists: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$

$i)$ If $L<1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

$ii)$ If $L>1$ or if the limit is infinite, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

$iii)$ If $L=1$, the Ratio Test is inconclusive.

(the series could be absolutely convergent, conditionally convergent, or divergent)

10.5 The Root Test

Let $\{a_n\}$ be a sequence and assume that the following limit exists: $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L$

$i)$ If $L<1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

$ii)$ If $L>1$ or if the limit is infinite, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

$iii)$ If $L=1$, the Root Test is inconclusive.

(the series could be absolutely convergent, conditionally convergent, or divergent)

Determine whether the series is convergent or divergent.

$i)\ \sum_{n=1}^{\infty} \frac{n^3}{4^n}$

$$i)\ \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)^3}{4^{n+1}}}{\frac{n^3}{4^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^3}{n^3}\cdot\frac{4^n}{4^{n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^3}{n^3}\cdot\frac{4^n}{4\cdot 4^n}\right| = \frac{1}{4}$$

$\nearrow 1$

$\sum_{n=1}^{\infty} \frac{n^3}{4^n}$ is convergent

Determine whether the series is convergent or divergent.

$ii)\ \sum_{n=1}^{\infty} \frac{n!}{4^n}$

$$ii)\ \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{4^{n+1}}}{\frac{n!}{4^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)!}{n!}\cdot\frac{4^n}{4^{n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)\,n!}{n!}\cdot\frac{4^n}{4\cdot 4^n}\right| = \lim_{n\to\infty}\frac{n+1}{4} = \infty$$

$\sum_{n=1}^{\infty} \frac{n!}{4^n}$ is divergent

Determine whether the series is convergent or divergent.

$iii)\ \sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}}$

$$iii)\ \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(2n+1)^n}{(n^2)^n}\right|}$$

$$= \lim_{n\to\infty} \left[\left(\frac{2n+1}{n^2}\right)^n\right]^{1/n} = \lim_{n\to\infty} \frac{2n+1}{n^2} = 0 \qquad \deg(num.) < \deg.(denom.)$$

$$\boxed{\sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}} \text{ is convergent}}$$

Determine whether the series is convergent or divergent.

$iv)\ \sum_{n=1}^{\infty} \frac{n^2}{(2n+1)!}$

$$iv)\ \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(n+1)^2}{(2[n+1]+1)!}}{\frac{n^2}{(2n+1)!}}\right|$$

$$= \lim_{n\to\infty} \left|\frac{(n+1)^2}{n^2} \cdot \frac{(2n+1)!}{(2n+3)!}\right| \qquad = \lim_{n\to\infty} \left|\frac{(n+1)^2}{n^2} \cdot \frac{(2n+1)!}{(2n+3)\cdot(2n+2)\,(2n+1)!}\right|$$

$$= \lim_{n\to\infty} \frac{1}{(2n+3)\cdot(2n+2)} = 0$$

$$\boxed{\sum_{n=1}^{\infty} \frac{n^2}{(2n+1)!} \text{ is convergent}}$$

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10.5 Ratio and Root Test

Determine whether the series is convergent or divergent.

$$v)\ \sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}$$

$$v)\ \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{[(n+1)!]^2}{(2[n+1])!}}{\frac{(n!)^2}{(2n)!}}\right| = \lim_{n\to\infty}\left|\frac{[(n+1)!]^2}{(2[n+1])!}\cdot\frac{(2n)!}{(n!)^2}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)!}{n!}\cdot\frac{(n+1)!}{n!}\cdot\frac{(2n)!}{(2n+2)!}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\cdot n!}{n!}\cdot\frac{(n+1)\cdot n!}{n!}\cdot\frac{(2n)!}{(2n+2)(2n+1)(2n)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)(n+1)}{(2n+2)(2n+1)}\right| = \frac{1}{4}$$

$$\boxed{\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}\text{ is convergent}}$$

Math 104 – Rimmer
10.5 Ratio and Root Test

Determine whether the series is convergent or divergent.

$$vi)\ \sum_{n=1}^{\infty}\frac{n!}{n^n}$$

$$vi)\ \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{n!}\cdot\frac{n^n}{(n+1)^{n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)\cdot n!}{n!}\cdot\frac{n^n}{(n+1)^n\,(n+1)}\right| = \lim_{n\to\infty}\frac{n^n}{(n+1)^n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \text{"}1^{\infty}\text{"}$$

$$y = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n$$

$$\ln y = \lim_{n\to\infty}\ln\left(\frac{n}{n+1}\right)^n$$

$$\ln y = \lim_{n\to\infty} n\cdot\ln\left(\frac{n}{n+1}\right) = \text{"}\infty\cdot 0\text{"}$$

$$\ln y = \lim_{n\to\infty}\frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{n}} = \frac{\text{"}0\text{"}}{0}$$

$$\ln y \overset{L'H}{=} \lim_{n\to\infty}\frac{\frac{1}{\frac{n}{n+1}}\cdot\frac{(n+1)\cdot 1 - n\cdot 1}{(n+1)^2}}{\frac{-1}{n^2}}$$

$$\ln y = \lim_{n\to\infty}\frac{\frac{n+1}{n}\cdot\frac{n+1-n}{(n+1)^2}}{\frac{-1}{n^2}}$$

$$\ln y = \lim_{n\to\infty}\frac{\frac{n+1}{n}\cdot\frac{1}{(n+1)^2}}{\frac{-1}{n^2}}$$

$$\ln y = \lim_{n\to\infty}\frac{1}{n(n+1)}\cdot\frac{-n^2}{1}$$

$$\ln y = \lim_{n\to\infty}\frac{-n^2}{n^2+n}$$

$$\ln y = -1$$

$$e^{\ln y} = e^{-1}$$

$$y = e^{-1}$$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{1}{e} < 1$$

$$\boxed{\sum_{n=1}^{\infty}\frac{n!}{n^n}\text{ is convergent}}$$