Alternating Series Estimation Theorem

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ $(\text{where } b_n > 0)$ satisfies:

$i)$ $\lim_{n\to\infty} b_n = 0$

$ii)$ $\{b_n\}$ is a decreasing sequence

then $|R_n| = |s - s_n| \le b_{n+1}$

The size of the error is at most the size of the first omitted term.

The actual sum is between $s_n - b_{n+1}$ and $s_n + b_{n+1}$.

The error has the same sign as the first omitted term.

example

Stop @ n=7

$s = s_7 + R_7$

$-s_7 \quad -s_7$

$R_7 = s - s_7$

$\boxed{|R_7| = |s - s_7| \le b_8}$

$|s - s_7| \le b_8$

$-b_8 \le s - s_7 \le b_8$

$+s_7 \quad +s_7 \quad +s_7$

$\boxed{s_7 - b_8 \le s \le s_7 + b_8}$

First omitted term a_8

a_8 negative

$\Rightarrow R_7$ negative

$s - s_7 < 0 \Rightarrow s < s_7$

$\Rightarrow s_7$ overestimates s

a_8 positive

R_7 positive

$s - s_7 > 0 \Rightarrow s > s_7$

s_7 underestimates s

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \overbrace{\underbrace{1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\frac{1}{36}+\frac{1}{49}-\frac{1}{64}+\frac{1}{81}}_{s_9}\underbrace{-\frac{1}{100}+\frac{1}{121}-\frac{1}{144}\cdots}_{R_9}}^{}$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{s}$$

The error committed in using the 9th partial sum to approximate the total sum is R_9

The size of this error is at most the size of the first omitted term.

$$|R_9| = |s - s_9| \le \frac{1}{100} \quad \Rightarrow \frac{-1}{100} \le s - s_9 \le \frac{1}{100}$$

$$s_9 - \frac{1}{100} \le s \le s_9 + \frac{1}{100}$$

The actual sum is between $s_n - b_{n+1}$ and $s_n + b_{n+1}$.

The sign of the error is the sign of the first omitted term.

$R_9 = s - s_9 < 0 \qquad \Rightarrow s_9 > s \qquad s_9$ is an overestimate

since $a_{10} = -\frac{1}{100}$

Fall 2011

9. Which of the following is the best approximation of $\ln(\frac{11}{10})$?

(A) $\cancel{0}$ (B) $\frac{1}{10}$ (C) $\cancel{\frac{5}{100}}$ (D) $\cancel{\frac{9}{100}}$ $\boxed{\text{(E) } \frac{95}{1000}}$ (F) $\cancel{\frac{99}{1000}}$ (G) $\cancel{\frac{109}{1000}}$ (H) $\cancel{\frac{115}{1000}}$

S_1 S_2

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \text{ with } R = 1$$

$1 - x = \frac{11}{10}$

$1 - \frac{11}{10} = x$

$-\frac{1}{10} = x$

$$\ln\left(1 - -\frac{1}{10}\right) = -\left(-\frac{1}{10}\right) - \frac{\left(-\frac{1}{10}\right)^2}{2} - \frac{\left(-\frac{1}{10}\right)^3}{3} - \ldots$$

$$\ln\left(\frac{11}{10}\right) = \frac{1}{10} - \frac{1}{200} + \frac{1}{3000} - \ldots$$

$S_1 = \frac{1}{10}$

$S_2 = \frac{1}{10} - \frac{1}{200} = \frac{20-1}{200} = \frac{19 \cdot 5}{200 \cdot 5} = \frac{95}{1000}$

$S_3 = \frac{1}{10} - \frac{1}{200} + \frac{1}{3000} = \frac{300 - 15 + 1}{3000} = \frac{286}{3000} = \frac{143}{1500}$

Taylor Series Estimation Theorem

Taylor's Formula

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x), \quad (1)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } c \text{ between } a \text{ and } x. \quad (2)$$

If $R_n(x) \to 0$ as $n \to \infty$ for all $x \in I$, we say that the Taylor series generated by f at $x = a$ **converges** to f on I, and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

a
pick c
L endpt
R endpt

$f^{(4)}(c) \le M$

fourth der. @ c smaller than M

The Remainder Estimation Theorem If there is a positive constant M such that $\boxed{|f^{(n+1)}(t)| \le M}$ for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \le M\frac{|x-a|^{n+1}}{(n+1)!}.$$

If this inequality holds for every n and the other conditions of Taylor's Theorem are satisfied by f, then the series converges to $f(x)$.

endpt

a

$|x-a|$ dist from a

$$|R_n(x)| = \frac{\boxed{|f^{(n+1)}(c)|}\cdot|x-a|^{n+1}}{(n+1)!} \le \frac{M\cdot|x-a|^{n+1}}{(n+1)!}$$

n=3

$$|R_3| \le \frac{M\cdot(\text{endpt})^4}{4!}$$

Consider the polynomial $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}$ as an approximation to e^x on the interval $-2 \le x \le 2$. What is the best bound on the error for this estimate that is given by Taylor's inequality?

(a) $1/24$ (b) $e/12$ (c) $2e^2/3$ (d) $e^3/4$ (e) $3e^4/2$ (f) e^5

$f(x)=e^x$ The Taylor series is centered at $a=0$

$$|R_n| \le M\frac{|x-a|^{n+1}}{(n+1)!}$$

$I=(-2,2)$

$n=3$ and $P_3=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}$

M is the upper bound on the 4th derivative of $f(x)$ choosing an x in I

$\left|f^{(4)}(c)\right| \le M$ c in $(-2,2)$

$f^{(4)}(c)=e^c$ choose c to make this as big as possible $\Rightarrow c=2$ and $M=e^2$

$|R_3| \le M\frac{|x-0|^4}{4!} \Rightarrow |R_3| \le \frac{e^2}{24}|x-0|^4$

choose x to make $|x|^4$ as big as possible $\Rightarrow x=2$ or -2

$\Rightarrow |R_3| \le \frac{e^2}{24}(2)^4 \Rightarrow |R_3| \le \frac{16e^2}{24}$ $\boxed{|R_3| \le \frac{2e^2}{3}}$