Math 312 Jerry L. Kazdan

Orthogonal Projection

Let V be an inner product space (that is, a linear space with an inner product) and let $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ be non-zero orthogonal vectors and let $\mathcal{S}$ be the subspace spanned by these $\vec{x}_j$'s. Given a vector $\vec{y} \in V$, we want to write

$$\vec{y} = a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + \vec{x}_k + \vec{w}, \tag{1}$$

where $\vec{w}$ is orthogonal to $\mathcal{S}$. This decomposes $\vec{y}$ as the sum of two orthogonal vectors, one in $\mathcal{S}$ and one orthogonal to $\mathcal{S}$. We often introduce the linear map $P_{\mathcal{S}}$ of orthogonal projection into $\mathcal{S}$

$$P_{\mathcal{S}}\vec{y} := a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + \vec{x}_k.$$

If we write $\mathcal{S}^\perp$ for the orthogonal complement of $\mathcal{S}$, then $\vec{w} = P_{\mathcal{S}^\perp}\vec{y}$, so $\vec{y} = P_{\mathcal{S}}\vec{y} + P_{\mathcal{S}^\perp}$. The problem is to find the coefficients a_j and the vector $\vec{w}$. Taking the inner product of both sides of this with $\vec{x}_1$ we find that $\langle\vec{y}, \vec{x}_1\rangle = a_1\langle\vec{x}_1, \vec{x}_1\rangle$ and similarly for the other a_j's. Thus

$$a_j = \frac{\langle\vec{y}, \vec{x}_j\rangle}{\|\vec{x}_j\|^2}.$$

We can now solve equation (1) for $\vec{w}$ and find

$$\vec{w} = \vec{y} - [a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + \vec{x}_v].$$

Since the $\vec{x}_j$'s and $\vec{w}$ are orthogonal, the Pythagorean theorem applied to (1) tells us that

$$\|\vec{y}\|^2 = |a_1|^2\|\vec{x}_1\|^2 + \cdots + |a_k|^2\|\vec{x}_k\|^2 + \|\vec{w}\|^2.$$

In particular, $\|\vec{w}\|^2$ gives the square of the distance from $\vec{y}$ to the subspace $\mathcal{S}$.

Examples

1. Find the distance between the point $\vec{y} = (1, 2, -3, 0) \in \mathbb{R}^4$ and the subspace of points $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ that satisfy $x_1 - x_2 + x_3 + 2x_4 = 0$.

2. Find the distance between the hyperplane of points $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ that satisfy $x_1 - x_2 + x_3 + 2x_4 = 2$ and the origin.

3. Find an orthogonal basis for the subspace of $\mathbb{R}^4$ spanned by $\vec{u}_1 = (1, 1, 0, 0)$ and $\vec{u}_2 = (0, 1, 1, 0)$

4. Find a vector in $\mathbb{R}^4$ that is orthogonal to the subapace spanned by $\vec{u}_1 = (1, 1, 0, 0)$ and $\vec{u}_2 = (0, 1, 1, 0)$.

5. Find an orthogonal basis for the subspace of $\mathbb{R}^4$ spanned by $\vec{u}_1 = (1, 1, 0, 0)$, $\vec{u}_2 = (0, 1, 1, 0)$, and $\vec{u}_3 = (0, 0, 1, 1)$.

6. Find an orthonormal basis for the subapace of $\mathbb{R}^4$ determined by $x_1 - x_2 + x_3 - 2x_4 = 0$.

7. Find a vector that is orthogonal to the above subspace.

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