Math 312, Spring 2013 Jerry L. Kazdan

Problem Set 8

DUE: In class Thursday, Apr. 4 *Late papers will be accepted until 1:00 PM Friday.*

1. Complex numbers, $z = x+iy$, can be represented perfectly as 2×2 using the observation that $J := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ has the property that $J^2 = -I$ (geometrically, J represents a rotation by $\pi/2$). We represent the complex number $z = x + iy$ as the 2×2 matrix

$$Z = xI + yJ = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

   a) If $W = uI + vJ$, where u and v are real numbers, show that complex multiplication of these special matrices is commutative: $ZW = WZ$.

   b) If $Z \neq 0$, show that Z is invertible. Compute Z^{-1} and verify that the result agrees with the usual formula for $1/z$.

2. Say a square matrix C has the property that $C^3 - C = 0$. What are the possible eigenvalues of C? Justify your answer.

3. For which real numbers a and b can the matrix $M := \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}$ be diagonalized? Justify your response.

4. [Bretscher Sec. 7.2 #32] Consider the matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ k & 3 & 0 \end{pmatrix}$, where k is an arbitrary real number. For which values of k does A have three real eigenvalues? [Suggestion: Graph the characteristic polynomial.]

5. Find the eigenvalues and eigenvectors of $B := \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

6. A certain real 4×4 matrix A has $\lambda_1 = 2 - 5i$ and $\lambda_2 = 1 + 2i$ as eigenvalues. What are the other two eigenvalues? Can you diagonalize A? Why or why not?

7. [Bretscher Sec. 7.3 #40, 41, 44] Let A and B be $n \times n$ matrices.

   a) Show that $\operatorname{trace}(AB) = \operatorname{trace}(BA)$.

   b) Use this to give another proof that if A and C are similar, then $\operatorname{trace}(A) = \operatorname{trace}(C)$.

c) Are there $n \times n$ matrices so that $AB - BA = I$?

8. [Bretscher (5^{th} edition, Sec. 7.4 #30a] Sketch the phase portrait for the dynamical system $\vec{x}(t+1) = A\vec{x}(t)$ where $A := \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$.

9. Multinational companies in the Americas, Asia, and Europe have assets of $4 trillion. At the start, $2 trillion are in the Americas and $2 trillion are in Europe. Each year 1/2 of the Americas money stays home and 1/4 goes to each of Asia and Europe. For Asia and Europe, 1/2 stays home and 1/2 is sent to the Americas.

   a) Let C_k be the column vector with the assets of the Americas, Asia, and Europe at the beginning of year k. Find the transition matrix T that gives the amount in year $k+1$: $C_{k+1} = TC_k$

   b) Find the eigenvalues and eigenvectors of T.

   c) Find the limiting distribution of the $4 trillion as the world ends

   d) Find the distribution of the $4 trillion at year k.

10. Let A and B be $n \times n$ matrices that can both be diagonalized by the *same* matrix S, so $A = SD_1S^{-1}$ and $B = SD_2S^{-1}$, where D_1 and D_2 are both diagonal matrices. Show that $AB = BA$.

11. Let $A := \begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix}$ and let $\vec{x}(t) := \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$. Solve the system of second order differential equations

$$\frac{d^2\vec{x}(t)}{dt^2} = A\vec{x}(t)$$

with the initial conditions $\vec{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\vec{x}\,'(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

[REMARK: This problem assumes you know how to solve scalar ordinary differential equations like $u'' + 25u = 0$ and $u'' - 25u = 0$. Review your Nath 240 text.]

12. If M is a square matrix, define e^M by the power series

$$e^M = I + M + \frac{M^2}{2!} + \cdots + \frac{M^k}{k!} + \cdots.$$

We will take the convergence of this series for granted (it is not difficult – but we skip this).

   a) If $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$, compute e^A.

b) For real t show that

$$e^{\left(\begin{smallmatrix} 0 & -t \\ t & 0 \end{smallmatrix}\right)} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

(The matrix on the right is a rotation of $\mathbb{R}^2$ through the angle t).

c) $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$, compute e^A. [Hint: diagonalize A.]

d) If A does not depend on t, show that $\dfrac{de^{At}}{dt} = Ae^{At}$.

e) If A is a diagonalizable constant square matrix, show that the solution of $\dfrac{d\vec{x}(t)}{dt} = A\vec{x}(t)$ with initial condition $\vec{x}(0) = \vec{b}$ is $\vec{x}(t) = e^{At}\vec{b}$.

Bonus Problem

[Please give this directly to Professor Kazdan]

1-B Let A be an $n \times n$ matrix all of whose elements are 1 (as in Problem Set 7 #5) and let $L := I + A$.

a) Why is L invertible?

b) Find and explicit formula for L^{-1}. [Suggestion: Let $\vec{v}$ be a column vector of all 1's and note that $\vec{v}$ is a basis for the image of A. Thus $A\vec{x} = c\vec{v}$ for some scalar c that depends on $\vec{x}$. But if $L\vec{x} = \vec{y}$, then $\vec{x} = \vec{y} - A\vec{x} = \vec{y} - c\vec{v}$ so all you need to do is find the scalar c.]

[Last revised: May 5, 2013]