Many Coupled Oscillators

A VIBRATING STRING

Say we have n particles with the same mass m equally spaced on a string having tension τ. Let y_k denote the vertical displacement if the kth mass. Assume the ends of the string are fixed; this is the same as having additional particles at the ends, but with zero displacement: $y_0 = 0$ and $y_{n+1} = 0$. Let ϕ_k be the angle the segment of the string between the kth and k+1st particle makes with the horizontal. Then Newton's second law of motion applied to the kth mass asserts that

$$m\ddot{y}_k = \tau \sin\phi_k - \tau \sin\phi_{k-1}, \qquad k = 1,\ldots,n. \tag{1}$$

If the particles have horizontal separation h, then $\tan\phi_k = (y_{k+1} - y_k)/h$. For the case of small vibrations we assume that $\phi_k \approx 0$; then $\sin\phi_k \approx \tan\phi_k = (y_{k+1} - y_k)/h$ so we can rewrite (1) as

$$\ddot{y}_k = p^2(y_{k+1} - 2y_k + y_{k-1}), \qquad k = 1,\ldots,n, \tag{2}$$

where $p^2 = \tau/mh$. This is a system of second order linear constant coefficient differential equations with the boundary conditions $y_0(t) = 0$ and $y_{n+1}(t) = 0$. As usual, one seeks special solutions of the form $y_k(t) = v_k e^{\alpha t}$. Substituting this into (2) we find

$$\alpha^2 v_k = p^2(v_{k+1} - 2v_k + v_{k-1}), \qquad k = 1,\ldots,n,$$

that is, α^2 is an eigenvalue of the matrix $p^2(T - 2I)$, where

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0 \end{pmatrix}. \tag{3}$$

From the work in the next section (see (9)), we conclude that

$$\alpha_k^2 = -2p^2(1 - \cos\tfrac{k\pi}{n+1}) = -4p^2 \sin^2\tfrac{k\pi}{2(n+1)}, \qquad k = 1,\ldots,n,$$

so

$$\alpha_k = 2ip\sin\tfrac{k\pi}{2(n+1)}, \qquad k = 1,\ldots,n.$$

The corresponding eigenvectors V_k are the same as for T (see (10)). Thus the special solutions are

$$Y_k(t) = V_k e^{2ipt\sin\frac{k\pi}{2(n+1)}}, \qquad k = 1,\ldots,n,$$

where $Y(t) = (y_1(t),\ldots,y_n(t))$.

A SPECIAL TRIDIAGONAL MATRIX

We investigate the simple $n \times n$ real tridiagonal matrix:

$$M = \begin{pmatrix} \alpha & \beta & 0 & 0 & \dots & 0 & 0 & 0 \\ \beta & \alpha & \beta & 0 & \dots & 0 & 0 & 0 \\ 0 & \beta & \alpha & \beta & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \alpha & \beta & 0 \\ 0 & 0 & 0 & 0 & \dots & \beta & \alpha & \beta \\ 0 & 0 & 0 & 0 & \dots & 0 & \beta & \alpha \end{pmatrix} . = \alpha I + \beta T,$$

where T is defined by (3). This matrix arises in many applications, such as n coupled harmonic oscillators (see the previous section) and solving the Laplace equation numerically. Clearly M and T have the same eigenvectors and their respective eigenvalues are related by $\mu = \alpha + \beta\lambda$. Thus, to understand M it is sufficient to work with the simpler matrix T.

EIGENVALUES AND EIGENVECTORS OF T

Usually one first finds the eigenvalues and then the eigenvectors of a matrix. For T, it is a bit simpler first to find the eigenvectors. Let λ be an eigenvalue (necessarily real) and $V = (v_1, v_2, \dots, v_n)$ be a corresponding eigenvector. It will be convenient to write $\lambda = 2c$. Then

$$0 = (T - \lambda I)V = \begin{pmatrix} -2c & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & -2c & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & -2c & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -2c & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & -2c & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & -2c \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-2} \\ v_{n-1} \\ v_n \end{pmatrix} \tag{4}$$

$$= \begin{pmatrix} -2cv_1 + v_2 \\ v_1 - 2cv_2 + v_3 \\ \vdots \\ v_{k-1} - 2cv_k + v_{k+1} \\ \vdots \\ v_{n-2} - 2cv_{n-1} + v_n \\ v_{n-1} - 2cv_n \end{pmatrix}$$

Except for the first and last equation, these have the form

$$v_{k-1} - 2cv_k + v_{k+1} = 0. \tag{5}$$

We can also bring the first and last equations into this same form by introducing new artificial variables v_0 and v_{n+1}, setting their values as zero: $v_0 = 0$, $v_{n+1} = 0$.

The result (5) is a *second order linear difference equation with constant coefficients* along with the *boundary conditions* $v_0 = 0$, and $v_{n+1} = 0$. As usual for such equations one seeks a solution with the form $v_k = r^k$. Equation (5) then gives $1 - 2cr + r^2 = 0$ whose roots are

$$r_\pm = c \pm \sqrt{c^2 - 1}$$

Note also

$$2c = r + r^{-1} \qquad \text{and} \qquad r_+ r_- = 1. \tag{6}$$

Case 1: $c \neq \pm 1$. In this case the two roots $r_\pm$ are distinct. Let $r := r_+ = c + \sqrt{c^2-1}$. Since $r_- = c - \sqrt{c^2 - 1} = 1/r$, we deduce that the general solution of (4) is

$$v_k = Ar^k + Br^{-k}, \qquad k = 2, \ldots, n-1 \tag{7}$$

for some constants A and B which.

The first boundary condition, $v_0 = 0$, gives $A + B = 0$, so

$$v_k = A(r^k - r^{-k}), \qquad k = 1, \ldots, n-1. \tag{8}$$

Since for a non-trivial solution we need $A \neq 0$, the second boundary condition, $v_{n+1} = 0$, implies

$$r^{n+1} - r^{-(n+1)} = 0, \qquad \text{so} \qquad r^{2(n+1)} = 1.$$

In particular, $|r| = 1$. Using (6), this gives $2|c| \leq |r| + |r|^{-1} = 2$. Thus $|c| \leq 1$. In fact, $|c| < 1$ because we are assuming that $c \neq \pm 1$.

Case 2: $c = \pm 1$. Then $r = c$ and the general solution of (4) is now

$$v_k = (A + Bk)c^k.$$

The boundary condition $v_0 = 0$ implies that $A = 0$. The other boundary condition then gives $0 = v_{n+1} = B(n+1)c^{n+1}$. This is satisfied only in the trivial case $B = 0$. Consequently the equations (4) have no non-trivial solution for $c = \pm 1$.

It remains to rewrite our results in a simpler way. We are in Case 1 so $|r| = 1$. Thus $r = e^{i\theta}$, $c = \cos\theta$, and $1 = r^{2(n+1)} = e^{2i(n+1)\theta}$. Consequently $2(n+1)\theta = 2k\pi$ for some $1 \leq k \leq n$ (we exclude $k = 0$ and $k = n+1$ because we know that $c \neq \pm 1$, so $r \neq \pm 1$). Normalizing the eigenvectors V by the choice $A = 1/2i$, we summarize as follows:

Theorem 1 *The $n \times n$ matrix T has the eigenvalues*

$$\lambda_k = 2c = 2\cos\theta = 2\cos\tfrac{k\pi}{n+1}, \qquad 1 \le k \le n \tag{9}$$

and corresponding eigenvectors

$$V_k = (\sin\tfrac{k\pi}{n+1}, \sin\tfrac{2k\pi}{n+1}, \ldots, \sin\tfrac{nk\pi}{n+1}). \tag{10}$$

REMARK 1. If $n = 2k+1$ is odd, then the middle eigenvalue is zero because $(k+1)\pi/(n+1) = (k+1)\pi/2(k+1) = \pi/2$.

REMARK 2. Since $2ab = a^2 + b^2 - (a-b)^2 \le a^2 + b^2$ with equality only if $a = b$, we see that for any $x \in \mathbb{R}^n$

$$\langle x, Tx\rangle = 2(x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n) \le x_1^2 + 2(x_2^2 + \cdots + x_{n-1}^2) + x_n^2 \le 2\|x\|^2$$

with equality only if $x = 0$. Similarly $\langle x, Tx\rangle \ge -2\|x\|^2$. Thus, the eigenvalues of T are in the interval $-2 < \lambda < 2$. Although we obtained more precise information above, it is useful to observe that we could have deduced this so easily.

REMARK 3. *Gershgorin's circle theorem* is also a simple way to get information about the eigenvalues of a square (complex) matrix $A = (a_{ij})$. Let D_i be the disk whose center is at a_{ii} and radius is $R_i = \sum_{j\ne i}|a_{ij}|$, so

$$|\lambda - a_{jj}| \le R_j.$$

These are the *Gershgorin disks*.

Theorem 2 (Gershgorin) *Each eigenvalues of A lies in at least one of these Gershgorin discs.*

Proof: Say $Ax = \lambda x$ and say $|x_i| = \max_j |x_j|$. The i^{th} component of $Ax = \lambda x$ is

$$(\lambda - a_{ii})x_i = \sum_{j\ne i} a_{ij}x_j$$

so

$$|(\lambda - a_{ii})x_i| \le \sum_{j\ne i}|a_{ij}||x_j| \le R_i|x_i|.$$

That is, $|\lambda - a_{ii}| \le R_i$, as claimed.

By Gershgorin's theorem, we observed immediately that all of the eigenvalues of T satisfy $|\lambda| \le 2$.

DETERMINANT OF $T-\lambda I$

We use recursion on n, the size of the $n\times n$ matrix T. It will be convenient to build on (4) and let $D_n=\det(T-\lambda I)$. As before, let $\lambda=2c$. Then, expanding by minors using the first column of (4) we obtain the formula

$$D_n=-2cD_{n-1}-D_{n-2} \qquad n=3,4,\ldots. \tag{11}$$

Since $D_1=-2c$ and $D_2=4c^2-1$, we can use (11) to define $D_0:=1$. The relation (11) is, except for the sign of c, is identical to (5). The solution for $c\neq\pm1$ is thus

$$D_k=As^k+Bs^{-k}, \qquad k=0,1,\ldots, \tag{12}$$

where

$$-2c=s+s^{-1} \qquad \text{and} \qquad s=-c+\sqrt{c^2-1}. \tag{13}$$

This time we determine the constants A, B from the *initial conditions* $D_0=1$ and $D_1=-2c$. The result is

$$D_k=\begin{cases}\dfrac{1}{2\sqrt{c^2-1}}\left(s^{k+1}-s^{-(k+1)}\right) & \text{if} \quad c\neq\pm1,\\ (-c)^k(k+1) & \text{if} \quad c=\pm1.\end{cases} \tag{14}$$

For many purposes it is useful to rewrite this.

Case 1: $|c|<1$. Then $s=-c+i\sqrt{1-c^2}$ has $|s|=1$ so $s=e^{i\alpha}$ and $c=-\cos\alpha$ for some $0<\alpha<\pi$. Therefore from (14),

$$D_k=\frac{\sin(k+1)\alpha}{\sin\alpha}. \tag{15}$$

Case 2: $c>1$. Write $c=\cosh\beta$ for some $\beta>0$. Since $-e^{\beta}-e^{-\beta}=-2c=s+s^{-1}$, write $s=-e^{\beta}$. Then from (14),

$$D_k=(-1)^k\frac{\sinh(k+1)\beta}{\sinh\beta}, \tag{16}$$

where we chose the sign in $\sqrt{c^2-1}=-\sinh\beta$ so that $D_0=1$.

Case 3: $c<-1$. Write $c=-\cosh\beta$ for some $\beta>0$. Since $e^t+e^{-t}=-2c=s+s^{-1}$, write $s=e^{\beta}$. Then from (14),

$$D_k=\frac{\sinh(k+1)\beta}{\sinh\beta}, \tag{17}$$

where we chose the sign in $\sqrt{c^2-1}=+\sinh t$ so that $D_0=1$.

Note that as $t\to0$ in (15)–(17), that is, as $c\to\pm1$. these formulas agree with the case $c=\pm1$ in (14).