Math 425, Spring 2011 **Jerry L. Kazdan**

Problem Set 6

DUE: Thursday March 17 [*Late papers will be accepted until 1:00 PM Friday*].

1. This problem concerns orthogonal projections into a subspace of a larger space.
   a) Let $\mathbf{U} = (1,1,0,1)$ and $\mathbf{V} = (-1,2,1,-1)$ be given orthogonal vectors in R^4 and let $\mathcal{S}$ be the two dimensional subspace they span. Write the vector $\mathbf{X} = (1,1,1,-1)$ in the form $\mathbf{X} = a\mathbf{U} + b\mathbf{V} + \mathbf{W}$, where a, b are scalars and $\mathbf{W}$ is a vector perpendicular to $\mathbf{U}$ and $\mathbf{V}$.
   We call $\mathbf{X}_1 = a\mathbf{U} + b\mathbf{V}$, which is in $\mathcal{S}$, the *orthogonal projection* of $\mathbf{X}$ *into* $\mathcal{S}$. The notation $P_{\mathcal{S}}X = a\mathbf{U} + b\mathbf{V}$ for the projection of X into $\mathcal{S}$ is sometimes helpful.
   b) Let $\mathcal{T}$ be the three dimensional space spanned by $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{W}$. Find the orthogonal projection of $\mathbf{Y} = (1,0,0,0)$ into $\mathcal{T}$.
   c) Find an orthonormal basis for $\mathcal{S}$ and then for $\mathcal{T}$.

2. Suppose f is a function of one variable that has a continuous second derivative. Show that for any constants a and b, the function $u(x,y) := f(ax+by)$ is a solution of the *Monge-Ampère Equation* $u_{xx}u_{yy} - u_{xy}^2 = 0$.

3. Let $u(x,t)$ be a solution of the wave equation $u_{tt} = c^2 u_{xx}$ for $-1 < x < 1$, $t > 0$ with

$$u(x,0) = (1-x^2), \quad u_t(x,0) = 0, \qquad u(-1,t) = u(1,t) = 0.$$

   Without solving the equation explicitly, explain why $u(x,t) = u(-x,t)$ for all $t > 0$ and $-1 < x < 1$.

4. Let $u(x,t)$ be the temperature in a rod of length L that satisfies the partial differential equation:

$$u_t = u_{xx}, \qquad 0 < x < L, \quad t > 0$$

   together with the initial condition

$$u(x,0) = \varphi(x)$$

   where φ satisfies $\varphi(0) = \varphi(L) = 0$. Assume u also satisfies the Neumann boundary conditions

$$u_x(0,t) = 0, \quad \text{and} \quad u_x(L,t) = 0,$$

   Let $A(t)$ be the average temperature in the rod at time t, which is given by

$$A(t) = \frac{1}{L}\int_0^L u(x,t)\,dx$$

   Show that $A(t)$ is a constant (so it is independent of t) and thus the average of the initial temperature.

   REMARK: The Bonus Problem below asks you to show that $\lim_{t\to\infty} u(x,t) = A(0)$.

5. Consider the wave equation with friction: $u_{tt} + 2bu_t = u_{xx}$, $b > 0$ a constant, for $0 < x < \pi$ and $t > 0$ with the initial and boundary conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \qquad u(0) = u(\pi) = 0.$$

a) Use separation of variables to find the solution as a Fourier series in x and say how you would use the initial conditions to determine the coefficients.

b) Assuming the series converges, show that $\lim_{t\to\infty} u(x,t) = 0$, just the effect we anticipate friction will give. [This is a generalization of the fact that if $b >$, $c > 0$ are any positive constants, then all the solutions of the damped oscillator equation, $u'' + bu' + cu = 0$, tend to zero as $t \to \infty$.]

6. In $\mathbb{R}^n$ the distance r from a point $x := (x_1, x_2, \ldots, x_n)$ to the origin is $r^2 = x_1^2 + \cdots + x_n^2$. Assume that the function $u(x)$ depends *only* on r.

a) Show that $\partial r/\partial x_j = x_j/r$ and use this to show that

$$\frac{\partial u(x)}{\partial x_j} = \frac{x_j}{r}\frac{\partial u}{\partial r}$$

b) Compute $\dfrac{\partial^2 u(x)}{\partial x_j^2}$ and consequently

$$\Delta u(r) = u_{rr} + \frac{n-1}{r} u_r.$$

c) Find all harmonic functions that is, solutions of $\Delta u = 0$, that depend only on r. Note the answer will depend on n and that the solution might blow-up at the origin.

d) Let $\mathcal{A} \subset \mathbb{R}^2$ be the annular region $1 < \|x\| < 2$. Find a solution of $\Delta u = 0$ in $\mathcal{A}$ that satisfies the boundary conditions $u(x) = \alpha$ when $\|x\| = 1$ and $u(x) = \beta$ when $\|x\| = 2$.

e) Generalize the previous part to a shell $\mathcal{A} = \{x \in \mathbb{R}^n \mid 1 < \|x\| < 2\}$.

Bonus Problem

1-B [CONTINUATION OF PROBLEM 4] Show that $\lim_{t\to\infty} u(x,t) = A(0)$.

[Last revised: March 9, 2011]