10.2 Series

We will now add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$

to get $\underbrace{a_1 + a_2 + a_3 + \cdots + a_n + a_{n+1} + \cdots}$

this is called an infinite **series**

Notation:

$$\sum_{n=1}^{\infty} a_n$$

Example:

$$2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81} + \cdots + \frac{2}{3^{n-1}} + \cdots$$

S_n = the sum of the first n terms

it is called the nth partial sum

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

$S_1 = 2$

$S_2 = 2 + \frac{2}{3} = \frac{8}{3}$

$S_3 = 2 + \frac{2}{3} + \frac{2}{9} = \frac{26}{9}$

$S_4 = 2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} = \frac{80}{27}$

The partial sums form a sequence $\{S_n\}_{n=1}^{\infty}$

$$\{S_n\}_{n=1}^{\infty} = \left\{2, \frac{8}{3}, \frac{26}{9}, \frac{80}{27}, \cdots\right\}$$

n	S_n
1	2
2	2.66666
3	2.88888
4	2.96296
5	2.98765
10	2.99995
15	2.99999
20	2.99999
25	**2.99999**

$\lim_{n\to\infty} S_n = s$ (the limit of the sequence of partial sums exists and is finite) $\Rightarrow$ We call s the **sum** of the infinite series

$$\sum_{n=1}^{\infty} a_n = s$$

and the series is called **convergent**

(by adding sufficiently many terms of the series, we can get as close as we like to the number s.)

otherwise the series is called **divergent**

The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
We will show this in 10.3

$\{S_n\}_{n=1}^{\infty} = \left\{2, \frac{8}{3}, \frac{26}{9}, \frac{80}{27}, \cdots\right\}$ It seems like $\lim_{n\to\infty} S_n = 3$

$$\Rightarrow 2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^{n-1}} + \cdots = \sum_{n=1}^{\infty} \frac{2}{3^{n-1}} = 3$$

We can show that the sum is 3 since this series is an example of a special type of series called a geometric series.

A **geometric series** is one in which each term is obtained from the preceding one by multiplying it by the common ratio r.

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

this only converges for certain values of r.

$\underline{r=1}$

$S_n = a + a + a + a + \cdots = na \Rightarrow \lim_{n\to\infty} na = \infty$ so it diverges for $r = 1$

$\underline{r=-1}$

$S_n = a - a + a - a + \cdots = (-1)^{n-1} a \Rightarrow \underset{\left(\substack{\text{it could be } a\text{, it could be 0} \\ \text{depending on the value of } n}\right)}{\lim_{n\to\infty} (-1)^{n-1} a}$ does not exist, so it diverges for $r = -1$

$\underline{r \neq \pm 1}$

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = a\left(1 + \cancel{r} + \cancel{r^2} + \cancel{r^3} + \cdots + \cancel{r^{n-1}}\right)$$

$$-rS_n = -ar - ar^2 - ar^3 - \cdots - ar^n = -a\left(\cancel{r} + \cancel{r^2} + \cancel{r^3} + \cdots + r^n\right)$$

$$S_n - rS_n = \qquad a\left(1 - r^n\right)$$

$$\Rightarrow S_n(1-r) = a\left(1-r^n\right)$$

$$\Rightarrow S_n = \frac{a\left(1-r^n\right)}{1-r}$$

$$\text{so, } \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a\left(1-r^n\right)}{1-r} = \frac{a\left(1-\lim_{n\to\infty} r^n\right)}{1-r} = \frac{a\left(1-\cancel{\lim_{n\to\infty} r^n}^{=0}\right)}{1-r} = \frac{a}{1-r}$$

if $-1 < r < 1$

We saw in section 10.1 :

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

$$\lim_{n\to\infty} r^n = \begin{cases} \text{undefined} & \text{if } r \in (-\infty, -1] \\ \infty & \text{if } r \in (1, \infty) \end{cases}$$

so, $\lim_{n\to\infty} S_n = \dfrac{a}{1-r}$ provided that $-1 < r < 1$ or $|r| < 1$.

The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges to the sum of $\dfrac{a}{1-r}$ if $|r| < 1$

The geometric series diverges for all other values of r

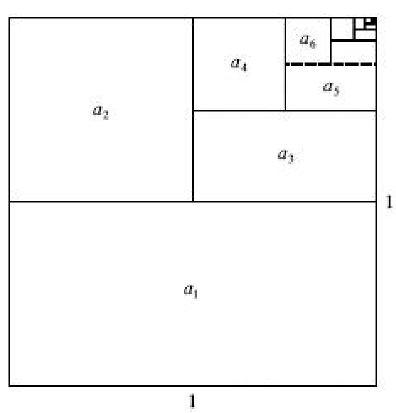
Back to our example:

$$2+\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\cdots+\frac{2}{3^{n-1}}+\cdots = 2\left(1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots+\frac{1}{3^{n-1}}+\cdots\right)$$

$a =$ the first term $\qquad a=2$

$r =$ the ratio b/w the terms $\qquad r=\frac{1}{3} \qquad |r|<1$

$$s=\frac{a}{1-r}=\frac{2}{1-\frac{1}{3}}=\frac{2}{\frac{2}{3}}=2\cdot\frac{3}{2}=\boxed{3}$$



Area of square $=1$

sum of the series should also be 1

Find $\sum_{n=1}^{\infty} a_n$. $\qquad a=\frac{1}{2}, r=\frac{1}{2} \qquad s=\frac{a}{1-r}=\frac{\frac{1}{2}}{1-\frac{1}{2}}=1$

$$\sum_{n=1}^{\infty} a_n=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^n}+\cdots=1$$

Represent $2.\overline{15}$ as an improper fraction by using a geometric series.

$$2.\overline{15} = 2.151515\ldots$$

$$= 2 + \underbrace{\frac{15}{100} + \frac{15}{10,000} + \frac{15}{1,000,000} +}\cdots$$

a geometric series with $a = \frac{15}{100}$ and $r = \frac{1}{100}$

$$s = \frac{a}{1-r} = \frac{\frac{15}{100}}{1-\frac{1}{100}} = \frac{\frac{15}{100}}{\frac{99}{100}} = \frac{15}{99} = \frac{5}{33}$$

$$2.\overline{15} = 2\frac{5}{33} = \frac{66+5}{33}$$

$$2.\overline{15} = \boxed{\frac{71}{33}}$$

A **telescoping series** is one in which the middle terms cancel and the sum collapses into just a few terms.

$$\frac{3}{n(n+3)} = \frac{A}{n} + \frac{B}{n+3}$$

$$3 = A(n+3) + Bn$$

$$n = 0 \Rightarrow A = 1$$

$$n = -3 \Rightarrow B = -1$$

Example:

$$\sum_{n=1}^{\infty} \frac{3}{n(n+3)} = \sum_{n=1}^{\infty}\left[\frac{1}{n} - \frac{1}{n+3}\right]$$

$$S_n = \underbrace{\left(1-\cancel{\frac{1}{4}}\right)}_{n=1} + \underbrace{\left(\frac{1}{2}-\cancel{\frac{1}{5}}\right)}_{n=2} + \underbrace{\left(\frac{1}{3}-\cancel{\frac{1}{6}}\right)}_{n=3} + \underbrace{\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{7}}\right)}_{n=4} + \underbrace{\left(\cancel{\frac{1}{5}}-\cancel{\frac{1}{8}}\right)}_{n=5} + \underbrace{\left(\cancel{\frac{1}{6}}-\cancel{\frac{1}{9}}\right)}_{n=6} + \cdots$$

$$+ \underbrace{\left(\cancel{\frac{1}{n-2}} - \frac{1}{n+1}\right)}_{(n-2)\text{ term}} + \underbrace{\left(\cancel{\frac{1}{n-1}} - \frac{1}{n+2}\right)}_{(n-1)\text{ term}} + \underbrace{\left(\cancel{\frac{1}{n}} - \frac{1}{n+3}\right)}_{\text{nth term}}$$

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}$$

so, $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[1 + \frac{1}{2} + \frac{1}{3} - \underbrace{\cancel{\frac{1}{n+1}}}_{0} - \underbrace{\cancel{\frac{1}{n+2}}}_{0} - \underbrace{\cancel{\frac{1}{n+3}}}_{0}\right] = \frac{6+3+2}{6} = \boxed{\frac{11}{6}}$

36. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{n^2}-\frac{3}{(n+1)^2}\right)$

$$S_n = \left(\boxed{3}-\cancel{\frac{3}{4}}\right)+\left(\cancel{\frac{3}{4}}-\cancel{\frac{3}{9}}\right)+\left(\cancel{\frac{3}{9}}-\cancel{\frac{3}{16}}\right)+\left(\cancel{\frac{3}{16}}-\cancel{\frac{3}{25}}\right)+\cdots$$

$n=1$ $n=2$ $n=3$ $n=4$

$$\left(\cancel{\frac{3}{(n-1)^2}}-\cancel{\frac{3}{n^2}}\right)+\left(\cancel{\frac{3}{n^2}}-\boxed{\frac{3}{(n+1)^2}}\right)$$

stop

$$S_n = 3-\frac{3}{(n+1)^2}$$

$$\lim_{n\to\infty} 3-\frac{3}{(n+1)^2}\to 0 = \boxed{3}$$

47. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{\ln(n+2)}-\frac{1}{\ln(n+1)}\right)$

$$S_n = \left(\cancel{\frac{1}{\ln 3}}-\frac{1}{\ln 2}\right)+\left(\cancel{\frac{1}{\ln 4}}-\cancel{\frac{1}{\ln 3}}\right)+\left(\cancel{\frac{1}{\ln 5}}-\cancel{\frac{1}{\ln 4}}\right)+\left(\frac{1}{\ln 6}-\cancel{\frac{1}{\ln 5}}\right)+\cdots$$

$n=1$ $n=2$ $n=3$ $n=4$

$$(\;)\;\;\boxed{+\left(\cancel{\frac{1}{\ln(n+1)}}-\cancel{\frac{1}{\ln n}}\right)+\left(\frac{1}{\ln(n+2)}-\cancel{\frac{1}{\ln(n+1)}}\right)}$$

$$S_n = \frac{-1}{\ln 2}+\frac{1}{\ln(n+2)} \qquad S=\lim_{n\to\infty}\frac{-1}{\ln 2}+\frac{1}{\ln(n+2)}\to 0 = \frac{-1}{\ln 2}$$

Math 104 – Rimmer
10.2 Series

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Converse :

If $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent. This is **False**!

$\left(\text{Just because } \lim_{n\to\infty} a_n = 0, \text{ you } \textbf{cannot} \text{ conclude that the series } \sum_{n=1}^{\infty} a_n \text{ is convergent.}\right)$

$$\boxed{\lim_{n\to\infty} \frac{1}{n} = 0, \text{ but } \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}}$$

Contrapositive :

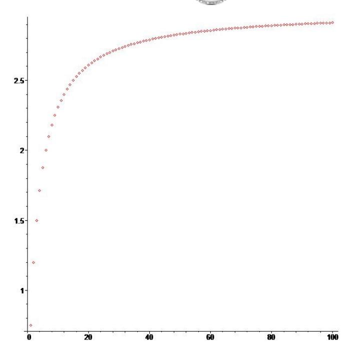
Test for Divergence :

If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

This is **True**!

Math 104 – Rimmer
10.2 Series

$$\sum_{n=1}^{\infty} \frac{3n^2}{n(n+3)} = \frac{3}{4} + \frac{6}{5} + \cdots + \underbrace{\frac{150}{53}}_{n=50} + \cdots + \underbrace{\frac{300}{103}}_{n=100} + \cdots$$



There is no way that this sum could converge to some finite number since the terms approach 3 as $n \to \infty$.

sum $= 0.75 + 1.2 + \cdots + 3 + 3 + 3 + \cdots + 3 + 3 + 3 + \cdots$

$$\sum_{n=1}^{\infty} \frac{3n^2}{n(n+3)} \qquad \lim_{n\to\infty} \frac{3n^2}{n(n+3)} = 3 \neq 0$$

so this series diverges by the **Test for Divergence**.

Just remember that if you get 0 for the limit, you **can't** conclude that the series converges. This just means that it has a chance to converge.

If $\sum a_n$ and $\sum b_n$ are convergent series,
then so are the series $\sum ca_n$ (where c is a constant),
$\sum(a_n + b_n)$, and $\sum(a_n - b_n)$,

i) $\sum ca_n = c\sum a_n$

example:

$$\sum_{n=1}^{\infty} 7 \cdot \frac{2}{3^{n-1}} = 7 \cdot \sum_{n=1}^{\infty} \frac{2}{3^{n-1}}$$
$$= 7 \cdot 3$$
$$= \boxed{21}$$

ii) $\sum(a_n + b_n) = \sum a_n + \sum b_n$

iii) $\sum(a_n - b_n) = \sum a_n - \sum b_n$

example:

$$\sum_{n=1}^{\infty}\left[\frac{2}{3^{n-1}} - \frac{3}{n(n+3)}\right] = \underbrace{\sum_{n=1}^{\infty}\frac{2}{3^{n-1}}}_{3} - \underbrace{\sum_{n=1}^{\infty}\frac{3}{n(n+3)}}_{\frac{11}{6}}$$
$$= 3 - \frac{11}{6}$$
$$= \frac{18-11}{6} = \boxed{\frac{7}{6}}$$