Math 504 | Exam | Jerry L. Kazdan
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DIRECTIONS This exam has two parts, the first has four short computations (*5 points each*) while the second has seven traditional problems (*10 points each*).

Part A: Short Computations (4 problems, 5 points each)

A–1. Find a real 2×2 matrix A (other than $A=\pm I$) such that $A^4 = I$.

A–2. Find a function $u(x,y)$ satisfying $\dfrac{\partial u}{\partial x} - 2u = 0$ with $u(0,y)=\sin(3y)$.

A–3. Say $T(x,y,z) = x^2+xy+y^3-z^2$ gives the temperature at the point (x,y,z). At the point $(1,1,1)$, in which direction should one move so that the temperature increases fastest?

A–4. Compute $J := \displaystyle\iint_{\mathbb{R}^2} \frac{1}{[1+(2x+y+1)^2+(x-y+3)^2]^2}\,dx\,dy$.

Part B: Traditional Problems (7 problems, 10 points each)

B–1. If $A = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$, find an invertible matrix C such that $D := C^{-1}AC$ is a diagonal matrix. Compute A^{50}.

B–2. Let $T:\mathbb{R}^n \to \mathbb{R}^k$ be a real matrix (not necessarily square). If the nullspace of T is $\{0\}$, show that the matrix T^*T is invertible and positive definite.

B–3. Let a_n be a bounded sequence of real numbers. If $c>1$, show that the series $\sum \dfrac{a_n}{n^x}$ converges uniformly in the region $x \ge c$.

B–4. Let $\gamma(t)$ define a smooth curve that does not pass through the origin. If the point $\mathbf{P} = \gamma(t_0)$ is a point on the curve that is closest to the origin (and *not* an end point of the curve), show that the position vector $\gamma(t_0)$ is perpendicular to the tangent vector $\gamma'(t_0)$.

B–5. Let A be an $n\times n$ real symmetric matrix with eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ and corresponding orthonormal eigenvectors $v_1,\ldots,v_n$. Show that

$$\lambda_2 = \min_{x\ne 0,\ x\perp v_1} \frac{\langle x,\, Ax\rangle}{\|x\|^2}.$$

B–6. Let $f(x)$ be a continuous function for $0\le x\le 1$. Compute $\displaystyle\lim_{n\to\infty} n\int_0^1 f(x)e^{-3nx}\,dx$.

B–7. Let $\mathcal{D}$ be a bounded region in the plane, and $\mathcal{B}$ be its boundary (assumed smooth). Let $u(x,y,t)$ be a solution of the heat equation $\dfrac{\partial u}{\partial t} = \Delta u$ for (x,y) in $\mathcal{D}$. Say that the temperature $u(x,y,t)=0$ for all points (x,y) on the boundary $\mathcal{B}$.

If $E(t) := \frac{1}{2}\iint_{\mathcal{D}} u^2(x,y,t)\,dx\,dy$, show that $dE/dt \le 0$.

More Problems *The following problems were in the first draft of this exam but were deleted since they made the exam way too long.*

M–1. Let $p(x) := \left(\dfrac{d}{dx}\right)^3 (1-x^2)^3$. Show that p is a polynomial of degree 3 and that it has 3 real distinct zeroes, all lying in the interval $-1<x<1$.

M–2. Let $v(x,t) := \displaystyle\int_{x-2t}^{x+2t} g(s)\,ds$, where g is a continuous function. Compute $\partial v/\partial t$.

M–3. Let $L:\mathbb{R}^n \to \mathbb{R}^k$ be a linear map. Show that $\dim\ker(L) - \dim\ker(L^*) = n-k$.

M–4. Let $v(x)$ be a smooth real-valued function for $0\le x\le 1$. If $v(0)=v(1)=0$ and $v''(x)>0$ for all $0\le x\le 1$, show that $v(x)\le 0$ for all $0\le x\le 1$.

M–5. Given any points (x_1,y_1), (x_2,y_2), (x_3,y_3), (x_4,y_4) in the plane $\mathbb{R}^2$ with the x_i's distinct, show there is a unique cubic polynomial $p(x)$ with the properties that $p(x_i)=y_i$.

M–6. Let $y = f(x,u)$ and $z = g(x,u)$ be smooth functions with, say, $f(x_0,u_0) = y_0$ and $g(x_0,u_0)=z_0$.

a) Under what condition(s) can one eliminate x from these equations to express z as $z = H(u,y)$ as a smooth function of u and y near $u=u_0$, $y=y_0$?

b) If $f_x(x_0,u_0)=1$, $f_u(x_0,u_0)=-2$, $g_x(x_0,u_0)=-3$, and $g_u(x_0,u_0)=4,$, compute $\partial z/\partial u$ in terms of the derivatives of f and g.

M–7. Let $h(x,y)$ and $f(x)$ be continuous for $0\le x\le 1$, $0\le y\le 1$. Let $u_0(x)\equiv 0$ and define $u_k(x)$, $k=1,2,\ldots$, recursively by the rule

$$u_{k+1}(x) = f(x) + \int_0^1 h(x,y)u_k(y)\,dy.$$

If $\max|h(x,y)| \le c < 1$ show that the $u_k(x)$ converge uniformly for $0\le x\le 1$ to a continuous function $u(x)$ that satisfies

$$u(x) = f(x) + \int_0^c h(x,y)u(y)\,dy.$$

M–8. Let $f(x) \in C^\infty(\mathbb{R})$ be a real valued function with $f(0) = 0$. Show there is a $g(x) \in C^\infty(\mathbb{R})$ such that $f(x) = xg(x)$.