# 10.3 The Integral Test



If $f(x)$ is: $a)$ continuous, on the interval $[k,\infty)$ (constant $k>0$)

$b)$ positive,

$c)$ and decreasing

,then the series $\sum_{n=k}^{\infty} a_n$ $\left(\text{with } a_n = f(n)\right)$

$i)$ is convergent when $\int_k^{\infty} f(x)\,dx$ is convergent.

$ii)$ is divergent when $\int_k^{\infty} f(x)\,dx$ is divergent.

Note:

the function does not necessarily have to be decreasing for all $x \in [k,\infty)$

as long as the function is decreasing "eventually"

(there is some number $N$ so that $f$ is decreasing for all $x > N$)

The next two slides give you a feeling of **how** the integral test works.

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10.3 Integral Test

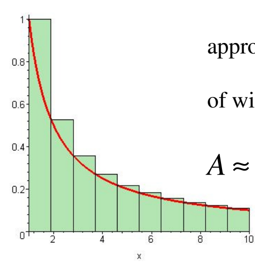
$f(x) = \dfrac{1}{x}$

on $[1,\infty)$

$a)$ continuous,

$b)$ positive,

$c)$ and decreasing




approximate the area $\int_1^{\infty} \frac{1}{x}\,dx$ with rectangles

of width 1 using the left endpoint

$$A \approx 1(1) + 1\left(\tfrac{1}{2}\right) + 1\left(\tfrac{1}{3}\right) + \cdots = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots$$

$$A \approx \sum_{n=1}^{\infty} \frac{1}{n}$$ but this is an **overestimate**

$$\Rightarrow \int_1^{\infty} \frac{1}{x}\,dx < \sum_{n=1}^{\infty} \frac{1}{n}$$

But $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x}\,dx = \lim_{b\to\infty} \ln x\Big|_1^b = \lim_{b\to\infty} \ln b = \infty$

The integral $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx$ diverges and $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx < \sum_{n=1}^{\infty} \frac{1}{n}$

$\Rightarrow$ The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ must also diverge

the **harmonic series**

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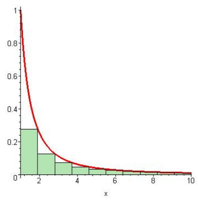
$f(x) = \frac{1}{x^2}$

on $[1, \infty)$

$a)$ continuous,
$b)$ positive,
$c)$ and decreasing




approximate the area $\int_1^\infty \frac{1}{x^2}\,dx$ with rectangles of width 1 using the right endpoint

$$A \approx 1\left(\tfrac{1}{4}\right) + 1\left(\tfrac{1}{9}\right) + 1\left(\tfrac{1}{16}\right) + \cdots = \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \cdots$$

$$A \approx \sum_{n=2}^{\infty} \frac{1}{n^2}$$ but this is an **underestimate**

$$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n^2} < \int_1^\infty \frac{1}{x^2}\,dx \quad \Rightarrow 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} < 1 + \int_1^\infty \frac{1}{x^2}\,dx$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + \int_1^\infty \frac{1}{x^2}\,dx$$

But $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b x^{-2}\,dx = \lim_{b\to\infty} \left.\frac{-1}{x}\right|_1^b = \lim_{b\to\infty} \frac{-1}{b} + 1 = 1$

The integral $\int_1^\infty \frac{1}{x^2}\,dx$ converges and $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} < 2$

(The sequence of partial sums $S_n$ is a bounded increasing sequence $\Rightarrow$ this sequence converges)

$\Rightarrow$ The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ also converges

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$f(x) = \frac{1}{x^p}$

on $[1, \infty)$

$a)$ continuous,
$b)$ positive,
$c)$ and decreasing

For what values of $p$ does the integral converge?

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b x^{-p}\,dx = \lim_{b\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right|_1^b$$

need $-p+1$ to be negative so that we can get convergence by moving the $x-$term to the denominator

$$-p+1 < 0 \Rightarrow \boxed{p > 1}$$

corresponding to this function is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

this is called a $p-$series

$i)$ $\sum_{n=1}^{\infty} \frac{1}{n^p}$ **converges** when $p > 1$

$ii)$ $\sum_{n=1}^{\infty} \frac{1}{n^p}$ **diverges** when $p \le 1$

Which of these converge?



$a)\ \sum_{n=1}^{\infty}\frac{1}{n^{5/2}}$ $\quad b)\ \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ $\quad c)\ \sum_{n=1}^{\infty}\frac{3}{2n^3}$ $\quad d)\ \sum_{n=1}^{\infty}n^{-e}$

$a)\ \sum_{n=1}^{\infty}\frac{1}{n^{5/2}}$ **converges** $p-$series with $p=2.5$

$b)\ \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ **diverges** $p-$series with $p=\frac{1}{2}$

$c)\ \sum_{n=1}^{\infty}\frac{3}{2n^3}=\frac{3}{2}\sum_{n=1}^{\infty}\frac{1}{n^3}$ **converges** $p-$series with $p=3$

$d)\ \sum_{n=1}^{\infty}n^{-e}=\sum_{n=1}^{\infty}\frac{1}{n^e}$ **converges** $p-$series with $p=e$

Which of these converge?



$a)\ \sum_{n=1}^{\infty}\frac{1}{n^2+4}$ $\quad b)\ \sum_{n=2}^{\infty}\frac{1}{n\ln n}$ $\quad c)\ \sum_{n=2}^{\infty}\frac{\ln n}{n^2}$

$a)\ \sum_{n=1}^{\infty}\frac{1}{n^2+4}$ $\quad f(x)=\frac{1}{x^2+4}$ continuous,positive, and decreasing on $[1,\infty)$

$$\int_1^{\infty}\frac{1}{x^2+4}dx=\lim_{b\to\infty}\int_1^{b}\frac{1}{x^2+4}dx=\lim_{b\to\infty}\frac{1}{2}\arctan\left(\frac{x}{2}\right)\Big|_1^b$$

$$=\lim_{b\to\infty}\frac{1}{2}\arctan\left(\frac{b}{2}\right)-\arctan\left(\frac{1}{2}\right) \qquad \left(\arctan x\to\frac{\pi}{2}\text{ as }x\to\infty\right)$$

$$=\frac{\pi}{4}-\frac{1}{2}\arctan\left(\frac{1}{2}\right) \quad \Rightarrow \text{ the integral converges}$$

so, $\sum_{n=1}^{\infty}\frac{1}{n^2+4}$ **converges** by the integral test.

Which of these converge?

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$$a)\ \sum_{n=1}^{\infty}\frac{1}{n^2+4} \qquad b)\ \sum_{n=2}^{\infty}\frac{1}{n\ln n} \qquad c)\ \sum_{n=2}^{\infty}\frac{\ln n}{n^2}$$

$b)\ \sum_{n=2}^{\infty}\frac{1}{n\ln n}$ $\quad f(x)=\frac{1}{x\ln x}$ continuous,positive, and decreasing on $[2,\infty)$

$\left(f'(x)=\frac{-[\ln(x)+1]}{(x\ln x)^2}\text{ always negative on }[2,\infty)\right)$

$$\int_2^{\infty}\frac{1}{x\ln x}dx=\lim_{b\to\infty}\int_2^b\frac{1}{x\ln x}dx=\lim_{b\to\infty}\ln(\ln x)\Big|_2^b=\lim_{b\to\infty}\ln(\ln b)-\ln(\ln 2)$$

$u=\ln x$
$du=\frac{1}{x}dx \quad \int\frac{1}{u}du=\ln|u|+C$

$=\infty$ $\quad (\ln x\to\infty \text{ as } x\to\infty)$

$\Rightarrow$ the integral diverges

so, $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ **diverges** by the integral test.

Which of these converge?

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$$a)\ \sum_{n=1}^{\infty}\frac{1}{n^2+4} \qquad b)\ \sum_{n=2}^{\infty}\frac{1}{n\ln n} \qquad c)\ \sum_{n=2}^{\infty}\frac{\ln n}{n^2}$$

$f'(x)=\frac{x-2x\ln(x)}{x^4}=\frac{1-2\ln(x)}{x^3}$
$1-2\ln x<0\Rightarrow\ln x>\frac{1}{2}\Rightarrow x>e^{1/2}$
$f$ is decreasing for $x>\sqrt{e}\sim1.65$

$c)\ \sum_{n=2}^{\infty}\frac{\ln n}{n^2}$ $\quad f(x)=\frac{\ln x}{x^2}$ continuous,positive, and decreasing on $[2,\infty)$

$$\int_2^{\infty}\frac{\ln x}{x^2}dx=\lim_{b\to\infty}\int_2^b\frac{\ln x}{x^2}dx=\lim_{b\to\infty}\frac{-(1+\ln x)}{x}\Bigg|_2^b=\lim_{b\to\infty}\frac{-(1+\ln b)}{b}+\frac{1+\ln 2}{2}=\frac{1+\ln 2}{2}$$

$u=\ln x \quad dv=\frac{1}{x^2}dx$
$du=\frac{1}{x}dx \quad v=\frac{-1}{x}$
$uv-\int vdu=\frac{-1}{x}\ln x+\int\frac{1}{x^2}dx$
$=\frac{-1}{x}\ln x+\frac{-1}{x}$
$=\frac{-(1+\ln x)}{x}$

since, $\lim_{b\to\infty}\frac{-(\ln b+1)}{b}=\frac{-\infty}{\infty}\overset{L'H}{=}\lim_{b\to\infty}\frac{-\frac{1}{b}}{1}=0$

$\Rightarrow$ the integral converges

so, $\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$ **converges** by the integral test.