

Fourier Series of $f(x) = x$

To write:

$$x = \sum_k c_k \frac{e^{ikx}}{\sqrt{2\pi}}.$$

The Fourier coefficients are

$$c_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x [\cos kx - i \sin kx] dx = \frac{-2i}{\sqrt{2\pi}} \int_0^{\pi} x \sin kx dx$$

But

$$\int_0^{\pi} x \sin kx dx = \left. -\frac{x \cos kx}{k} \right|_0^{\pi} + \frac{1}{k} \int_0^{\pi} \cos kx dx = \frac{-\pi \cos k\pi}{k} = -\frac{\pi}{k} (-1)^k.$$

Thus

$$c_k = -\frac{2i}{\sqrt{2\pi}} \left[-\frac{\pi}{k} (-1)^k \right] = i\sqrt{2\pi} \left[\frac{(-1)^k}{k} \right].$$

Consequently

$$\begin{aligned} x &= i\sqrt{2\pi} \sum_{k \neq 0} \frac{(-1)^k}{k} \frac{e^{ikx}}{\sqrt{2\pi}} = i \sum_{k \neq 0} \frac{(-1)^k}{k} e^{ikx} \\ &= -2 \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sin kx = 2 \left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots \right] \end{aligned}$$

Finally we compute $\|x\|^2 = \sum |c_k|^2$. Since

$$\|x\|^2 = \int_{-\pi}^{\pi} |x|^2 dx = \frac{2}{3} \pi^3,$$

and

$$\sum |c_k|^2 = 2\pi \left[\sum_{-\infty}^{-1} \frac{1}{k^2} + \sum_1^{\infty} \frac{1}{k^2} \right] = 4\pi \sum_1^{\infty} \frac{1}{k^2}$$

Therefore

$$\frac{2}{3} \pi^3 = 4\pi \sum_1^{\infty} \frac{1}{k^2}, \quad \text{that is,} \quad \sum_1^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$