Limit Problems

1) $\lim\limits_{x\to4}\frac{\sqrt{x+5}-3}{x-4}$

2) $\lim\limits_{x\to\frac{5}{2}}\frac{4x^2-25}{6x^2-7x-20}$

3) $\lim\limits_{x\to-\infty}\frac{3-2x^2+5x-\sqrt{x}}{\sqrt[3]{x}+4-3x+8x^2}$

4) $\lim\limits_{x\to\infty}\frac{5+x-4x^2}{6-18x+7x^{\pi}}$

5) $\lim\limits_{x\to\infty}\frac{(x+2)(x^3-64)}{16-x^3}$

6) $\lim\limits_{x\to0}\frac{\ln(1-x)-\sin x}{1-x-\cos^2 x}$

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1) $\lim\limits_{x\to4}\frac{\sqrt{x+5}-3}{x-4}\cdot\frac{\sqrt{x+5}+3}{\sqrt{x+5}+3}=\lim\limits_{x\to4}\frac{x+5-9}{(x-4)(\sqrt{x+5}+3)}$

$$=\lim_{x\to4}\frac{\cancel{x-4}}{(\cancel{x-4})(\sqrt{x+5}+3)}=\lim_{x\to4}\frac{1}{\sqrt{x+5}+3}=\frac{1}{\sqrt{4+5}+3}$$

$$=\boxed{\frac{1}{6}}$$

2) $\lim\limits_{x\to\frac{5}{2}}\frac{4x^2-25}{6x^2-7x-20}=\lim\limits_{x\to\frac{5}{2}}\frac{(2x+5)\cancel{(2x-5)}}{\cancel{(2x-5)}(3x+4)}$

$$=\lim_{x\to\frac{5}{2}}\frac{2x+5}{3x+4}=\frac{2(5/2)+5}{3(5/2)+4}=\frac{10}{\frac{15}{2}+4}=\frac{10\cdot2}{23}$$

$$\frac{15+8}{2}$$

$$=\boxed{20/23}$$

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3) $\lim_{x\to -\infty} \frac{3-2x^2+5x-\sqrt{x}}{\sqrt[3]{x}+4-3x+8x^2} \sim \lim_{x\to -\infty}\frac{-2x^2}{8x^2} = \boxed{-4}$

behaves like

4) $\lim_{x\to \infty} \frac{5+x-4x^2}{6-18x+7x^{\pi}} \sim \lim_{x\to \infty}\frac{-4x^2}{7x^{\pi}} = \boxed{0}$

deg. num > deg. denom

5) $\lim_{x\to \infty} \frac{(x+2)(x^3-64)}{16-x^3} \sim \lim_{x\to \infty}\frac{x^4}{-x^3} = \lim_{x\to \infty} -x = \boxed{-\infty}$

6) $\lim_{x\to 0} \frac{\ln(1-x)-\sin x}{1-x-\cos^2 x} = \text{"}\frac{0}{0}\text{"}$

$f = [\cos x]^2$

$f' = 2\cos x\cdot(-\sin x)$

$$\overset{L'H}{=} \lim_{x\to 0}\frac{\frac{1}{1-x}(-1)-\cos x}{-1-2\cos x\cdot(-\sin x)} = \frac{-1-1}{-1-0}$$

$$= \boxed{2}$$

Derivative Problems

7) Let $f(x) = 4x - 7x^2$. Find the derivative of the function **using the definition of the derivative**.

8) Let $y = x^2 e^{-x}$. Find the local minimum value of the function.

9) $y = xy^2 - 2x^2$ is a function that is defined implicitly. Find the slope of the tangent line at the point $(1,-1)$.

10) Let $f(x) = \dfrac{x^2 - 8x}{x+1}$. Find the absolute minimum value of f on $[0,4]$.

11) Let $f(x) = x^2 - 8x^{3/2}$. Find the $x-$value of the inflection point of f.

12) Let $y = \arcsin\left(\sqrt{x}\right)$. Find $y'\left(\dfrac{1}{2}\right)$.

7) Let $f(x) = 4x - 7x^2$. Find the derivative of the function **using the definition of the derivative**.

$$f(x+h) = 4(x+h) - 7(x+h)^2$$

$$f(x+h) = 4x + 4h - 7(x^2 + 2xh + h^2)$$

$$f(x+h) = 4x + 4h - 7x^2 - 14xh - 7h^2$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{4x} + 4h - \cancel{7x^2} - 14xh - 7h^2 - (\cancel{4x} - \cancel{7x^2})}{h}$$

$$= \lim_{h \to 0} \frac{4h - 14xh - 7h^2}{h} = \lim_{h \to 0} \frac{\cancel{h}(4 - 14x - 7h)}{\cancel{h}}$$

$$= \lim_{h \to 0} 4 - 14x - 7h = \boxed{4 - 14x}$$

8) Let $y = x^2e^{-x}$. Find the local minimum value of the function.

$$y' = 2x \cdot e^{-x} + x^2 \cdot (-1 \cdot e^{-x})$$

$$y' = xe^{-x}(2 - x) \overset{set}{=} 0$$

$x = 0$ never zero $x = 2$

y' $- - - \;|\; + + + \;|\; - - -$

0 2

local min. local max.

$x = 0$ $x = 2$

$y = 0$ $y = 4e^{-2}$

$$\boxed{\text{local min. value} = 0}$$

local max. value $= \frac{4}{e^2}$

9) $y = xy^2 - 2x^2$ is a function that is defined implicitly.

Find the slope of the tangent line at the point $(1,-1)$.

$$y' = 1 \cdot y^2 + x \cdot 2yy' - 4x$$

$$y'(1 - 2xy) = y^2 - 4x \quad @(1,-1)$$

$$y'(1 + 2) = 1 - 4$$

$$3y' = -3$$

$$\boxed{y' = -1}$$

10) Let $f(x)=\frac{x^2-8x}{x+1}$. Find the absolute minimum value of f on $[0,4]$.

$$f'(x) = \frac{(x+1)(2x-8)-(x^2-8x)(1)}{(x+1)^2}$$

$$f'(x) = \frac{2x^2-6x-8-x^2+8x}{(x+1)^2} = \frac{x^2+2x-8}{(x+1)^2} = \frac{(x+4)(x-2)}{(x+1)^2}$$

$f'(x)=0$ @ $x=-4, 2$

x	y
0	0
2	-4
4	-3.2

$f(0)=0$

$f(2)=\frac{4-16}{3}=\frac{-12}{3}=-4$

$f(4)=\frac{16-32}{5}=\frac{-16}{5}=-3.2$

11) Let $f(x)=x^2-8x^{3/2}$. Find the $x-$value of the inflection point of f.

$$f'(x) = 2x-8\cdot\frac{3}{2}x^{1/2} = 2x-12x^{1/2}$$

$$f''(x) = 2-12\cdot\frac{1}{2}x^{-1/2} = 2-\frac{6}{\sqrt{x}}$$

f'' DNE ------ +++
0 9

change in concavity } ⇒ infl. pt.

f'' undef @ $x=0$

$f''=0 \Rightarrow 2=\frac{6}{\sqrt{x}}$

$\sqrt{x}=3$

$x=9$

$\boxed{(9,-135)}$

$f(9) = 9^2-8\cdot 9^{3/2} = 81-8\cdot 27=-135$

12) Let $y = \arcsin\left(\sqrt{x}\right)$. Find $y'\left(\frac{1}{2}\right)$.

$$f = \arcsin x$$

$$f' = \frac{1}{\sqrt{1-x^2}}$$

$$y' = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \underbrace{\frac{1}{2\sqrt{x}}}_{\text{Chain Rule}}$$

$$y' = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$$

$$y'(1/2) = \frac{1}{\sqrt{1-1/2}} \cdot \frac{1}{2\sqrt{1/2}} = \frac{1}{\sqrt{1/2}} \cdot \frac{1}{2\sqrt{1/2}} = \frac{1}{2 \cdot \frac{1}{2}} = \boxed{1}$$

Integral Problems

13) Use $n = 4$ rectangles and left endpoints to estimate $\int_0^8 \left(\frac{2}{2+x}\right) dx$

14) Evaluate $\int \left(\sqrt{x} + \frac{1}{x} + \frac{2}{x^3}\right) dx.$

15) Evaluate $\int_3^0 \left(2x^3 - 6x^2\right) dx.$

16) Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos^3(t)\sin(t)\, dt.$

17) Evaluate $\int_e^{e^2} \frac{(\ln x)^2}{x}\, dx.$

18) Evaluate $\int_0^4 x\sqrt{x^2+9}\, dx.$

13) Use $n = 4$ rectangles and left endpoints to estimate $\int_0^8 \left(\frac{2}{2+x}\right) dx$

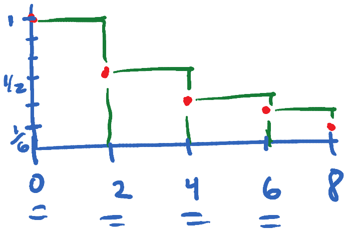
$$\text{Area} \approx 2 \cdot f(0) + 2f(2) + 2f(4) + 2f(6)$$

$$\approx 2[f(0) + f(2) + f(4) + f(6)]$$

$$\approx 2\left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{8}\right]$$

$$\approx 2\left(\frac{24 + 12 + 8 + 3}{24}\right)$$

$$\approx 2\left(\frac{47}{24}\right) = \frac{47}{12} = \boxed{3\tfrac{11}{12}}$$



14) Evaluate $\int \left(\sqrt{x} + \frac{1}{x} + \frac{2}{x^3}\right) dx.$

$$= \int \left(x^{1/2} + \frac{1}{x} + 2x^{-3}\right) dx$$

$$= \frac{x^{3/2}}{3/2} + \ln|x| + 2\frac{x^{-2}}{-2} + C$$

$$= \boxed{\frac{2}{3}x^{3/2} + \ln|x| - \frac{1}{x^2} + C}$$

15) Evaluate $\int_{3}^{0} \left(2x^3 - 6x^2\right) dx.$

$$= -\int_0^3 \left(2x^3 - 6x^2\right) dx$$

$$= -\left[\frac{2x^4}{4} - \frac{6x^3}{3}\right]_0^3 = -\left[\frac{x^4}{2} - 2x^3\right]_0^3$$

$$= -\left[\left(\frac{81}{2} - 2(27)\right) - (0)\right]$$

$$= -\left[\frac{81}{2} - 54\right] = -\left(\frac{81-108}{2}\right) = \boxed{\frac{27}{2}}$$

16) Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos^3(t)\sin(t)\; dt.$

$$= -\frac{1}{4}(\cos t)^4 \Big|_{\pi/6}^{\pi/4}$$

$$= -\frac{1}{4}\left[\left(\cos\frac{\pi}{4}\right)^4 - \left(\cos\frac{\pi}{6}\right)^4\right]$$

$$= -\frac{1}{4}\left[\left(\frac{\sqrt{2}}{2}\right)^4 - \left(\frac{\sqrt{3}}{2}\right)^4\right]$$

$$= -\frac{1}{4}\left[\frac{1}{4} - \frac{9}{16}\right] = -\frac{1}{4}\left[\frac{\overset{-5}{4-9}}{16}\right] = \boxed{\frac{5}{64}}$$

$u = \cos t$

$du = -\sin t\, dt$

$-du = \sin t\; dt$

$= -\int u^3 du$

$= -\frac{u^4}{4}$

$\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$ $\quad$ $\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}$

$\left(\frac{\sqrt{2}}{2}\right)^4 = \frac{1}{4}$ $\quad$ $\left(\frac{\sqrt{3}}{2}\right)^4 = \frac{9}{16}$

17) Evaluate $\int_{e}^{e^2} \frac{(\ln x)^2}{x}\,dx.$

$$= \frac{1}{3}\left[(\ln x)^3\right]\Big|_{e}^{e^2}$$

$$= \frac{1}{3}\left[(\ln e^2)^3 - (\ln e)^3\right]$$

$$= \frac{1}{3}\left[2^3 - 1^3\right] = \frac{1}{3}\left[8-1\right]$$

$$= \boxed{7/3}$$

$u = \ln x$

$du = \frac{1}{x}dx$

$= \int u^2\,du = \frac{u^3}{3}$

$\ln e^2 = 2\underbrace{\ln e}_{1} = 2$

$\ln e = 1$

18) Evaluate $\int_{0}^{4} x\sqrt{x^2+9}\,dx.$

$$= \frac{1}{3}\left[(x^2+9)^{3/2}\right]_{0}^{4}$$

$$= \frac{1}{3}\left[(16+9)^{3/2} - (0+9)^{3/2}\right]$$

$$= \frac{1}{3}(125-27)$$

$$= \frac{1}{3}(98) = \boxed{98/3}$$

$u = x^2+9$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$= \frac{1}{2}\int u^{1/2}\,du$

$= \frac{1}{2}\,\frac{u^{3/2}}{3/2}$

$= \frac{1}{2}\cdot\frac{2}{3}u^{3/2} = \frac{1}{3}u^{3/2}$

$25^{3/2} = (25^{1/2})^3 = 5^3 = 125$

$9^{3/2} = 3^3 = 27$