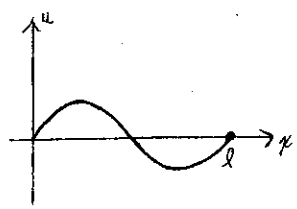
3. The Vibrating String.

Waves. You have been hearing about them your whole life. Waves are the term used to describe the oscillatory behavior of continuous media; water waves and sound waves being the most familiar. We shall give a mathematical description of a very simple type of wave - those in an oscillating violin string. The resulting mathematical model will be a second order linear partial differential equation - the wave equation - with both initial and boundary conditions.

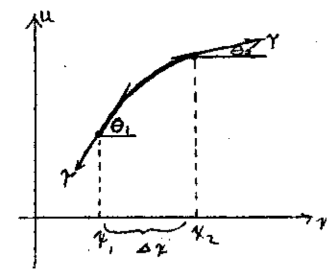
A. The Mathematical Model

Consider a string of length ℓ stretched along the x axis. Imagine the string vibrating in the plane of the paper and let $u(x, t)$ denote the vertical displacement of the point x at time t. In order to end up with a tractable mathematical model several reasonable simplying assumptions will be made. We assume the tension T and density ρ of the string are constant throughout the motion, while the string is taken to be perfectly flexible so the tension force in the string acts along the tangential direction. Dissipative effects (air resistance, heating,



etc). are entirely neglected. One more assumption will be made when needed. It essentially states that the oscillations are small in some sense.

Newton's second law, $ma = \sum F$, is where we begin. Draw your attention to a small segment of the string whose length, at rest, is $\Delta x = x_2 - x_1$. The mass of the segment is $\rho \Delta x$. By Newton's second law the segment moves in such a way that the product of its mass with the acceleration of its center of gravity equals the resultant of the forces acting on it. For the vertical component, this means



$$\rho \Delta x \frac{\partial^2 u}{\partial t^2} (\tilde{x}, t) = F_v ,$$

where $\tilde{x} \in (x, x+\Delta x)$ is the horizontal coordinate of the center of gravity of the segment, and F_v means the vertical component of the resultant force.

There are two types of forces. One is the tension acting at both ends of the segment. The other is gravity acting down with a force equal to the weight of the segment, $- \rho g \Delta x$. To evaluate the tension forces, let θ_1 and θ_2 be the angles the string makes with the horizontal at either end of the segment (see figure above). Then the vertical component of the tension force is

$$\tau \sin\theta_2 - \tau \sin\theta_1,$$

The signs indicate one force is up while the other is down. Adding the tension force to the gravitational force and substituting into Newton's second law, we find

$$\rho\Delta x \frac{\partial^2 u}{\partial t^2}(\tilde{x}, t) = \tau(\sin\theta_2 - \sin\theta_1) - \rho g \Delta x .$$

The dependence of θ_1 and θ_2 on the displacement can be brought out by using the relation

$$\sin\theta = \frac{u_x}{\sqrt{1+u_x^2}} ,$$

which follows from the relation $u_x = \tan\theta$ for the slope of the string. Using this, we obtain the equation

$$\rho\Delta x \frac{\partial^2 u}{\partial t^2}(\tilde{x}, t) = \tau\left[\left.\frac{u_x}{\sqrt{1+u_x^2}}\right|_{x=x_2} - \left.\frac{u_x}{\sqrt{1+u_x^2}}\right|_{x=x_1} \right] - \rho g \Delta x .$$

A simplifying assumption is badly needed. If the function $u_x/\sqrt{1+u_x^2}$ is expanded in a Taylor series,

$$\frac{u_x}{\sqrt{1+u_x^2}} = u_x - \frac{1}{2}u_x^3 + \cdots ,$$

we see that if the slope u_x is small, essentially only the linear term in this series counts. Therefore, we do assume the slope u_x is small (this is the same assumption made in treating the simple pendulum). With this simplification, the equation of motion is

$$\rho\Delta x \frac{\partial^2 u}{\partial t^2}(\tilde{x}, t) = \tau[u_x(x_2, t) - u_x(x_1, t)] - \rho g \Delta x .$$

Divide both sides of this equation by $\Delta x = x_2 - x_1$ and let the length of the interval shrink to zero. Since

$$\lim_{(x_2 - x_1)\to 0} \frac{u_x(x_2, t) - u_x(x_1, t)}{x_2 - x_1} = \frac{\partial}{\partial x} u_x(x, t) = \frac{\partial^2 u}{\partial x^2}(x, t),$$

where x is the limiting value of x_1 and x_2, we find

$$\rho \frac{\partial^2 u}{\partial t^2}(x, t) = \tau \frac{\partial^2 u}{\partial x^2}(x, t) - \rho g$$

Because the length of the interval has been shrunk to one point x, the center of gravity is now at x too.

It is customary to let $\tau/\rho = c^2$. The constant c has units of velocity, and, in fact, is just the speed with which waves travel along the string. Thus

$$Lu \equiv u_{tt} - c^2 u_{xx} = -g .$$

This is the wave equation, a second order linear inhomogeneous partial differential equation. As was the case with linear ordinary differential equations, it is easier to attempt first to solve the homogeneous equation

$$Lu \equiv u_{tt} - c^2 u_{xx} = 0 .$$

On physical grounds, we expect the motion $u(x, t)$ of the string will be determined if the initial position $u(x, 0)$ and initial velocity $u_t(x, 0)$ are known, along with the motion of both end points $u(0, t)$ and $u(\ell, t)$. However the mathematical model must be examined to see if these four facts do determine the subsequent motion (which it should if the model is to be of any use). Thus we must prove that given the

initial position	$u(x, 0) = f(x)$,	$x \in [0, \ell]$
initial velocity	$u_t(x, 0) = g(x)$,	$x \in [0, \ell]$
motion of left end	$u(0, t) = \phi(t)$	$t \geq 0$
motion of right end	$u(\ell, t) = \psi(t)$,	$t \geq 0$,

then a solution $u(x, t)$ of the wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

does exist which has these properties, and there is only one such solution. Existence and uniqueness theorems must therefore be proved.

B. Uniqueness.

This is almost identical to all uniqueness theorems encountered earlier, especially that for the simple harmonic oscillator in Chapter 4, Section 2.

Theorem 9 (Uniqueness). There exists at most one twice continuously differentiable function $u(x, t)$ which satisfies the inhomogeneous wave equation

$$Lu \equiv u_{tt} - c^2 u_{xx} = F(x, t)$$

and the subsidiary

initial conditions: $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$, $x \in [0, \ell]$

boundary conditions: $u(0, t) = \phi(t)$, $u(\ell, t) = \psi(t)$, $t \geq 0$,

where F, f, g, ϕ, and ψ are given functions.

Proof. Assume $u(x, t)$ and $v(x, t)$ both satisfy the same equation and the same subsidiary conditions. Let $w(x, t) = u(x, t) - v(x, t)$. Then $Lw = Lu - Lv = F - F = 0$, so w satisfies the homogeneous equation

$$Lw \equiv w_{tt} - c^2 w_{xx} = 0$$

and has zero subsidiary data

initial conditions: $w(x, 0) \equiv 0$, $w_t(x, 0) \equiv 0$, $x \in [0, \ell]$

boundary conditions: $w(0, t) \equiv 0$, $w(\ell, t) \equiv 0$, $t \geq 0$

We want to prove $w(x, t) \equiv 0$. Notice that w satisfies the equation for a vibrating string which is initially at rest on the x axis, and whose ends never move. Therefore our desire to prove the string never moves, $w(x, t) \equiv 0$, is certain physically reasonable.

For this function w, define the new function $E(t)$

$$E(t) = \frac{1}{2}\int_0^\ell [w_t^2 + c^2 w_x^2]\, dx .$$

We have named the function $E(t)$ since it actually happens to be the energy in the string associated with the motion $w(x, t)$ at time t, except for a factor of ρ. Assume it is "legal" to differentiate under the integral sign (it is). Upon doing so, we get

$$\frac{dE}{dt} = \int_0^\ell [w_t w_{tt} + c^2 w_x w_{xt}]\, dx .$$

But an integration by parts reveals that

$$\int_0^\ell w_x w_{xt} dx = w_x w_t \Big|_0^\ell - \int_0^\ell w_t w_{xx} dx .$$

Because the end points are held fixed, $w(0, t) = 0$ and $w(\ell, t) = 0$,

the velocity at those points is zero too, $w_t(0, t) = 0$ and $w_t(\ell, t) = 0$. This drops out the boundary terms in the integration by parts. Substituting the last expression into that for dE/dt, we find that

$$\frac{dE}{dt} = \int_0^\ell w_t [w_{tt} - c^2 w_{xx}]\, dx .$$

But w satisfies the homogeneous wave equation $w_{tt} - c^2 w_{xx} = 0$. Therefore $dE/dt \equiv 0$, so

$$E(t) \equiv \text{constant} = E(0),$$

that is, energy is conserved. Now

$$E(0) = \frac{1}{2}\int_0^\ell [w_t^2(x, 0) + c^2 w_x^2(x, 0)]\, dx .$$

Since the initial position is zero, $w(x, 0) = 0$, its slope is also zero, $w_x(x, 0) = 0$. The initial velocity $w_t(x, 0)$ is also zero, $w_t(x, 0) = 0$. Thus

$$E(t) \equiv E(0) \equiv 0 ,$$

that is,

$$0 = E(t) = \frac{1}{2}\int_0^\ell [w_t^2(x, t) + c^2 w_x^2(x, t)]\, dx .$$

Because the integrand is positive, we conclude $w_t(x, t) \equiv 0$ and

$w_x(x, t) \equiv 0$. Consequently $w(x, t) \equiv$ constant. Since $w(0, t) = 0$, that constant is the zero constant,

$$w(x, t) \equiv 0 .$$

Therefore

$$u(x, t) - v(x, t) \equiv w(x, t) \equiv 0 ,$$

so $u(x, t) \equiv v(x, t)$: the solution is unique.

C. Existence.

For the simple one (space) dimension wave equation, there are many ways to prove a solution exists. The one to be given here is not the simplest (see Exercise 6 for the result of that method), but it does generalize immediately to many other problems. It makes no difference how we find a solution, for once found, by the uniqueness theorem it is the only possible solution. To avoid complications, we shall consider only the homogeneous equation and assume the end points are tied down. Thus, we want to solve

Wave equations: $u_{tt} - c^2 u_{xx} = 0$.

Initial conditions: $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$.

Boundary conditions: $u(0, t) = 0$, $u(\ell, t) = 0$.

The idea is first to find special solutions $u_1(x, t)$, $u_2(x, t)$, ..., which satisfy the boundary conditions but do not necessarily satisfy the initial conditions. Then, as was done for linear O.D.E.'s, we build the solution which does satisfy the given initial conditions as a linear combination of these special solutions,

$$u(x, t) = \sum A_j u_j(x, t),$$

that is, by superposition.

Let us seek special solutions in the form of a standing wave,

$$u(x, t) = X(x)T(t).$$

Here $X(x)$ and $T(t)$ are functions of one variable. Our procedure is reasonably called separation of variables. Substitution of this into the wave equation gives

$$\ddot{T}(t)\,X(x) - c^2 X''(x)\,T(t) = 0,$$

or

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{\ddot{T}(t)}{T(t)} .$$

Since the left side depends only on x, while the right depends only on t, both sides must be constant (a somewhat tricky remark; think it over). Let that constant be $-\gamma$ (using $-\gamma$ instead of γ is the result of hindsight, as you shall see).

$$\frac{X''}{X} = \frac{1}{c^2}\frac{\ddot{T}}{T} = -\gamma .$$

This leads us to the two ordinary differential equations

$$X''(x) + \gamma X(x) = 0, \qquad \ddot{T}(t) + \gamma c^2 T(t) = 0.$$

Since $u(0, t) = 0$ and $u(\ell, t) = 0$ and $u(x, t) = X(x)\,T(t)$, the function $X(x)$ must also satisfy the boundary conditions

$$X(0) = 0, \qquad X(\ell) = 0.$$

There are several ways to show γ must be positive. Perhaps the simplest is to observe that if $\gamma < 0$ or $\gamma = 0$, the only function $X(t)$ which satisfies the differential equation $X'' + \gamma X = 0$ and boundary conditions $X(0) = X(\ell) = 0$ is the zero function $X(x) \equiv 0$. Since for this function $u(x, t) = X(x)\,T(t) \equiv 0$, it is devoid of further interest.

Another way to show γ is positive is to multiply the ordinary differential equation $X'' + \gamma X = 0$ by $X(x)$ and integrate over the length of the string,

$$\int_0^\ell [X(x)\,X''(x) + \gamma X^2(x)]\,dx = 0.$$

Upon integrating by parts, we find that

$$\int_0^\ell X(x)\,X''(x)dx = XX'\Big|_0^\ell - \int_0^\ell X'^2(x)dx .$$

Since $X(0) = X(\ell) = 0$, the boundary terms drop out. Substituting this into the above equation, we find that

$$\int_0^\ell X'^2(x)dx = \gamma \int_0^\ell X^2(x)dx .$$

If $X(x)$ is not identically zero, this can be solved for γ

$$\gamma = \frac{\int_0^\ell X'^2(x)dx}{\int_0^\ell X^2(x)\,dx} ,$$

and clearly shows $\gamma > 0$.

Enough for that. The solution of $X'' + \gamma X = 0$, $\gamma > 0$, is

$$X(x) = A \cos\sqrt{\gamma}\,x + B \sin\sqrt{\gamma}\,x .$$

The boundary condition $X(0) = 0$ implies $A = 0$, while the boundary condition at the other end point $X(\ell) = 0$, implies

$$0 = B \sin\sqrt{\gamma}\,\ell .$$

If $B = 0$ too, then $X(x) \equiv 0$, so $u(x,t) \equiv 0$. This is of no use to us. The only alternative is to restrict γ so that $\sin\sqrt{\gamma}\,\ell = 0$. This means $\sqrt{\gamma}\,\ell$ is a multiple of π, $\sqrt{\gamma}\,\ell = n\pi$, $n = 1, 2, \ldots$,

$$\sqrt{\gamma} = \frac{n\pi}{\ell}, \quad n = 1, 2, \ldots .$$

There is then one possible solution $X(x)$ for each integer n,

$$X_n(x) = B_n \sin \frac{n\pi}{\ell} x ,$$

where the constants B_n are arbitrary.

<u>Remark:</u> There is a similarity of deep significance for mathematics and physics between the work in these last few paragraphs and that done for the coupled oscillators in Chapter 6. There (p. 528-9), we had an operator A and wanted to find nonzero vectors S_n and numbers λ such that

$$AS_n = \lambda_n S_n .$$

The numbers found λ_n were called the eigenvalues of A, and S_n the corresponding eigenvectors.

Here, we were given the operator $A = -\frac{d^2}{dx^2}$ and wanted to find nonzero functions $X_n(t) \in \{X \in C^2[0, \ell] \mid X(0) = X(\ell) = 0\}$ which satisfy the equation

$$AX_n = \gamma_n X_n$$

The numbers found, $\gamma_n = n^2\pi^2/\ell^2$, are also called the <u>eigenvalues</u> of A, and the function $X_n(t) = \sin \frac{n\pi}{\ell} x$, the <u>eigenfunction</u> of A corresponding to the eigenvalue γ_n.

Associated with each possible eigenvalue γ_n, there is a

solution of the time equation, $\ddot{T} + \gamma c^2 T = 0$,

$$T_n(t) = C_n \cos \frac{nc\pi}{\ell} t + D_n \sin \frac{nc\pi}{\ell} t .$$

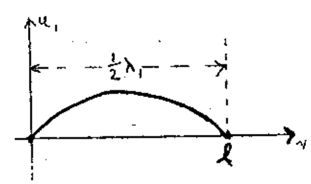
We therefore have found one special solution, $u_n(x, t) = X_n(t)\, T_n(t)$, for each value of the index n,

$$u_n(x, t) = \sin \frac{n\pi x}{\ell} \left(\alpha_n \cos \frac{nc\pi t}{\ell} + \beta_n \sin \frac{nc\pi t}{\ell}\right).$$

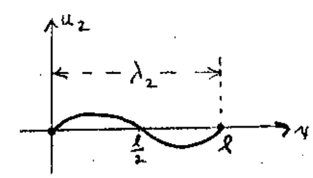
The arbitrary constants have been lumped in this equation. These special solutions are the "natural" vibrations of the string, or <u>normal modes of vibration.</u> A snapshot at $t = t_0$ of the string moving in the n^{th} normal mode would reveal the sine curve

$$u_n(x, t_0) = C \sin \frac{n\pi x}{\ell} ,$$

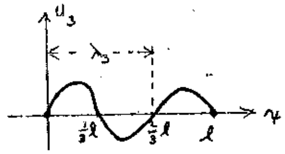
the constant C accounting for the remaining terms, which are constant for t fixed. In music, the integer n refers to the octave. The fundamental tone is the case $n = 1$, while the tone for $n = 2$, the <u>second harmonic</u> or <u>first overtone,</u> is one octave higher.



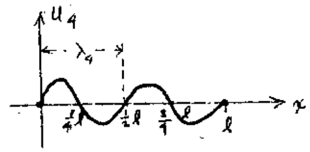
Fundamental Tone



First Overtone



Second Overtone



Third Overtone

The time frequency ν_n of the n^{th} normal mode is $\nu_n = \frac{nc\pi}{\ell}$, this is the number of oscillations in 2π units of time. It is the time frequency which we usually associate with musical pitch. The (time) period τ_n of the n^{th} uormal mode is $2\pi/\nu_n$, that is $\tau_n = 2\ell/nc$. Another name you will want to know is the wave length λ_n of the n^{th} normal mode, $\lambda_n = 2\ell/n$ (see figures above). Notice that $\nu_n \lambda_n = c$, an important relationship.

Having found the special normal mode solutions, $u_n(x, t)$, we hope that arbitrary constants α_n and β_n can be chosen so a linear combination

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} (\alpha_n \cos \frac{nc\pi t}{\ell} + \beta_n \sin \frac{nc\pi t}{\ell}) \sin \frac{n\pi x}{\ell}$$

will satisfy the given initial conditions. Every function $u(x, t)$ of this form automatically satisfies the boundary conditions $u(0, t) = 0$, $u(\ell, t) = 0$ since each of the u_n's satisfy them.

If $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$, then from the above equation, we must have

$$f(x) = \sum_{n=1}^{\infty} u_n(x, 0) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{n\pi x}{\ell}$$

and

$$g(x) = \sum_{n=1}^{\infty} \frac{\partial u_n}{\partial t}(x, 0) = \sum_{n=1}^{\infty} \frac{n\pi c}{\ell} \beta_n \sin \frac{n\pi x}{\ell}.$$

Thus, the coefficients α_n are the coefficients in the Fourier sine series for f, while the β_n are essentially the coefficients in the Fourier sine series for g. In fact, this is how Fourier was led to the series bearing his name. These formulas for $u(x, y)$, $f(x)$, and $g(x)$ become easier on the eye if the length of the string is π, $\ell = \pi$. Then

$$u(x, y) = \sum_{n=1}^{\infty} (\alpha_n \cos nct + \beta_n \sin nct) \sin nx,$$

while

$$f(x) = \sum_{n=1}^{\infty} u_n(x, 0) = \sum_{n=1}^{\infty} \alpha_n \sin nx,$$

and

$$g(x) = \sum_{n=1}^{\infty} \frac{\partial u_n}{\partial t}(x, 0) = \sum_{n=1}^{\infty} nc\beta_n \sin nx. \tag{3}$$

Finding the coefficients α_n and β_n is particularly simple if f and g can be represented by finite series.

<u>Examples.</u> Find the solution $u(x, t)$ of the wave equation for a string of length π, $\ell = \pi$, which is pinned down at its end points, $u(0, t) = u(\pi, t) = 0$, and satisfies the given initial conditions.

1. $u(x, 0) = f(x) = 2 \sin 3x$, $u_t(x, 0) = g(x) = \frac{1}{2} \sin 4x$. We have to find α_n and β_n for the two series

$$2 \sin 3x = \sum_{n=1}^{\infty} \alpha_n \sin nx$$

$$\frac{1}{2} \sin 4x = \sum_{n=1}^{\infty} nc\, \beta_n \sin nx .$$

For these simple functions, just match coefficients, giving

$$\alpha_3 = 2, \ \alpha_n = 0, \ n \neq 3, \text{ and } \beta_4 = \frac{1}{8c}, \ \beta_n = 0, \ n \neq 4.$$

Therefore, the sum of the two waves

$$u(x, t) = 2 \cos 3\,ct \sin 3x + \frac{1}{8c} \sin 4ct \sin 4x$$

is the (unique!) solution of this example.

2. $u(x, 0) = f(x) = \frac{1}{2} \sin 3x - \sin 17x$ and

$$u_t(x, 0) = g(x) = -9 \sin x + 13 \sin 973x .$$

We have to find α_n and β_n for the two series

$$\frac{1}{2} \sin 3x - \sin 17x = \sum_{n=1}^{\infty} \alpha_n \sin nx$$

and

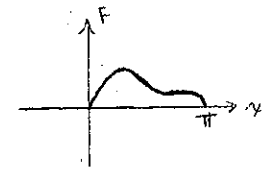
$$-9 \sin x + 13 \sin 973x = \sum_{n=1}^{\infty} nc\beta_n \sin nx .$$

By matching again, we find $\alpha_3 = \frac{1}{2}$, $\alpha_{17} = -1$, and $\alpha_n = 0$ for $n \neq 3$ or 17. Also, $\beta_1 = \frac{9}{c}$, $\beta_{973} = \frac{13}{973c}$, and $\beta_n = 0$ for $n \neq 1$ or 973. The (unique) solution is then a sum of four waves

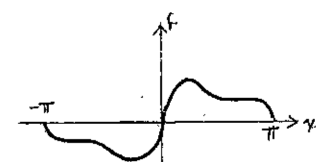
$$u(x, t) = -\frac{9}{3} \sin ct \sin x + \frac{1}{2} \cos 3\,ct \sin 3x$$

$$- \cos 17ct \sin 17x + \frac{13}{973c} \sin 973ct \sin 973x .$$

Since f and g are not usually given in the simple form of these example, the full Fourier series is needed. Recall that the string is pinned down at both ends. Therefore both the initial position function $f(x)$ and velocity function $g(x)$ have the property $f(0) = f(\pi) = 0$, and $g(0) = g(\pi) = 0$, where we have taken the length of the string to be π. It is now possible to extend both f and g, assumed continuous in $[0, \pi]$, to the whole interval $[-\pi, \pi]$ as continuous odd functions,



f before extension



f after extension as an odd function

that is, if $x \in [0, \pi]$, we can define

$$f(-x) = -f(x) \quad \text{and} \quad g(-x) = -g(x),$$

since the right sides, $-f(x)$ and $-g(x)$, are known functions for $x \in [0, \pi]$.

As odd functions now on the whole interval $[-\pi, \pi]$, the functions f and g have Fourier sine series (cf. p. 252, Exercise 3a).

$$f(x) = \sum_{n=1}^{\infty} b_n \frac{\sin nx}{\sqrt{\pi}}$$

$$g(x) = \sum_{n=1}^{\infty} \tilde{b}_n \frac{\sin nx}{\sqrt{\pi}}$$

where

$$b_n = 2\int_0^{\pi} f(x) \frac{\sin nx}{\sqrt{\pi}}\, dx, \quad \tilde{b}_n = 2\int_0^{\pi} g(x) \frac{\sin nx}{\sqrt{\pi}}\, dx \tag{4}$$

Comparing with the previous formulas (3) for f and g, we find

$$\alpha_n = b_n/\sqrt{\pi}, \quad \text{and} \quad \beta_n = \tilde{b}_n/nc\sqrt{\pi}$$

Consequently

$$u(x, t) = \sum_{n=1}^{\infty} \left(b_n \frac{\cos nct}{\sqrt{\pi}} + \frac{\tilde{b}_n}{nc} \frac{\sin nct}{\sqrt{\pi}}\right) \sin nx \tag{5}$$

the coefficients b_n and $\tilde{b}_n$ being determined from the initial conditions by equation (4).

Thus, we have almost proved

<u>Theorem 10.</u> If $f(x)$ is twice continuously differentiable and $g(x)$ once continuously differentiable for $x \in [0, \pi]$ and both functions vanish at $x = 0$ and $x = \pi$, then the function $u(x, t)$ defined by equation (5) is a solution of the homogeneous wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

and satisfies the

initial conditions: $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$, $x \in [0, \pi]$,

as well as the

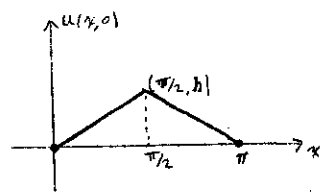
boundary conditions: $u(0, t) = 0$, $u(\pi, t) = 0$, $t \geq 0$,

where b_n and $\tilde{b}_n$ are determined from f and g through equations (4). Moreover, this solution is unique (by Theorem 9).

<u>Outline of Proof.</u> If it is possible to differentiate the infinite series (5) term by term, $u(x, t)$ would satisfy the wave equation since each special solution $u_n(x, t)$ does. In any case, the initial condition $u(x, 0) = f(x)$ is clearly satisfied. However, checking the other initial condition $u_t(x, 0) = g(x)$ also involves differentiating the infinite series term by term.

Thus, we must only justify the term by term differentiation of an infinite Fourier series. For power series, we found (p. 82-3, Theorem 16) we can always differentiate term by term within its disc of convergence. Such is not the case with Fourier series. For example, the Fourier series $\sum_{n=1}^{\infty} \frac{\sin n^2 x}{n^2}$ converges for all x, but the series obtain by differentiating formally, $\sum_{n=1}^{\infty} \cos n^2 x$ diverges at $x = 0$. However, if a function is sufficiently smooth, its Fourier series can be differentiated term by term and does converge to the derivative of the function. Since the details of a complete proof are but a rehash of the proof carried out for power series (p. 82 ff), we omit it.

Example. Find the displacement $u(x, t)$ of a violin string of length π with fixed end points which is plucked at its midpoint to height h. The initial position is then



$$f(x) = \begin{cases} xh, & x \in [0, \pi/2] \\ (\pi - x)h, & x \in [\pi/2, \pi], \end{cases}$$

and the initial velocity, $g(x)$, is zero.

We must find the coefficients b_n and $\tilde{b}_n$ in the series (5). After mentally continui ng f and g to the interval $[-\pi, \pi]$ as odd

functions, the formulas (4) give us b_n and $\tilde{b}_n$,

$$b_n = 2\int_0^{\pi} f(x)\frac{\sin nx}{\sqrt{\pi}}\,dx = \frac{2h}{\sqrt{\pi}}\left\{\int_0^{\pi/2} x \sin nx\,dx + \int_{\pi/2}^{\pi} (\pi - x)\sin nx\,dx\right\}.$$

Integrating and simplifying, we find that

$$b_n = \frac{4h}{\sqrt{\pi}\,n^2}\sin\frac{n\pi}{2} = \begin{cases} 0, & n \text{ even} \\ 1, & n = 1, 5, 9, 13, \ldots \\ -1, & n = 3, 7, 11, 15 \end{cases}.$$

From $g(x) \equiv 0$, it is immediate that $\beta_n = 0$ for all n. Thus,

$$u(x, t) = \frac{4h}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\cos n\,ct\,\sin nx$$

$$= \frac{4h}{\pi}\left[\frac{\cos 3ct \sin x}{1} - \frac{\cos ct \sin 3x}{3^2} + \frac{\cos 5ct \sin 5x}{5^2} + \ldots\right]$$

is the desired solution.

Exercises.

1. a) Find a solution $u(x, t)$ of the homogeneous wave equation for a string of length π whose end points are held fixed if the initial position function is

$$u(x, 0) = \frac{1}{2}\sin 4x - \sin 7x,$$

while the initial velocity is

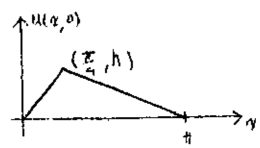
$$u_t(x, 0) = \sin 3x + \sin 73x.$$

b). Same problem as a), but

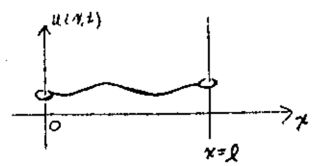
$$u(x, 0) = \sin 5x + 12 \sin 6x - 7 \sin 9x$$

$$u_t(x, 0) = -\sin x + 91 \sin 273x .$$

2. Find a solution $u(x, t)$ of the homogeneous wave equation for a string of length π whose end points are held fixed if the string is initially plucked at the point $x = \pi/4$ to the height h.



3. Consider a vibrating string of length ℓ whose end points are on rings which can slide freely on poles at 0 and ℓ. Then the boundary conditions at the end points are



$$u_x(0,t) = 0, \quad u_x(\ell, t) = 0$$

that is, zero slope.

a). Use the method of separation of variables to find the form of special standing wave solutions.

[Answer: $u_n(x, t) = \cos \frac{n\pi x}{\ell} \left(\alpha_n \cos \frac{nc\pi t}{\ell} + \beta_n \sin \frac{nc\pi t}{\ell}\right)$].

b). Use these to find a solution with the initial conditions

$$u(x, 0) = \cos x - 6 \cos 3x \qquad (\text{let } \ell = \pi)$$

$$u_t(x, 0) = \frac{1}{2} \cos 2x .$$

4. Let $u(x, t)$ satisfy the homogeneous wave equation. Instead of keeping the end points fixed, we either put them on rings (cf. Exercise 3) or attach them by elastic bands, in which case the boundary conditions become

$$u_x(0, t) - c_1 u(0, t) = 0, \quad u_x(\pi, t) + c_2 u(\pi, t) = 0, \quad c_1, c_2 \geq 0.$$

a). Define the energy as before, and prove that energy is dissipated with these boundary conditions, unless c_1 and c_2 vanish.

b). Prove there is at most one function $u(x, t)$ which satisfies the inhomogeneous wave equation $u_{tt} - c^2 u_{xx} = F(x, t)$ with initial conditions as before, but with elastic boundary conditions

$$u_x(0, t) - c_1 u(0, t) = \phi(t), \quad u_x(\pi, t) + c_2 u(\pi, t) = \psi(t),$$

where c_1 and c_2 are non-negative constants.

5. To account for the effect of air resistance on a vibrating string, one common assumption is that the resistance on a segment of length Δx is proportional to the velocity of its center of gravity,

$$F_{res} = -k \Delta x \, u_t(\tilde{x}, t), \quad k > 0,$$

where k is a numerical constant. This is analogous to the standard viscous resistance force on a harmonic oscillator.

a). Find the equation of motion ignoring gravity.

[Answer: $\frac{1}{c^2} u_{tt} + ku_t = u_{xx}$]

b). Find the form of the special standing wave solutions, assuming, the end points are held fixed.

c). Write a formula giving the probable form for the general solution $u(x, t)$.

d). If the end points are pinned down, what do you expect the behavior of the string will be as $t \to \infty$? Does the formula found in part c) verify your belief (it should).

e). Define the energy $E(t)$ as before and show that energy is dissipated if the ends are held fixed.

f). Use the result of e) to prove $\dot{E}(t) + 2kE(t) \geq 0$, and conclude that $E(t) \geq E(0)e^{-2kt}$ for $t \geq 0$. This shows that the energy is not dissipated too rapidly.

6. It is possible to write the solution of the homogeneous wave equation for a string of length π with fixed end points in a simple closed form by using the trigonometric identities

$$2 \sin nx \cos nct = \sin n(x-ct) + \sin n(x+ct).$$
$$2 \sin nx \sin nct = \sin n(x-ct) - \cos n(x+ct).$$

a). Do this and obtain <u>d'Alembert's</u> <u>formula</u>

$$u(x, t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)d\xi .$$

b). Solve the example of a plucked string (p. 641) again using this formula. Draw two sketches, one indicating the position of the string at time $t = \frac{\pi}{2c}$ and another at $t = \frac{\pi}{c}$.

7. a). Prove the wave operator $L \equiv \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}$, c a constant, is translation invariant, that is, if $T : u(x, t) \longrightarrow u(x + x_0, t + t_0)$, prove $(LT)u = (TL)u$ for all values of x_0 and t_0, and for all functions u for which the operators make sense.

b). Find the function $\phi(a, b)$ in the formula

$$Le^{ax+bt} = \phi(a, b)\, e^{ax+bt} .$$

c). Use part b) to show that if a is any constant, the four functions

$$e^{a(x+ct)},\ e^{-a(x+ct)},\ e^{a(x-ct)},\ e^{-a(x-ct)}$$

are solutions of the homogeneous wave equation $Lu = 0$.

d). Use the fact that each of the above functions satisfies the ordinary differential equation $v''(x) = a^2 v(x)$ to conclude that if linear combinations of these d). Use the method of separation of variables to find an infinite number of special solutions of the heat equation for a thin rod whose end points have zero temperature for all $t \geq 0$. [Answer: $u_n(x, t) = c_n e^{\frac{-n^2k^2\pi^2}{\ell^2}t} \sin \frac{n\pi}{\ell} x$, $n = 1, 2, \ldots$]

e). If the ends of a rod have zero temperature for all $t \geq 0$, what do you intuitively expect the temperature $u(x, t)$ will be as $t \longrightarrow \infty$? Is this born out by the formulas for the special solutions?

f). Find the temperature distribution in a rod of length π if the ends have zero temperature and if the initial temperature distribution in the rod is

$$u(x, 0) = \sin x - 4 \sin 7x,$$

10 . If the temperature at the ends of the bar of length ℓ is constant but not necessarily zero, say

$$u(0, t) = \theta_1, \quad u(\ell, t) = \theta_2,$$

the temperature distribution can be found by splitting the solution into two parts, $u(x, t) = \tilde{u}(x, t) + u_p(x, t)$, where $u_p(x, t)$ is a particular solution having the correct temperatures at the ends of the bar and $u(x, t)$ is a general solution which has zero temperature at the ends.

a). Find a particular solution of the homogeneous heat equation $u_t = k^2 u_{xx}$ which satisfies $u(0, t) = 20^\circ$, $u(\ell, t) = 50^\circ$, but does not necessarily satisfy any prescribed initial condition. [Answer: Many possible solutions - for example $u_p(x, t) = 20 + 30 \frac{x}{\ell}$, or $u_p(x, t) = 20 + 30 \sin \frac{\pi x}{2\ell}$].

b). Find the temperature distribution in a rod of length π if the initial temperature is $u(x, 0) = 2 \sin x - \sin 4x$, while the boundary conditions are as in part a).

11 . If the ends of a bar of length ℓ are insulated instead of being kept at zero, the boundary conditions are

$$u_x(0, t) = u_x(\ell, t) = 0.$$

a). Use the method of separation of variables to find an infinite number of special solutions for the homogeneous heat equation with insulated ends. [Answer: $u_n(x, t) = c_n e^{\frac{-n^2k^2\pi^2}{\ell^2}t} \cos \frac{n\pi x}{\ell}$, $n = 0, 1, 2, \ldots$].

b). What is the temperature distribution in a rod whose ends are insulated if the initial temperature distribution is

$$u(x, t) = 3 \cos \frac{2\pi x}{\ell} - \frac{1}{5} \cos \frac{5\pi x}{\ell}.$$

12. In this exercise you will find a quantative estimate for the rate of decrease of energy for the heat in a rod of length ℓ with zero temperature at the ends.

a). Use the result of Exercise 9a to prove the differential inequality

$$\frac{dE}{dt} \leq -cE(t),$$

where c is a positive constant. [Hint: Look at p. 227 Exercise 15c].

b). Conclude that

$$E(t) \leq E(0)e^{-ct}, \quad t \geq 0.$$

This is the desired estimate for the decrease of energy in the rod.

13. The linear partial differential equation

$$u_{xx} - u = u_t$$

governs the temperature distribution in a rod of length ℓ made up of a material which uses up heat to carry out a chemical process. Define the energy $E(t)$ in the rod as in Exercise 9.

a). Prove that if the ends of the rod have zero temperature, then the energy is dissipated, $\dot{E}(t) \leq 0$.

b). Given a rod whose ends have zero temperature and whose initial temperature $u(x, 0)$ is zero, use a) to prove that the temperature remains zero, $u(x, t) \equiv 0$, $t \geq 0$.

c). Use part b) to prove that the temperature of the rod described above is uniquely determined if the following three data are known

$$u(x, 0) \text{ for } x \in [0, \ell], \; u(0, t) \text{ and } u(\ell, t) \text{ for } t \geq 0.$$

14. In setting up the mathematical model for the vibrating string, we never examined the horizontal components of the forces.

a). Show that the net horizontal force is

$$F_h = \tau \cos \theta_2 - \tau \cos \theta_1$$

b). Under our assumption u_x is small, show that the net horizontal force is zero - so there is no horizontal motion of the string. This justifies the statement that the motion of the string is entirely vertical.

15. Use the formula $\nu_n = n\pi c/\ell$ (page 635) for the frequency and the relationship between c, T and ρ (page 624) to derive a formula for ν_n in terms of the physical constants ℓ, T, and ρ for a vibrating string. Interpret the effect on the frequency, ν_n, if the physical constants are changed. Does this agree with your experience in tuning stringed instruments?