# Fourier Series of $f(x) = x$

Given a real periodic function $f(x)$, $-\pi < x < \pi$, one can find its Fourier series in two (equivalent) ways: using trigonometric functions:

$$f(x) = \frac{a_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty}\left[a_k \frac{\cos kx}{\sqrt{\pi}} + b_k \frac{\sin kx}{\sqrt{\pi}}\right]$$

or using the complex exponential

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \frac{e^{ikx}}{\sqrt{2\pi}}.$$

Note that if $f(x)$ is a real-valued function, we can take the real part of the complex exponential version to get the trigonometric version (caution: the coefficiants $c_k$ will probably be complex numbers).

Here we will use complex exponentials. The Fourier coefficients are

$$c_k = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} x e^{-ikx}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} x[\cos kx - i\sin kx]\,dx = \frac{-2i}{\sqrt{2\pi}}\int_{0}^{\pi} x\sin kx\,dx$$

But

$$\int_0^{\pi} x\sin kx\,dx = \left.\frac{-x\cos kx}{k}\right|_0^{\pi} + \frac{1}{k}\int_0^{\pi}\cos kx\,dx = \frac{-\pi\cos k\pi}{k} = -\frac{\pi}{k}(-1)^k.$$

Thus

$$c_k = -\frac{2i}{\sqrt{2\pi}}\left[-\frac{\pi}{k}(-1)^k\right] = i\sqrt{2\pi}\left[\frac{(-1)^k}{k}\right].$$

Consequently

$$x = i\sqrt{2\pi}\sum_{k\neq 0}\frac{(-1)^k}{k}\frac{e^{ikx}}{\sqrt{2\pi}} = i\sum_{k\neq 0}\frac{(-1)^k}{k}e^{ikx}$$

$$= -2\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\sin kx = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right]$$

Finally we compute what the Phthagorean Theorem tells us: $\|x\|^2 = \sum |c_k|^2$. Since

$$\|x\|^2 = \int_{-\pi}^{\pi}|x|^2\,dx = \frac{2}{3}\pi^3,$$

and

$$\sum |c_k|^2 = 2\pi\left[\sum_{-\infty}^{-1}\frac{1}{k^2} + \sum_{1}^{\infty}\frac{1}{k^2}\right] = 4\pi\sum_{1}^{\infty}\frac{1}{k^2}$$

Therefore

$$\frac{2}{3}\pi^3 = 4\pi\sum_{1}^{\infty}\frac{1}{k^2}, \qquad \text{that is,} \qquad \sum_{1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}.$$

Interesting! – and not obvious at all.