Math 104 – Rimmer
10.1 Sequences

10.1 Sequences

A **sequence** is an ordered list of numbers.

A sequence can be finite or infinite.

- finite: countably many numbers in the list
- infinite: infinitely many numbers in the list

In this class we will deal with infinite sequences

Notation:

$$\{a_1, a_2, a_3, \ldots, a_n, a_{n+1}, \ldots\}$$

a_1: 1st term; a_2: 2nd term; a_n: nth term; a_{n+1}: (n+1)st term

$$\{a_n\} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty}$$

a_n: a formula for the nth term

Note: the sequence doesn't have to start at n = 1

$$\left\{\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}, \frac{5}{36}, \ldots\right\}$$

$n=1 \quad n=2 \quad n=3 \quad n=4 \quad n=5$

$$\left\{\frac{n}{(n+1)^2}\right\}$$

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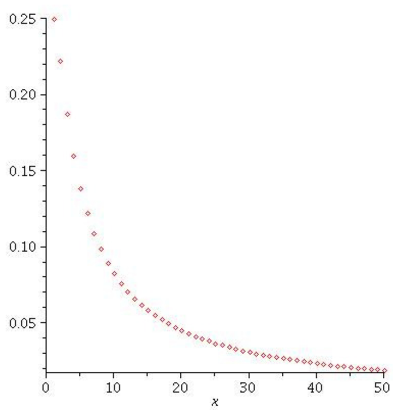
$$a_n = \frac{n}{(n+1)^2}$$

input : positive integers
output : terms of the sequence

$$\left(1, \frac{1}{4}\right), \left(2, \frac{2}{9}\right), \left(3, \frac{3}{16}\right), \left(4, \frac{4}{25}\right), \left(5, \frac{5}{36}\right), \ldots$$

These isolated points make up the graph of the sequence.



It seems as though the terms of the sequence are approaching 0 as $n \to \infty$

$$\lim_{n\to\infty} \frac{n}{(n+1)^2} = \lim_{n\to\infty} \frac{n}{n^2 + 2n + 1}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n} \to 0}{1 + \frac{2}{n} + \frac{1}{n^2}} = 0 \qquad \left(\frac{2}{n} \to 0, \ \frac{1}{n^2} \to 0\right)$$

$$\lim_{n\to\infty} \frac{p(n)}{q(n)} = 0 \qquad (p \text{ and } q \text{ polynomials})$$

when $\deg(num.) < \deg.(denom.)$

In general, if the terms of the sequence are approaching L as $n \to \infty$, then $\lim_{n\to\infty} a_n = L$

When this limit exists and is finite, we say the sequence is **convergent**.

When the $\lim_{n\to\infty} a_n$ does not exist or is infinite, the sequence is called **divergent**.

$$\left\{\cos\left(\frac{n\pi}{2}\right)\right\} = \{0,-1,0,1,0,-1,0,1,\ldots\}$$

$\Rightarrow \left\{\cos\left(\frac{n\pi}{2}\right)\right\}$ is divergent since the $\lim_{n\to\infty}\cos\left(\frac{n\pi}{2}\right)$ does not exist.

$\left\{\frac{n^2}{n+2}\right\}$ $\quad \lim_{n\to\infty}\frac{n^2}{n+2} = \infty$

$\Rightarrow \left\{\frac{n^2}{n+2}\right\}$ is divergent

$\lim_{n\to\infty}\frac{p(n)}{q(n)} = \infty$ $\quad$ (p and q polynomials)

when $\deg(num.) > \deg.(denom.)$

So basically, finding the limit of a sequence boils down to being able to find limits at infinity.

Tools:

Section 2.2 Limit Laws $\quad$ **Section 2.6 Limits at Infinity**

Section 4.5 Indeterminate forms and L'Hopitals Rule

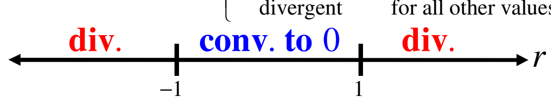
Thoerems:

1. Squeeze Theorem:

$$\left.\begin{array}{c} a_n \le c_n \le b_n \text{ for all } n > N \\ and \\ \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L \end{array}\right\} \Rightarrow \lim_{n\to\infty} c_n = L$$

2. $\lim_{n\to\infty}|a_n| = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$

3. The sequence $\{r^n\}$ is $\begin{cases} \text{convergent to } 0 & \text{if } -1 < r < 1 \\ \text{convergent to } 1 & r = 1 \\ \text{divergent} & \text{for all other values of } r \end{cases}$



4. $$\left.\begin{array}{c} \lim_{n\to\infty} a_n = L \\ and \\ f \text{ is contin. at } L \end{array}\right\} \Rightarrow \lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L)$$

bring the limit inside

5. Every **bounded** and **increasing** sequence and every **bounded** and **decreasing** sequence is **convergent**.

Determine whether the sequence converges or diverges. If it converges, find the limit.

$$a_n = \frac{3+5n^2}{n+n^2}$$

$$\lim_{x\to\infty} \frac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0} = \frac{a_m}{b_m}$$
when $\deg(num.) = \deg.(denom.)$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3+5n^2}{n+n^2} = \frac{5}{1} = 5$$

$\{a_n\}$ is convergent with $\lim_{n\to\infty} a_n = 5$

Determine whether the sequence converges or diverges. If it converges, find the limit.

$$a_n = \left(\frac{1}{\pi}\right)^n$$

3. The sequence $\{r^n\}$ is $\begin{cases} \text{convergent} & \text{if } -1 < r \le 1 \\ \text{divergent} & \text{for all other values of } r \end{cases}$

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{1}{\pi}\right)^n \qquad r = \frac{1}{\pi} \approx \frac{1}{3}$$

$\{a_n\}$ is convergent with $\lim_{n\to\infty} a_n = 0$

Determine whether the sequence converges or diverges.
If it converges, find the limit.

$$a_n = \sqrt{\frac{n+1}{9n+1}}$$

4. $\left.\begin{array}{c} \lim_{n\to\infty} a_n = L \\ \textit{and} \\ f \text{ is contin. at } L \end{array}\right\} \Rightarrow \lim_{n\to\infty} f(a_n) = L$

$$\lim_{x\to\infty} \frac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0} = \frac{a_m}{b_m}$$
when $\deg(num.) = \deg.(denom.)$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{\frac{n+1}{9n+1}} = \sqrt{\lim_{n\to\infty} \frac{n+1}{9n+1}} = \sqrt{\frac{1}{9}} = \frac{1}{3}$$

$\{a_n\}$ is convergent with $\lim_{n\to\infty} a_n = \frac{1}{3}$

$$a_n = \frac{(-1)^{n-1} n}{n^2+1} = \left\{\frac{1}{2}, -\frac{2}{5}, \frac{3}{10}, -\frac{4}{17}, \frac{5}{26}, \ldots\right\}$$

$n=1 \quad n=2 \quad n=3 \quad n=4 \quad n=5$

alternating sign term

2. $\lim_{n\to\infty} |a_n| = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$

$\lim_{n\to\infty} \frac{p(n)}{q(n)} = 0 \quad (p \text{ and } q \text{ polynomials})$
when $\deg(num.) < \deg.(denom.)$

$$|a_n| = \frac{n}{n^2+1} \qquad \lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n}{n^2+1} = 0$$

$\{a_n\}$ is convergent with $\lim_{n\to\infty} a_n = 0$

Determine whether the sequence converges or diverges.
If it converges, find the limit.

$$a_n = \left(1+\frac{1}{n}\right)^n \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \sim \text{"}1^\infty\text{"} \text{ Indeterminate form}$$

$$\text{Let } y = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \Rightarrow \ln y = \ln\left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right] \Rightarrow \ln y = \lim_{n\to\infty}\left[\ln\left(1+\frac{1}{n}\right)^n\right]$$

$$\Rightarrow \ln y = \lim_{n\to\infty}\left[n\ln\left(1+\frac{1}{n}\right)\right] \sim \text{"}\infty\cdot 0\text{"}$$

Indeterminate form

$$\ln y = \lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} \sim \text{"}\frac{0}{0}\text{"}$$

Indeterminate form that
we can use L'Hopitals Rule on

$$\ln y = \lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}$$

$$\stackrel{L'H}{=} \lim_{n\to\infty}\frac{\frac{1}{1+\frac{1}{n}}\cdot\left(\frac{-1}{n^2}\right)}{-\frac{1}{n^2}} = \lim_{n\to\infty}\frac{\frac{1}{\frac{n+1}{n}}\cdot\left(\frac{-1}{n^2}\right)}{-\frac{1}{n^2}} = \lim_{n\to\infty}\frac{n}{n+1}$$

$$\Rightarrow \ln y = \lim_{n\to\infty}\frac{n}{n+1}$$

$$\Rightarrow \ln y = 1$$

$$\Rightarrow e^{\ln y} = e^1$$

$$\Rightarrow \boxed{y = e} \qquad \text{so, } \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \qquad \left\{\left(1+\frac{1}{n}\right)^n\right\} \text{ is convergent}$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

$$\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^n = e^3$$

In general,

$$\lim_{n\to\infty}\left(1+\frac{m}{n}\right)^n = e^m$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n} = e^{-1}$$

In general,

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{kn} = e^k$$

Putting these together,

$$\lim_{n\to\infty}\left(1+\frac{m}{n}\right)^{kn} = e^{mk}$$

Determine whether the sequence converges or diverges.
If it converges, find the limit.

$$a_n = \frac{(-1)^n \sin\left(n^2\right)}{n}$$

Consider $\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\frac{\left|(-1)^n \sin\left(n^2\right)\right|}{|n|} = \lim_{n\to\infty}\frac{1\cdot\left|\sin\left(n^2\right)\right|}{n}$

since n is positive $|n|=n$

$$0 \le \left|\sin\left(n^2\right)\right| \le 1 \qquad \frac{0}{n} \le \frac{\left|\sin\left(n^2\right)\right|}{n} \le \frac{1}{n} \qquad \lim_{n\to\infty} 0 = 0 \text{ and } \lim_{n\to\infty}\frac{1}{n} = 0$$

so by the Squeeze theorem, $\lim_{n\to\infty}\frac{\left|\sin\left(n^2\right)\right|}{n} = 0$

$$\lim_{n\to\infty}|a_n| = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$$