Problem Set 7

DUE: In class Thursday, Mar. 28 *Late papers will be accepted until 1:00 PM Friday.*

1. [Bretscher (5^{th} edition, Sec. 5.5 #39] The following table lists the estimated number of genes and the estimated number of cell types for various organisms:

Organism	**Number of Genes,** g	**Number of Cell Types**
Humans	600,000	250
Annelid worms	200,000	60
Jellyfish	60,000	25
Sponges	10,000	12
Yeasts	2,500	5

   a) Fit a function of the form $\log z = c_0 + c_1 \log g$ to the data points $(\log g_i, \log z_i)$, using least squares.

   b) Use this to fit a *power function* $z = kg^n$ to the data points (g_i, z_i).

2. Say $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map with the property that $A^2 - 3A + 2I = 0$. If $\vec{v} \neq 0$ is an eigenvector of A with eigenvalue λ, so $A\vec{v} = \lambda\vec{v}$, what are the possible values of λ?

3. Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and corresponding eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$. Say $\vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$.

   a) Compute $A\vec{x}$, $A^2\vec{x}$, and $A^3\vec{x}$ in terms of the c_i, λ_i and $\vec{v}_i$, $i = 1, \ldots, n$.

   b) If $\lambda_1 = 1$ and the remaining λ_j satisfy $|\lambda_j| < 1$, $j = 2, \ldots, n$, compute $\lim_{k\to\infty} A^k\vec{x}$. [This arises in the study of *Markov Chains*].

4. [Bretscher Sec. 7.1 #68 (=#50 in the 4th Edition)] Two interacting populations of hares and foxes can be modeled by the recursive equations

$$h(t+1) = 4h(t) - 2f(t)$$
$$f(t+1) = h(t) + f(t).$$

   For each of the initial populations given in parts a)-c) below, find closed formulas for $h(t)$ and $f(t)$.

   a) $h(0) = f(0) = 100$.

   b) $h(0) = 200$, $f(0) = 100$.

   c) $h(0) = 600$, $f(0) = 500$.

5. Let $A := \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$ and $B := \begin{pmatrix} 3 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 3 \end{pmatrix}$.

   a) Compute the dimension of the image of A and of the kernel of A.
   b) Find a basis for the image of A and the kernel of A.
   c) Compute the trace of A.
   d) Find the eigenvalues and eigenvectors of A
   e) Find the trace, determinant, eigenvalues and eigenvectors of B.
   f) Is B invertible? If so, compute B^{-1}.

6. Combine the rank-nullity Theorem 3.3.7 with Theorem 5.4.1, which says $(\operatorname{im} A)^{\perp} = \ker(A^*)$, to show that $\operatorname{rank} A = \operatorname{rank} A^*$, that is, $\dim \operatorname{im}(A) = \dim \operatorname{im}(A^*)$.

7. Let $A : \mathbb{R}^k \to \mathbb{R}^n$ be a linear map. Show that $\dim \ker A - \dim \ker A^* = k - n$.

Bonus Problem

[Please give this directly to Professor Kazdan]

1-B Let U, V, and W be finite dimensional vector spaces with inner products. If $A : U \to V$ and $B : V \to W$ are linear maps with adjoints A^* and B^*, define the linear map $C : V \to V$ by

$$C = AA^* + B^*B.$$

If the sequence of maps $U \xrightarrow{A} V \xrightarrow{B} W$ has the property $\operatorname{image}(A) = \ker(B)$, show that C maps V to V and that it is invertible. [SUGGESTION: Use Theorem 5.4.1]

[Last revised: May 5, 2013]