Problem Set 5

DUE: Thurs. Feb. 24 *Late papers will be accepted until 1:00 PM Friday.*

1. In $\mathbb{R}^4$ the vectors

$$U_1 := (1,1,1,1), \quad U_2 := (1,1,-1,-1), \quad U_3 := (2,-2,2,-2), \quad U_4 := (1,-1,-1,1)$$

are orthogonal, as you can easily verify.

a) Use these to find an orthonormal basis $e_k := \alpha_k U_k$, $k = 1,\ldots,4$.

b) Write the vector $v := (0,-2,2,5)$ using this basis: $v = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4$.

c) Find the projection, Pv, of v into the plane spanned by U_2 and U_3.

d) Compute $\|Pv\|$.

2. Let X be a linear space with an inner product (not necessarily $\mathbb{R}^n$) and let $P : X \to X$ be an *orthogonal projection*, so $P^2 = P$ and $P = P^*$. Write V for the image of P; it is the space into which vectors are projected. Given $x \in X$, write $x = v + w$, where $v = Px$ is the projection of x into V. Show that w is orthogonal to V.

3. Let $f(x)$ be a 2π periodic function. Use Fourier series to investigate finding 2π periodic solutions of

$$-u''(x) + u = f(x),$$

so we want u and all of its derivatives to be 2π periodic.

This is routine – and short. Expand f in a Fourier series, so $f(x) = \sum_{k=-\infty}^{\infty} a_k e^{ikx}$ and seek the solution as a Fourier series $u(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$. So all you need do is determine the c_k's in terms of the a_k's.

4. Consider the wave equation $u_{tt} = u_{xx}$, $0 \le x \le \pi$ with the boundary conditions

$$u(0,t) = 0, \quad u(\pi,t) = 0, \quad (t \ge 0).$$

a) Find all solutions of the special form $u(x,t) = \phi(x)T(t)$ (*standing wave solutions*).

b) Use this to solve the wave equation with the above boundary conditions and the initial conditions

$$u(x,0) = 2\sin(3x) - 7\sin(19x), \quad u_t(x,0) = 0.$$

5. Consider the wave equation $u_{tt} = u_{xx}$, $0 \le x \le \pi$ with the mixed boundary conditions

$$u(0,t) = 0, \quad \frac{\partial u}{\partial x}(\pi,t) = 0, \quad (t \ge 0).$$

   a) Find all solutions of the special form $u(x,t) = \phi(x)T(t)$ (*standing wave solutions*).
   b) Use this to solve the wave equation with the above boundary conditions and the initial conditions

$$u(x,0) = 4\sin(5x/2) - 7\sin(9x/2), \qquad u_t(x,0) = 0.$$

6. LORENTZ TRANSFORMATIONS Let $u(x,t)$ be a given function. Find all linear changes of variable

$$\tau = \alpha x + \beta t, \qquad z = \gamma x + \delta t$$

that keep the wave operator invariant, that is

$$u_{tt} - c^2 u_{xx} = u_{\tau\tau} - c^2 u_{zz}.$$

SUGGESTION: You will be led to three equations for the four coefficients. Try to find a cleaner way to write these in terms of some other parameter. Here is a related example. Say a, b, c, and d satisfy

$$a^2 + b^2 = 1, \quad c^2 + d^2 = 1, \quad ac + bd = 0. \tag{1}$$

In this example, try writing $a = \cos\theta$. Then $b = \pm\sin\theta$ etc and you'll get equations for the four coefficients in terms of the one parameter θ (with some choices for $\pm$ a few places?). Upshot, the equations (1) just describe a rotation (and possibly also a reflection) around the origin in the plane $\mathbb{R}^2$.

7. [INTEGRATION BY PARTS FOR MULTIPLE INTEGRALS] Let $u(x,y)$ be a scalar function and $\mathbf{F}(x,y)$ a vector field in a bounded region $\mathcal{D}$ in $\mathbb{R}^2$ and let the closed curve C be the boundary of $\mathcal{D}$ with $\mathbf{N}$ be the unit outer normal vector field on this boundary.
   a) Prove the identity $\nabla \cdot (u\mathbf{F}) = \nabla u \cdot \mathbf{F} + u\nabla \cdot \mathbf{F}$. Compare this with the special case of a function of one variable.
   b) Use the divergence theorem to obtain the following generalization of *integration by parts* for multiple integrals:

$$\iint_{\mathcal{D}} u\nabla \cdot \mathbf{F}\, dA = \oint_C u\mathbf{F} \cdot \mathbf{N}\, ds - \iint_{\mathcal{D}} \nabla u \cdot \mathbf{F}\, dA.$$

   Notice that for a function of one variable with $\mathcal{D}$ being the interval $\{a < x < b\}$, this reduces precisely to the usual formula for integration by parts.
   c) Generalize this formula to the case where $\mathcal{D}$ is a bounded (solid) region in three dimensional space.

d) One frequently uses this with $\mathbf{F} = \nabla v$. Show the above formula for integration by parts becomes (say in two dimensions)

$$\iint_{\mathcal{D}} u \Delta v \, dA = \oint_C u \nabla v \cdot \mathbf{N} \, ds - \iint_{\mathcal{D}} \nabla u \cdot \nabla v \, dA.$$

To what does this reduce for functions on one variable?

e) As a short application using this, say $u(x,y)$ is a *harmonic function* in a bounded region $\mathcal{D}$, so $\nabla \cdot \nabla u = 0$. One can think of $u(x,y)$ as being the equilibrium temperature of $\mathcal{D}$. Let C is the boundary of $\mathcal{D}$. If $u = 0$ on C, it is plausible that one must have $u(x,y) = 0$ throughout $\mathcal{D}$. Show how this follows from the above formula. What is the analogous assertion for functions of one variable, where a harmonic function is just a solution of $u'' = 0$?

Bonus Problem

1-B [FOURIER SERIES IN SEVERAL VARIABLES] . Fourier series extends immediately to functions of several variables. Let T^2 be the square $\{(x,y) \in \mathbb{R}^2 \mid -\pi \le x \le \pi, \ -\pi \le y \le \pi\}$ and consider functions $f(x,y)$ that are 2π periodic in both variables with the $L_2(T^2)$ inner product

$$\langle f, g \rangle := \iint_{T^2} f(x,y)\overline{g(x,y)}\, dx\, dy.$$

a) Show that the functions

$$\varphi_{jk} := e^{i(jx+ky)} \qquad j,k = 0, \pm 1, \pm 2, \ldots$$

are orthogonal. How should you modify these to get orthonormal functions?

b) If $f(x,y)$ is 2π periodic in both variables, use Fourier series to investigate finding periodic solutions $u(x,y)$ of

$$-\Delta u(x,y) + u = f(x,y).$$

[This is almost identical to Problem 3 above.]

c) If $f(x,y)$ is 2π periodic in both variables, use Fourier series to investigate finding periodic solutions of

$$-\Delta u(x,y) = f(x,y).$$

[Last revised: February 23, 2011]