Problem Set 2

DUE: In class Thursday, Jan. 24 *Late papers will be accepted until 1:00 PM Friday.*

Lots of problems. Most are really short.

In addition to the problems below, you should also know how to solve *all* of the problems in Chapters 1 and 2 of the text, particularly those at the beginning of each of the problem sets for each section. Most are simple mental exercises.

1. Consider the system of equations

$$\begin{aligned} x + y - \ z &= a \\ x - y + 2z &= b \\ 3x + y \quad &= c \end{aligned}$$

   a) Find the general solution of the homogeneous equation.

   b) If $a = 1$, $b = 2$, and $c = 4$, then a particular solution of the inhomogeneous equations is $x = 1$, $y = 1$, $z = 1$. Find the most general solution of these inhomogeneous equations.

   c) If $a = 1$, $b = 2$, and $c = 3$, show these equations have *no* solution.

2. [Bretscher, Sec.2.2 #10] Let $\mathcal{L}$ be the line in $\mathbb{R}^2$ that consists of all scalar multiples of the vector $(4, 3)$. Find the matrix of the orthogonal projection onto this line $\mathcal{L}$.

3. [Bretscher, Sec.2.2 #17] Let $A := \left(\begin{smallmatrix} a & b \\ b & -a \end{smallmatrix}\right)$, where $a^2 + b^2 = 1$. Find two perpendicular non-zero vectors $\vec{v}$ and $\vec{w}$ so that $A\vec{v} = \vec{v}$ and $A\vec{w} = -\vec{w}$ (write the entries of $\vec{v}$ and $\vec{w}$ in terms of a and b). Conclude that thinking of A as a linear map it is an orthogonal reflection across the line $\mathcal{L}$ spanned by $\vec{v}$.

4. [Bretscher, Sec.2.2 #31] Find a nonzero 3×3 matrix A so that $A\vec{x}$ is perpendicular to $\vec{v} := (1, 2, 3)$ for all vectors $\vec{x} \in \mathbb{R}^3$. [There are many examples. You are only asked to find one example.]

5. [Deleted. This was already on Homework Set 1]

6. [Bretscher, Sec.2.3 #48]

   a) If $A := \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ and $B := \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$, compute AB and A^{10}.

   b) Find a 2×2 matrix A so that $A^{10} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

7. Which of the following sets are linear spaces?

a) $\{\vec{x} = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ with the property $x_1 - 2x_3 = 0\}$

b) The set of solutions x of $Ax = 0$, where A is an $m \times n$ matrix.

c) The set of polynomials $p(x)$ with $\int_{-1}^{1} p(x)\,dx = 0$.

d) The set of solutions $y = y(t)$ of $y'' + 4y' + y = 0$ (you are *not* being asked to actually find these solutions).

8. Proof or counterexample. In these L is a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$, so its representation will be as a 2×2 matrix.

   a) If L is invertible, then L^{-1} is also invertible.

   b) If $L\vec{v} = 5\vec{v}$ for all vectors $\vec{v}$ and $M\vec{w} = (1/5)\vec{w}$ for all vectors $\vec{w}$, then L is invertible and $L^{-1} = M$.

   c) If L is a rotation of the plane by 45 degrees *counterclockwise*, then L^{-1} is a rotation by 45 degrees *clockwise*.

   d) If L is a rotation of the plane by 45 degrees counterclockwise, then L^{-1} is a rotation by 315 degrees counterclockwise.

   e) The zero map ($0\vec{v} := 0$ for all vectors $\vec{v}$) is invertible.

   f) The identity map ($I\vec{v} := \vec{v}$ for all vectors $\vec{v}$) is invertible.

   g) If L is invertible, then $L^{-1}0 = 0$.

   h) If $L\vec{v} = 0$ for some non-zero vector $\vec{v}$, then L is not invertible.

   i) The identity map (say from the plane to the plane) is the only linear map that is its own inverse: $L = L^{-1}$.

9. Think of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as mapping one plane to another.

   a) If two lines in the first plane are parallel, show that after being mapped by A they are also parallel – although they might coincide.

      [REMARK: The simplest way to describe a straight line in $\mathbb{R}^2$ (or even $\mathbb{R}^n$) that passes through two points, say P and Q, is to think of this line as the path $\vec{x}(t)$ of a particle at time t with $\vec{x}(t) = P + t\vec{v}$ where $\vec{v} = Q - P$ and $-\infty < t < \infty$. For any point $\hat{P}$ the line $\vec{y}(t) = \hat{P} + t\vec{w}$ is *parallel* to $\vec{x}(t)$ if $\vec{w} = c\vec{v}$ for some scalar c. Here we are assuming $\vec{v} \neq 0$ and $\vec{w} \neq 0$.]

   b) Let Q be the unit square: $0 < x < 1, 0 < y < 1$ and let Q' be its image under this map A. Give a geometric argument to show that the area$(Q') = |ad - bc|$. [For simplicity in your figure assume the points (a, c) and (b, d) are in the first quadrant with $a > b$ and $c < d$ anduse the rectangle with vertices at $(0, 0)$, $(a + b, 0)$, $(a + b, c + d)$, and $(0, c + d)$.] [More generally, the area of any region is magnified by $|ad - bc|$, which is called the *determinant* of A.]

10. Let A be a matrix, not necessarily square. Say $\vec{v}$ and $\vec{w}$ are particular solutions of the equations $A\vec{v} = \vec{y}_1$ and $A\vec{w} = \vec{y}_2$, respectively, while $\vec{z} \neq 0$ is a solution of the homogeneous equation $A\vec{z} = 0$. Answer the following in terms of $\vec{v}$, $\vec{w}$, and $\vec{z}$.

    a) Find some solution of $A\vec{x} = 3\vec{y}_1$.
    b) Find some solution of $A\vec{x} = -5\vec{y}_2$.
    c) Find some solution of $A\vec{x} = 3\vec{y}_1 - 5\vec{y}_2$.
    d) Find another solution (other than $\vec{z}$ and 0) of the homogeneous equation $A\vec{x} = 0$.
    e) Find *two* solutions of $A\vec{x} = \vec{y}_1$.
    f) Find another solution of $A\vec{x} = 3\vec{y}_1 - 5\vec{y}_2$.
    g) If A is *any* square matrix, for any given vector $\vec{w}$ can one always find at least one solution of $A\vec{x} = \vec{w}$? Why?

11. Let V be the linear space of smooth real-valued functions and $L : V \to V$ the linear map defined by $Lu := u'' + u$.

    a) Compute $L(e^{2x})$ and $L(x)$.
    b) Find particular solutions of the inhomogeneous equations

$$a).\ u'' + u = 7e^{2x}, \qquad b).\ w'' + w = 4x, \qquad c).\ z'' + z = 7e^{2x} - 3x$$

12. Let $A : \mathbb{R}^3 \to \mathbb{R}^2$ and $B : \mathbb{R}^2 \to \mathbb{R}^3$, so $BA : \mathbb{R}^3 \to \mathbb{R}^3$ and $AB : \mathbb{R}^2 \to \mathbb{R}^2$.

    a) Why must there be a non-zero vector $\vec{x} \in \mathbb{R}^3$ such that $A\vec{x} = 0$.
    b) Show that the 3×3 matrix BA can *not* be invertible.
    c) Give an example showing that the 2×2 matrix AB might be invertible.

[Last revised: May 5, 2013]