Final Exam: Wednesday, May 1, 12:00-2:00 in DRL A-1. Closed book, no calculators, but you may use one 3×5 card with notes on both sides.

Problem Set 10

DUE: **Tuesday, April 23** [*Late papers will be accepted until 1:00 on Wednesday*].

1. Let A be a positive definite (symmetric) matrix. Show there is a change of variables $\vec{y} = P\vec{x}$, where P is an invertible symmetric matrix, so that $\langle \vec{x}, A\vec{x} \rangle = \langle \vec{y}, \vec{y} \rangle$.

2. a) In $\mathbb{R}^2$, for which values of the constant c is the line $x_1 + 2x_2 = c$ tangent to the circle $x_1^2 + x_2^2 = 4$?
   b) In $\mathbb{R}^2$, for which values of the constant c is the line $x_1 + 2x_2 = c$ tangent to the ellipse $x_1^2 + 4x_2^2 = 4$?
   c) In $\mathbb{R}^2$, for which values of the constant c is the line $x_1 + 2x_2 = c$ tangent to the ellipse $x_1^2 + 4x_1x_2 + 6x_2^2 = 4$?
   d) Let A be a positive definite 2×2 matrix and $\vec{N}$ a unit vector in $\mathbb{R}^2$. For which values of the constant c is the line $\langle \vec{N}, \vec{x} \rangle = c$ tangent to the ellipse $\langle \vec{x}, A\vec{x} \rangle = 1$?

3. Compute
   a) $\displaystyle\iint_{\mathbb{R}^2} \frac{dx\,dy}{(1 + 4x^2 + 9y^2)^2}$.
   b) $\displaystyle\iint_{\mathbb{R}^2} \frac{dx\,dy}{(1 + x^2 + 2xy + 5y^2)^2}$.
   c) Compute $\displaystyle\iint_{\mathbb{R}^2} \frac{dx_1\,dx_2}{[1 + \langle x, Cx \rangle]^2}$, where C is a positive definite (symmetric) 2×2 matrix, and $x = (x_1, x_2) \in \mathbb{R}^2$.

4. a) Compute $\displaystyle\iint_{\mathbb{R}^2} e^{-(x^2 - 4xy + 5y^2)}\,dx\,dy$.
   b) Compute $\displaystyle\iint_{\mathbb{R}^2} e^{-(x^2 - 4xy + 5y^2 - 2x + 4y + 3)}\,dx\,dy$. [SUGGESTION: Use Homework Set 6 #9.]
   c) Let A be a positive definite 2×2 matrix and $\vec{b} \in \mathbb{R}^2$. Generalize the previous part to obtain a formula for
   $$\iint_{\mathbb{R}^2} e^{-(\langle \vec{x}, A\vec{x} \rangle + 2\langle \vec{b}, \vec{x} \rangle + c)}\,dx_1\,dx_2.$$

5. Let A be an $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$. Show that
   $$\langle \vec{x}, A\vec{x} \rangle \le \lambda_n \|\vec{x}\|^2 \quad \text{for any } \vec{x} \in \mathbb{R}^n.$$

6. [Bretscher, Sec. 8.3 #12] Find the singular value decomposition (SVD) of the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$.

7. Consider the matrix $\begin{pmatrix} 2 & -1 \\ 2 & 2 \end{pmatrix}$.

   a) If $\vec{x} \in \mathbb{R}^2$ is a unit vector, what is the largest $\|A\vec{x}\|$ can be?

   b) Find the SVD of A.

   c) Using part (b), find a rank one approximation to A

8. [Bretscher, Sec. 8.3 #24] If A is a symmetric $n \times n$ matrix, what is the relationship between the eigenvalues and singular values of A?

[Last revised: May 5, 2013]