Problem Set 7

DUE: Thursday March 24 [*Late papers will be accepted until 1:00 PM Friday*].

1. Suppose u is a twice differentiable function on $\mathbb{R}$ which satisfies the ordinary differential equation
$$u'' + b(x)u' - c(x)u = 0,$$
where $b(x)$ and $c(x)$ are continuous functions on $\mathbb{R}$ with $c(x) > 0$ for every $x \in (0,1)$.
   a) Show that u cannot have a positive local maximum in the interval $(0,1)$, that is, have a local maximum at a point p where $u(p) > 0$. Also show that u cannot have a negative local minimum in $(0,1)$.
   [The example $u'' + u = 0$ has $u(x) = \sin x$ as a solution, which does have posivive local maxima and negative local minima. This shows that some assumption, such as our $c(x) > 0$ is needed.]
   b) If $u(0) = u(1) = 0$, prove that $u(x) = 0$ for every $x \in [0,1]$.
   c) If u satisfies
$$4u_{xx} + 3u_{yy} - 5u = 0$$
   in a region $\mathcal{D} \subset \mathbb{R}^2$, show that it cannot have a local positive maximum. Also show that u cannot have a local negative minimum.
   d) Repeat the above for a solution of
$$4u_{xx} - 2u_{xy} + 3u_{yy} + 7u_x + u_y - 5u = 0.$$
   e) If a function $u(x,y)$ satisfies the above equation in a bounded region $\mathcal{D} \in \mathbb{R}^2$ and is zero on the boundary of the region, show that $u(x,y)$ is zero throughout the region.

2. Let $Q \subset \mathbb{R}^2$ be the rectangle $Q := \{0 < x < 2,\ 0 < y < 1\}$. Solve the Dirichlet problem
$$u_{xx} + u_{yy} = 0 \text{ in } Q,$$
with
$$u(x,0) = u(x,1) = 0, \qquad u(0,y) = 0, \quad u(2,y) = \sin(4\pi y).$$

3. Solve $\Delta u = 0$ inside the disk of radius 2 with boundary condition $u(2,\theta) = \sin(3\theta)$.

4. Find a version of the Poisson formula for solving $\Delta u = 0$ in the *exterior* of the unit disk, so $r > 1$. Assume we require that the solution remains bounded as $r \to \infty$.
   Find the solution explicitly with the boundary condition $u(1,\theta) = \sin(2\theta)$.

5. Let Ω be the annulus $\Omega := \{(x,y) \in \mathbb{R}^2 \,|\, 1 < x^2 + y^2 < 4\}$. Solve the following Dirichlet problem for this using polar coordinates $u(r,\theta)$.

$$\Delta u = 0 \quad \text{in } \Omega, \quad u(1,\theta) = 0, \quad u(2,\theta) = \cos\theta.$$

Bonus Problem

1-B a) Find the eigenvalues and eigenfunctions for the boundary value problem

$$-u'' = \lambda u, \qquad u(0) = 0, \quad u'(3) + u(3) = 0$$

b) If we number the eigenvalues in increasing order, $\lambda_1 < \lambda_2 < \cdots$, find constants A amd B so that

$$\lim n \to \infty[\lambda_n - (An + B)^2] = 0.$$

[Last revised: March 18, 2011]