Problem Set 1

DUE: In class Thursday, Jan. 27. *Late papers will be accepted until 1:00 PM Friday.*

1. Find Green's function $g(x,s)$ to get a formula $u(x) = \int_0^x g(x,s)f(s)\,ds$ for a particular solution of $u''(x) = f(x)$.

2. In class we considered the oscillations of a weight attached to a spring hanging from the ceiling. If $u(t)$ is the displacement of the mass m we were let to solve $mu''(t) = -ku$, where $k > 0$ is a constant that depends on the stiffness of the spring. But this model neglected gravity. If we include gravity the equation becomes

$$mu'' = -ku + mg,$$

   where g is the gravitational constant,

   Solve this equation assuming you know the initial conditions $u(0) = A$ and $u'(0) = B$.

3. Let $a(x)$ and $f(x)$ be periodic functions with period P, so, for instance, $a(x+P) = a(x)$. This problem investigates periodic solutions $u(x)$ (with period P) of $Lu := u'(x) + a(x)u = f(x)$.
   a) Show there is a periodic solution of $u'(x) = f(x)$ if and only if $\int_0^P f(x)\,dx = 0$.
   b) Show that the homogeneous equation $Lu = 0$ has a non-trivial P-periodic solution $u(x)$ if and only if $\int_0^P a(x)\,dx = 0$.
   c) If $\int_0^P a(x)\,dx \neq 0$, show that the inhomogeneous equation $Lu = f$ always has a unique P-periodic solution $u(x)$.
      On the other hand, if $\int_0^P a(x)\,dx = 0$, find a necessary and sufficient condition for $Lu = f$ to have a P-periodic solution. If it has a P periodic solution, is this solution unique?

4. In class we obtained a simpler general formula for a particular solution of the inhomogeneous first order system $U' + AU = F$, where $U(x)$ and $F(x)$ are vectors with $n-$components and $A(x)$ is an $n \times n$ matrix. In addition we showed how much of the theory for a second order equation was in fact a special case of that for a first order system.

   Use this to re-derive Lagrange's formula for a particular solution of the inhomogeneous equation $u'' + u = f$.

5. Show that the *boundary value problem* $u'' + u = f$ on $0 \le x \le \pi$ with boundary conditions $u(0) = 0$ and $u(\pi) = 0$ has a solution if and only if $\int_0^\pi f(x)\sin x\,dx = 0$.

Bonus Problems (Due Jan. 27)

1-B Let $a(t)$ and $f(t)$ be periodic continuous functions with period 2π.

a) Show that the equation $u'' = f$ has a 2π-periodic solution (so both u and u' are 2π-periodic) if and only if

$$\int_0^{2\pi} f(t)\,dt = 0.$$

b) Show that the equation $u'' + u = f$ has a 2π-periodic solution if and only if *both* $\int_0^{2\pi} f(t)\sin t\,dt = 0$ and $\int_0^{2\pi} f(t)\cos t\,dt = 0$.

c) More generally, show that the equation $Lu := u'' + a(t)u = f$ has a 2π-periodic solution if and only if $\int_0^{2\pi} f(t)z(t)\,dt = 0$ for all 2π-periodic solutions of $z'' + a(t)z = 0$. [REMARK: These are special cases of the *Fredholm alternative*: the image of L is the orthogonal complement of the kernel of the adjoint operator L^*.]

[Last revised: January 25, 2011]