Math 425 **Exam 1** Jerry L. Kazdan

March 3, 2011 12:00 – 1:20

DIRECTIONS This exam has three parts, Part A, short answer, has 1 problem (12 points). Part B has 5 shorter problems (7 points each, so 35 points). Part C has 3 traditional problems (15 points each so 45 points). Total is 92 points.
Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides.

Part A: Short Answer (1 problems, 12 points).

1. Let S and T be linear spaces and $A : S \to T$ be a linear map. Say $\mathbf{V}$ and $\mathbf{W}$ are particular solutions of the equations $A\mathbf{V} = \mathbf{Y}_1$ and $A\mathbf{W} = \mathbf{Y}_2$, respectively, while $\mathbf{Z} \neq 0$ is a solution of the homogeneous equation $A\mathbf{Z} = 0$.

   Answer the following in terms of $\mathbf{V}$, $\mathbf{W}$, and $\mathbf{Z}$.

   a) Find some solution of $A\mathbf{X} = 3\mathbf{Y}_1$.
   b) Find some solution of $A\mathbf{X} = -5\mathbf{Y}_2$.
   c) Find some solution of $A\mathbf{X} = 3\mathbf{Y}_1 - 5\mathbf{Y}_2$.
   d) Find another solution (other than $\mathbf{Z}$ and 0) of the homogeneous equation $A\mathbf{X} = 0$.
   e) Find *two* solutions of $A\mathbf{X} = \mathbf{Y}_1$.
   f) Find another solution of $A\mathbf{X} = 3\mathbf{Y}_1 - 5\mathbf{Y}_2$.

Part B: Short Problems (5 problems, 7 points each so 35 points)

B–1. $\mathbf{U} = (1, 1, 0, 1)$ and $\mathbf{V} = (-1, 2, 1, -1)$ are orthogonal vectors in R^4.

Write the vector $\mathbf{X} = (1, 1, 1, 2)$ in the form $\mathbf{X} = a\mathbf{U} + b\mathbf{V} + \mathbf{W}$, where a, b are scalars and $\mathbf{W}$ is a vector perpendicular to $\mathbf{U}$ and $\mathbf{V}$.

B–2. Find $u(x, t)$ that satisfies $u_x - 2u_t = 1$ with $u(x, 0) = 0$.

B–3. Let $u(x, t)$ be a solution of the wave equation

$$u_{tt} = 4u_{xx}, \quad \text{for} \quad -\infty < x < \infty,\ t \geq 0,$$

with the (continuous) initial conditions

$$u(x, 0) = f(x), \qquad u_t(x, 0) = g(x).$$

Find the largest interval $J = \{a \leq x \leq b\}$ where changing $f(x)$ or $g(x)$ at any point of J can change ("influence") the value of $u(0, 3)$. In other words, in the (x, t) plane, find all the points on the x-axis that are in the domain of dependence of $(0,\ 3)$.

B–4. Find the general solution $u(x, y)$ of $u_{xy} = 4y$.

B–5. Let $u(x, y)$ and $v(x, y)$ be a solutions of the Laplace equation $\Delta u = 0$, $\Delta v = 0$ in a bounded region Ω in the plane. If $u > v$ on the boundary of Ω, what, if anything, can you conclude about the relationship between u and v inside Ω? Justify your assertion.

Part C: Traditional Problems (3 problems, 15 points each so 45 points)

C–1. Find the motion $u(x, t)$ of a clamped string $\{0 \le x \le \pi\}$

$$u_{tt} = u_{xx},$$

with initial and boundary conditions:

$$u(x, 0) = 0, \quad u_t(x, 0) = 15 \sin 5x, \quad \text{and} \quad u(0, t) = u(\pi, t) = 0.$$

C–2. Let $u(x, y)$ satisfy $\Delta u - u = 0$ in a bounded region $\Omega \subset \mathbb{R}^2$ with $u(x, y) = 0$ on the boundary of Ω. Use Green's identity to show that $u(x, y) = 0$ throughout Ω.

C–3. Let $u(x, t)$ be the temperature of a rod of length L that satisfies

$$u_t = u_{xx} - ru \qquad \text{for } 0 < x < L, \quad t > 0,$$

where $r > 0$ is a constant [this is related to the heat equation but assumes that heat radiates out into the air along the rod]. Assume u satisfies the initial condition $u(x, 0) = f(x)$.

Define the total heat energy by $E(t) = \frac{1}{2} \int_0^L u^2(x, t)\, dx$.

a) If u also satisfies the Dirichlet boundary conditions

$$u(0, t) = 0, \qquad u(L, t) = 0$$

(the ends of the rod are held at temperature 0), show that $E(t)$ is a decreasing function of t.

b) Show that even if u satisfies Neumann boundary conditions

$$u_x(0, t) = 0, \qquad u_x(L, t) = 0$$

(the ends of the rod are insulated), $E(t)$ is still a decreasing function of t.

c) [Extra credit!] Show that in either of the above cases $\lim_{t \to \infty} E(t) = 0$.