*Signature* PRINTED NAME

Math 504 **Exam 2** Jerry L. Kazdan

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DIRECTIONS: Part A has 5 shorter problems (5 points each) while Part B has 6 traditional problems (10 points each). To receive full credit your solution should be clear and correct. You have 1 hour 20 minutes. Closed book, no calculators, but you may use one 3 $\times$5 card with notes on both sides. If you use any standard theorems, please state them.

PART A: FIVE SHORTER PROBLEMS, 25 POINTS (5 POINTS EACH)

| *Score* | |
|---|---|
| A-1 | |
| A-2 | |
| A-3 | |
| A-4 | |
| A-5 | |
| B-1 | |
| B-2 | |
| B-3 | |
| B-4 | |
| B-5 | |
| B-6 | |
| *Total* | |

A-1. Let $G$ be a group with the property that every element has order $2$. Show that $G$ is commutative,

A-2. Give an example of a bounded continuous function $f(x)$ on the interval $\{0 < x < 1\}$, that is *not* uniformly continuous. You do not need to justify your assertions.

A-3. Let $f(x) = \frac{x}{1+x^2}$. Compute the value of its $100^{th}$ derivative at $x = 0$.

A-4. Give an example of a sequence of continuous functions $f(x)$, $x \in [0, 1]$, that converge to $0$ at every point of this interval but the convergence is *not* uniform (a sketch is adequate). You do not need to justify your assertions.

A-5. Give an example of a group of order 10 that is *not* commutative. You do not need to justify your assertions.

## Part B: Six standard problems, 60 points (10 points each)

B-1. A square $n \times n$ matrix $M$ is *diagonalized* by an invertible matrix $S$ if $SMS^{-1}$ is a diagonal matrix. Of the following three matrices, one can be diagonalized by an orthogonal matrix, one can be diagonalized but not by any orthogonal matrix, and one cannot be diagonalized. State which is which and why.

$$A = \begin{pmatrix} 1 & -2 \\ 2 & -5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 2 & -5 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix}$$

B-2. Let $x_k$, $k = 0, 1, 2, \ldots$ be a sequence of real numbers with the property that

$$|x_{k+1} - x_k| \leq \tfrac{1}{2}|x_k - x_{k-1}|, \quad k = 1, 2, \ldots.$$

Show that this sequence converges to some real number $\bar{x}$.

B-3. Let $f(x)$ be differentiable for all real $x$ with $|f'(x)| \le \frac{1}{2}$.

a) For any $x_0$, show that the sequence defined recursively by $x_{k+1} := f(x_k)$ converges to a fixed point $\bar{x}$ of $f$, that is, $f(\bar{x}) = \bar{x}$.

b) Show that this function $f$ has exactly *one* fixed point.

B-4. Let $a_n$ be a bounded sequence of real numbers. If $0 < r < 1$, show that the power series $\sum_0^\infty a_n x^n$ converges *uniformly* in the interval $\{|x| \leq r\}$.

B-5. Suppose $u$ is a twice differentiable function on $\mathbb{R}$ which satisfies the differential equation

$$\frac{d^2u}{dx^2} + b(x)\frac{du}{dx} - c(x)u = 0,$$

where $b(x)$ and $c(x)$ are continuous functions on $\mathbb{R}$ with $c(x) > 0$ for every $x \in (0, 1)$.

a) Show that $u$ cannot have a positive local maximum in the interval $(0, 1)$. Also show that $u$ cannot have a negative local minimum in $(0, 1)$.

b) If $u(0) = u(1) = 0$, prove that $u(x) = 0$ for every $x \in [0, 1]$.

B-6. Let $f(x, y) = e^{xy}$.

a) Explain why this function attains its minimum value on the set

$$S = \{(x, y) \in \mathbb{R}^2 \mid 2x^2 + y^2 \leq 1\}.$$

b) Determine this minimum value and the point(s) of $S$ where it is achieved.